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STABLE EXTRAPOLATION OF SCATTERING AMPLITUDES USING UNITARITY

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A B S T R A C T

Unitarity in all three channels for the πN system is used to show that a) one can, given the first L πN partial waves and the πN total cross-sections, extrapolate to find the $\bar{N}N \rightarrow \pi\pi$ waves, the whole amplitude, and the πN waves, in the unphysical regions; and this within absolute error bands, and b) from either of those quantities, we construct Padé approximants which are then shown to converge to the cuts ($4\mu^2 \leq t \lesssim 2M^2$) again here within an explicit error band. It is shown that the method is stable under perturbations, giving also the explicit form in which such perturbations propagate. The analysis is carried first for amplitudes with positivity, and then extended to any amplitude. This paper is the first of a series, and only the theory is presented.

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1. INTRODUCTION

It is a well-known fact that, because of crossing symmetry, it is in principle possible to extract, from knowledge in a finite energy range of the collision matrix for πN scattering (say, the considerations of this article apply equally well to processes of the type πA , A being any elementary particle ^{*}), the collision matrix for the process $\bar{N}N \rightarrow \pi\pi$. It is also well known that, unfortunately, this problem is of the kind that the mathematicians call "ill-posed" in that one necessitates to extrapolate analytic functions, and, therefore, the solution is plagued by ambiguities, instabilities, etc. In spite of these difficulties, the extrapolation of πN to $\bar{N}N \rightarrow \pi\pi$ has been carried over using a variety of methods that shall be reviewed briefly in a moment.

A problem related to this extrapolation is the extrapolation of the πN scattering matrix from physical to unphysical energies and angles. With respect to unphysical angles, analyticity in the momentum transfer makes it possible to use the Legendre polynomial expansion and we shall not comment any further on this point. However, if one wants to extrapolate to unphysical energies in order to use the output to (for example) dispersion relations determinations of partial πN or $\bar{N}N \rightarrow \pi\pi$ waves, the above mentioned difficulties pop up again.

The methods that have been devised to perform the aforementioned extrapolations are of different nature. We review only a few of these, and refer to the articles of Donnachie et al. ¹⁾ and of Hamilton ²⁾ for a more complete survey and lists of references. A first type of methods exploits the fact that the t channel forces contribute to the left-hand singularities of the s channel partial waves. Therefore, partial wave dispersion relations for πN scattering yield information about the process $\bar{N}N \rightarrow \pi\pi$ [Hamilton, Donnachie, etc., cf., Ref. 1) for details and bibliography]. In fact, it is in this way that the first evidence leading to the replacement of the σ by the ϵ was gathered ¹⁾. This method, which in principle also gives the partial πN waves, $f_{\ell\pm}(s)$ in the unphysical region ^{**)}, has all the problems inherent to any extrapolation by fits: it is impossible to put reliable errors and one does not know to what extent the results are model dependent.

A second category of methods ³⁾ (the first in chronological order) uses rather dispersion relations for the $\bar{N}N \rightarrow \pi\pi$ waves, $g_J^{\pm}(t)$. These waves can be computed for $-28\mu^2 < t < 0$ through the u channel πN amplitude, and the rest fitted.

^{*}) All that is needed for this generalization is to extend the relation

$$\frac{\partial}{\partial z} [Q_{n+k}(z)/Q_n(z)] < 0, \quad z > 1, \quad \text{to the } Q_n^{(\alpha, \beta)}.$$

This is done in Appendix A.

^{**)} The notations we shall use are the standard ones [e.g., Refs. 1), 2)] in that s, u, t are the Mandelstam variables, μ and M the pion and nucleon mass, A, B the standard invariant amplitudes, etc.

The same objections can be made to this method as to the previous one. This method is supplemented at times with the information about the $g_J^\pm(t)$ obtained from the πN scattering amplitude using the Ball-Wang method ⁴⁾. [See also Ref. 5) for an up-to-date application of this.]

Thirdly, one can use backward dispersion relations ⁶⁾, and saturate the $\bar{N}N \rightarrow \pi\pi$ contribution with a few resonances, which one then fits ⁷⁾. Apart from diseases common to all methods so far described, this one presents the further disadvantage of giving the $\bar{N}N \rightarrow \pi\pi$ amplitude at one single value of the scattering angle, making the fits still less reliable.

Several improvements have been made on the above-mentioned methods. First, since, clearly the quantities one is trying to compute are the same, one can impose consistency of the various approaches. Then, one can use the so-called discrepancy function (first introduced by Hamilton and collaborators ⁸⁾), defined as the difference between the piece of the function coming from contributions from what one can compute, and the exact function. This discrepancy function is known in some regions, and the problem becomes that of extrapolating it. This is then treated with the help of the methods developed by Cutkosky, Ciulli, etc., ⁹⁾ of stable extrapolations. We shall not give more details here, but refer to the review by Hamilton ²⁾ and the very comprehensive article of Nielsen et al., ¹⁰⁾ for details.

In spite of the obvious progress that these methods represent, it is clear that there are some as yet unsolved problems. To quote only the ones that are relevant for us here, let me point out that a) the calculations of the input functions are still model dependent, b) partly as a consequence of this, the input errors (whose knowledge is essential in the extrapolation procedure) are also model dependent and very difficult to assess, c) finally, for all the merits of the extrapolation procedure ⁹⁾, one does not have any guarantee that the actual functions are sufficiently smooth to apply them. To be precise, let me give an example of this: the methods of stable extrapolation, say, roughly speaking, that, given the function $\Delta(z)$ in the interval $0 \leq z \leq 1$, there exists an extrapolating function, $\Delta_N(z)$, such that if small perturbations ϵ are applied to $\Delta(z)$ in $(0,1)$, then if $\Delta_{N,\epsilon}(z)$ is the corresponding approximant to $\Delta(z)$, as $\epsilon \rightarrow 0$, $\Delta_{N,\epsilon}(z) \rightarrow \Delta_N(z)$. But it is a priori impossible to know how fast this limit is reached. This is best exemplified by the way in which, for example, Nielsen et al. ¹⁰⁾ estimate their errors.

It is the purpose of this article, and of the ones that will follow it, to give partial solutions to these just quoted problems. The clinch is the systematic use of unitarity. It has been shown in several previous articles by various people ¹¹⁾⁻¹⁴⁾ that positivity puts strong constraints on partial waves in the unphysical region. It then turns out that similar methods can be employed to give estimates for such partial waves, and, what is more, to put absolute bounds on the errors one is committing. Once this is done, we are left with the problem of extrapolating these waves to physically interesting regions. In a general case, one would

have to resort to above described extrapolation procedures; however, suitable combinations of waves turn out to have sufficient positivity properties that much better methods, based on suitable generalizations of the techniques of the moment problem, can be applied. These methods do not only give extrapolations, but permit one to keep explicit control of the errors that are being committed: in the example discussed before, this means that one is able to find explicitly quantities B_N and $B_{N,\epsilon}$ such that

$$|\Delta_N(z) - \Delta(z)| < B_N, \quad |\Delta_{N,\epsilon}(z) - \Delta_N(z)| < B_{N,\epsilon}.$$

Certainly, these methods also have the corresponding drawbacks. First, the necessary positivity is only enjoyed by some combinations of the partial waves, say. This is not too big a problem because, at a later stage, we can show how to bound errors to the waves without positivity in terms of the waves with positivity. A more serious difficulty is that, as always happens in rigorous treatments, the bound for the errors are fairly larger than what one expects these errors to be in actual fact. Therefore, our methods are to be used in conjunction with the standard ones, Refs. 2), 10), their main practical interest being in that they give bands on which the actual solution has necessarily to lie, thereby facilitating enormously (or so, at least, is hoped) the conventional extrapolation procedures. Two last drawbacks of our analysis are that, first, we can say nothing about the t channel S waves, which, therefore, have to be taken as given by the standard methods, and second, in order to obtain our bounds it is necessary for us to know the t channel D wave. This is not too bad, as independent determinations of this quantity exist ^{14), 15)} and, since this quantity is only needed to put bounds on errors, indeterminacies in its value are practically irrelevant.

The general outline of our method is the following. Suppose we have at our disposal a partial wave analysis of πN , giving the first $L+1$ waves, up to the energy $\sqrt{s_0}$. For certain spin and isospin combinations ^{*}), one can write the Froissart-Gribov representation,

$$g_L(t) = \int_{(M+\mu)^2}^{\infty} ds (F_s + F_u) Q_L$$

$$L = \text{even} \geq 2, \tag{1.1}$$

for the $\bar{N}N \rightarrow \pi\pi$ waves in the range $0 \leq t \leq 4\mu^2$; here F_s/F_u are the absorptive parts of the πN scattering amplitude, F , in the s/u channels. Further, one can expand

^{*}) We shall in this sketchy discussion, disregard spin and isospin together with irrelevant kinematical factors. Also the nucleon pole will be supposed to be absent. Exact formulae will be found later in this paper.

$$F_s + F_u = \sum_0^{\infty} (2l+1) P_l(\cos \theta_s) \operatorname{Im} f_l(s),$$

$$\operatorname{Im} f_l(s) \geq 0. \quad (1.2)$$

We then define a "truncated" amplitude:

$$F_s(s, t; L, s_0) + F_u(s, t; L, s_0) = \begin{cases} \sum_0^L (2l+1) P_l(\cos \theta_s) \operatorname{Im} f_l(s), & s \leq s_0 \\ \sqrt{s} q_s [\sigma_{\text{tot}}(s) + \overline{\sigma}_{\text{tot}}(s)], & s > s_0 \end{cases} \quad (1.3)$$

(actually, we will much improve this simple formula) where $\sigma_{\text{tot}}/\overline{\sigma}_{\text{tot}}$ are the total πN cross-sections in the s/u channels, and the errors

$$\Delta_s F(s, t) = \begin{cases} 0, & s \leq s_0 \\ F_s(s, t) + F_u(s, t) - \sqrt{s} q_s [\sigma_{\text{tot}}(s) + \overline{\sigma}_{\text{tot}}(s)], & s > s_0; \end{cases} \quad (1.4a)$$

$$\Delta_L F(s, t) = \begin{cases} \sum_{L+1}^{\infty} (2l+1) P_l(\cos \theta_s) \operatorname{Im} f_l(s), & s \leq s_0 \\ 0, & s > s_0. \end{cases} \quad (1.4b)$$

Since $F_s(s, t=0) = q_s \sqrt{s} \sigma_{\text{tot}}(s)$, $\Delta_{s_0} F$ is finite. What is more, due to (1.2) both $\Delta_{s_0} F$ and $\Delta_L F$ are positive.

From this, one can define truncated waves, and the corresponding errors:

$$g_\ell(t; L, s_0) = \int_{(M+\mu)^2}^{\infty} ds \{ F_s(s, t; L, s_0) + F_u(s, t; L, s_0) \} Q_\ell, \quad (1.5)$$

$$\Delta_{s_0} g_\ell(t) = \int_{s_0}^{\infty} ds (\Delta_{s_0} F) Q_\ell \quad (1.6a)$$

$$\Delta_L g_\ell(t) = \int_{(M+\mu)^2}^{s_0} ds (\Delta_L F) Q_\ell. \quad (1.6b)$$

If, as is the case, we assume that the total cross-sections are given exactly, for $s < s_0$, by the truncated partial wave series, then $g_\ell(t; L, s_0)$ will coincide exactly with $g_\ell(t)$ at $t=0$; because of (1.2) it follows that also $(\partial/\partial t) \Delta_{s_0} F > 0$,

$(\partial/\partial t)\Delta_{\text{L}}^{\text{F}} > 0$: hence, the errors (1.6) will increase up to $t = 4\mu^2$. At this point, the errors will be due to a) the partial wave tail $\Delta_{\text{L}}^{\text{F}}$, which should be small because the series (1.2) should be convergent up to the border of Mandelstam's double spectral function, which, when $s < s_0$, is away from $t = 4\mu^2$; and b) the contribution of the difference $\Delta_{s_0}^{\text{F}}$, which is also expected to be very small because the Pomeranchuk slope is small [in units of μ^2 , $\alpha_{\text{P}}'(0) \lesssim 10^{-2}$] and the Q_{ℓ} acts as a very strong damping factor of the type $\ll s_0^{-\ell-1}$. The actual value of the errors can here be computed by remarking that the D wave scattering length which is supposed to be known is given by $\lim_{t \rightarrow 4\mu^2} g_2(t)/(t - 4\mu^2)$, while its truncated value will be

$$\lim_{t \rightarrow 4\mu^2} \frac{g_2(t)}{t - 4\mu^2} = \int_{(M+\mu)^2}^{\infty} ds \left\{ F_s(s, t=4\mu^2; L, s_0) + F_u(s, t=4\mu^2; L, s_0) \right\} \frac{1}{(s + \mu^2 - M^2)^3},$$

which can be computed so that one knows $\lim_{t \rightarrow 4\mu^2} \Delta_{\text{S}} g_2(t)/(t - 4\mu^2)$ and $\lim_{t \rightarrow 4\mu^2} \Delta_{\text{L}} g_2(t)/(t - 4\mu^2)$, from which, following the methods of Refs. 14) and 15), bounds can be found for all the $\Delta_{\text{S}} g_{\ell}$ and $\Delta_{\text{L}} g_{\ell}$.

It is clear that all these considerations are directly applicable only if we have the combination $F_{\text{S}} + F_{\text{U}}$ in Eq. (1.1). Therefore, they do not hold directly for, e.g., waves $g_{\ell}^{(1)}(t)$ corresponding to isospin one, for here the relevant combination is $F_{\text{S}} - F_{\text{U}}$. However, defining likewise truncated amplitudes, and errors, we shall have that

$$\begin{aligned} \Delta_{s_0} F^{(1)} &= \Delta_{s_0} F_s + \Delta_{s_0} F_u \\ \Delta_L F^{(1)} &= \Delta_L F_s + \Delta_L F_u \end{aligned}$$

so that, obviously, $|\Delta_{s_0} F^{(1)}| < \Delta_{s_0} F$, $|\Delta_L F^{(1)}| < \Delta_L F$, and thus, it follows that the errors for these waves will be at most equal to the errors for the waves with positivity.

These are already interesting results, but we can go much further. First, it is possible, using techniques like in Ref. 14), to sum up the t channel partial wave series, for $(M - \mu)^2 \leq s \leq (M + \mu)^2$, getting

$$F(s, t; L, s_0) = \sum (2\ell + 1) P_{\ell}(\cos \theta_t) g_{\ell}(t; L, s_0) \quad (1.7)$$

with known (and small!) errors, got by summing likewise the errors. (It is here that one has to suppose the S wave, which is not obtainable by our analysis, to be given to us.) This result can be already used to test consistency of the πN phase shifts

comparing it with the output of a direct dispersive calculation, in the range in which the last is likely to hold ¹⁶⁾ ($0 \leq t \leq 2\mu^2$). Perhaps it would be good to remark here that such a method cannot be substituted to ours because firstly, it is not likely to work much above $t \approx 2\mu^2$, and, mainly, because, to put errors, one should be able to compute $F(s; t=4\mu^2)$ independently, which we do not know how to do.

Knowing $F(s, t)$ inside the Mandelstam eye, one can project it into partial waves in the s channel. Since the errors are known, it is not difficult to get absolute bounds to the errors induced in the $f_\ell(s)$. Hence, we shall have

$$|f_\ell(s; L, s_0) - f_\ell(s)| < b_\ell(L, s_0), \quad (M-\mu)^2 \leq s \leq (M+\mu)^2 \quad (1.8)$$

where $b_\ell(L, s_0)$ is known. Such relation should be useful for dispersion relations determinations of πN partial waves.

At this point, the usual procedures, to pass to the physically interesting "pseudo-physical" range ^{*)} $4\mu^2 \leq t \leq 2M^2$, would be to write t channel dispersion relations (we skip eventual subtractions) either for partial waves,

$$g_\ell(t) = \frac{1}{\pi} \int_{-\infty}^0 dt' \frac{\text{Im} g_\ell(t')}{t' - t} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\text{Im} g_\ell(t')}{t' - t}, \quad (1.9)$$

or for the whole amplitude,

$$F(s, t) = \frac{1}{\pi} \int du \frac{F_u(s, t)}{u' - u} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{F_t(s, t')}{t' - t}. \quad (1.10)$$

Let me enumerate the difficulties this presents.

- i) The $\text{Im} g$ of the left-hand cut in (1.9) can only be computed, from πN phase shifts, for $t > -28\mu^2$. The best one can do is hope that the far away contribution is small, or fit it. We have no proposal to improve this.
- ii) F_u in (1.10) can be computed from πN phase shifts, but there is again nothing one can do about the large t region of the right-hand cut integral in (1.10), $t > 2M^2$. Again, here we have nothing to say.
- iii) Leaving aside the above problems, one is left with equations of the type

$$\varphi(t) = \frac{1}{\pi} \int_{4\mu^2}^{2M^2} dt' \frac{\text{Im} \varphi(t')}{t' - t}, \quad 0 \leq t \leq 4\mu^2. \quad (1.11)$$

It is here that extrapolation procedures are used, which encounter the problems mentioned at the beginning of this Introduction.

^{*)} Most authors consider only the range up to $\sqrt{t} \approx 900$ MeV. We shall give reasons why, with our methods, it is possible to go beyond, up to, at least, $t \approx 2M^2$.

What we propose here is to use Padé approximants ¹⁷⁾ to continue $\varphi(t)$ from the region where it is known to the cut. The reason why Padé approximants are used is that we prove here that these approximants converge also inside the cut, and this convergence is stable against perturbations. What is more important is that their rate of convergence can be explicitly calculated. Thus, we shall be able to say that, for instance, in the range $4\mu^2 \leq t \leq 2M^2$, $\text{Im } g_\ell(t) \simeq \text{Im } g_\ell(t; N)$, $|\text{Im } g_\ell(t) - \text{Im } g_\ell(t; N)| < B(t, N)$, where both $\text{Im } g_\ell(t; N)$ and $B(t, N)$ are known. This is possible because, for suitable spin and isospin combinations we are able to prove positivity properties of the $\text{Im } g_\ell$, F_t ; specifically, we prove that

$$(-1)^{\ell/2} \text{Im } g_\ell(t) > 0; F_t(s, t) > 0, (M-\mu)^2 \leq t \leq (M+\mu)^2 \quad (1.12)$$

rigorously for $4\mu^2 \leq t \leq 16\mu^2$, most likely (though not rigorously) up to $t \simeq 2M^2$ at least. For amplitudes without positivity the analysis can still be made, although the results may be less sharp.

The plan of this article is as follows. The spinology, conventions, etc., are presented in Section 2. Then, in Section 3 we describe the projections onto $\pi\pi \rightarrow \bar{N}N$ partial waves in the unphysical region, with known errors. From this we reconstruct the whole amplitude in the Mandelstam region, again within fixed error band (Section 4). The stable projection of this on πN waves in the unphysical region is described in Section 5.

Up to here, only some combinations of waves with positivity have been considered. The separation of waves, and the extension of the analysis to waves without positivity is carried over in Section 6. In Section 7 we conjecture some positivity properties in the t channel, both of partial waves and of the full absorptive part. These positivity properties are used in the first half of Section 8 to extrapolate to the absorptive part in the quasi physical cut, $4\mu^2 \leq t \lesssim 2M^2$, again keeping bounds on the errors. The second half of Section 8 is then devoted to extrapolation to the real part, and to extend the analysis to the cases where one does not have positivity. The article is finished in Section 9 with some comments and discussions as well as by three Appendices where some mathematical estimates are described.

To be consistent, and keep the article within finite bounds, we have presented only the theory here. Numerical applications will be presented separately, and also separately will be published the rigorous proofs (only sketched here) of the results on Padé approximants that are useful to obtain stable extrapolations.

PART I: STABLE EXTRAPOLATION TO UNPHYSICAL REGIONS

2. NOTATION, SPINOLOGY, FROISSART-GRIBOV REPRESENTATION

We start by considering a scattering amplitude with positivity both in the s and u channels. It will turn out later (Section 7) that it will also possess some positivity in the t channel. Of course, there are several amplitudes with this property: Singh's $A^{(+)}(s,t)$, Bell's amplitudes¹⁸⁾, etc. We shall choose the one usually denoted by $F^{(+)}(s,t)$ given by

$$F^{(+)}(s,t) = \left\{ A^{(+)}(s,t) + \frac{s-M^2-\mu^2}{2M} B^{(+)}(s,t) \right\}. \quad (2.1)$$

This has positivity both in the s and u channels¹⁹⁾, and is free from kinematical singularities and zeros in all three channels²⁰⁾. In terms of isospin, we have

$$F^{(+)} = \frac{1}{3} F^{(\frac{1}{2}; \frac{1}{2})} + \frac{2}{3} F^{(\frac{1}{2}; \frac{3}{2})} = F^{(t;0)} \quad (2.2)$$

where $F^{(x;I)}$ means isospin I in channel x .

The partial wave expansions of $F^{(+)}$ read^{20),21)}

$$F^{(+)}(s,t) = \frac{4\pi\sqrt{s}}{Mq_s} \left\{ \sum_0^{\infty} [P'_{\ell+1}(\cos\theta_s) - P'_{\ell}(\cos\theta_s)] f_{\ell+}^{(+)}(s) + \sum_1^{\infty} [P'_{\ell}(\cos\theta_s) - P'_{\ell-1}(\cos\theta_s)] f_{\ell-}^{(+)}(s) \right\}, \quad (2.3)$$

in the s channel;

$$F^{(+)}(u,t) = \frac{4\pi\sqrt{u}}{Mq_u} \left\{ \sum_0^{\infty} [K_+(u,t) P'_{\ell+1}(\cos\theta_u) - K_-(u,t) P'_{\ell}(\cos\theta_u)] f_{\ell+}^{(+)}(u) + \sum_1^{\infty} [K_-(u,t) P'_{\ell}(\cos\theta_u) - K_+(u,t) P'_{\ell-1}(\cos\theta_u)] f_{\ell-}^{(+)}(u) \right\}, \quad (2.4)$$

with

$$K_{\pm}(u,t) = 1 + t / [(u''^2 \pm M)^2 - \mu^2],$$

in the u channel, and

$$F^{(+)}(s, t) = \frac{2\pi}{Q_t} \left\{ \left(\frac{t}{q_t Q_t} \right)^{1/2} \sum_{J=\text{even}} (2J+1) P_J(\cos \theta_t) g_J^{(+)}(t) \right. \\ \left. + \frac{(q_t \cos \theta_t - Q_t) t}{2 q_t^{3/2} Q_t^{1/2} M} \sum_{J=\text{even}} \frac{2J+1}{\sqrt{J(J+1)}} P_J'(\cos \theta_t) g_J^{(-)}(t) \right\}, \quad (2.5)$$

in the t channel. Only even waves contribute because of generalized Bose statistics and isospin, Eq. (2.2). We have used the standard notation

$$q_x^2 = \frac{[x - (M+\mu)^2][x - (M-\mu)^2]}{4x}, \quad x = s \text{ or } u, \\ q_t^2 = t/4 - \mu^2, \quad Q_t^2 = t/4 - M^2, \quad (2.6a)$$

and

$$\cos \theta_x = 1 + t/2q_x^2, \quad x = s \text{ or } u, \\ \cos \theta_t = \frac{s + Q_t^2 + q_t^2}{2Q_t q_t}. \quad (2.6b)$$

We recall here, for future reference, the threshold behaviour of the g waves ³⁾ (that of the f is not needed):

$$g_J^{(+)}(t) \simeq - \frac{2q_t^{1/2} (Q_t q_t)^J}{t^{1/2} Q_t^{1/2}} \alpha_J^{(+)} \\ g_J^{(-)}(t) \simeq - \frac{q_t^{1/2} (Q_t q_t)^J}{Q_t^{1/2}} \alpha_J^{(-)}, \quad (2.7)$$

the α being the scattering lengths at $t = 4\mu^2$. Besides (2.5) one can use three other expansions of $F^{(+)}$ in the t channel ¹⁴⁾:

$$F^{(+)}(s, t) = \sum (2l+1) P_l(\cos \theta_t) h_l^{(+)}(t), \quad (2.8)$$

$$F^{(+)}(s, t) = \frac{1}{\pi} \sum (\cos \theta_t)^l \mu_l^{(+)}(t), \quad (2.9)$$

$$F^{(+)}(j, t) = \sum \rho_\ell^{-1} P_\ell^{(1,1)}(\cos \theta_t) k_\ell^{(+)}(t);$$

$$\rho_\ell \equiv 8(\ell+1)/(\ell+2)(2\ell+3).$$
}

(2.10)

For ℓ = even (the only values of interest for us now) one can easily re-express these waves in terms of the g :

$$h_\ell^{(+)}(t) = 2\pi \frac{t^{1/2}}{q_t^{1/2} Q_t^{3/2}} \left\{ g_\ell^{(+)}(t) \right.$$

$$- \frac{t^{1/2}}{2M} \left[\sqrt{\frac{\ell}{\ell+1}} g_\ell^{(-)}(t) - \frac{2\ell+5}{\sqrt{(\ell+2)(\ell+3)}} g_{\ell+2}^{(-)}(t) \right.$$

$$\left. - \sum_{\substack{J=\ell+4 \\ J=\text{even}}}^{\infty} \frac{2J+1}{\sqrt{J(J+1)}} g_J^{(-)}(t) \right] \Big\},$$

(2.11)

$$h_\ell^{(+)}(t) = \frac{2^{\ell+1}}{\ell! \Gamma\pi} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\Gamma(\ell+\nu+3/2)}{\nu!} h_{\ell+2\nu}^{(+)}(t),$$

(2.12)

and

$$k_\ell^{(+)}(t) = 8\pi \frac{\ell+1}{2\ell+3} \cdot \frac{t^{1/2}}{q_t^{1/2} Q_t^{3/2}} \left\{ g_\ell^{(+)}(t) - g_{\ell+2}^{(+)}(t) \right.$$

$$\left. - \frac{t^{1/2}}{2M} \left[\sqrt{\frac{\ell}{\ell+1}} g_\ell^{(-)}(t) - \sqrt{\frac{\ell+2}{\ell+3}} g_{\ell+2}^{(-)}(t) \right] \right\}.$$

(2.13)

One can work with either of these waves. However, due to the fact that the k are finitely related to the g , we shall work explicitly only with them, noting that the calculations are totally similar for the others.

Then, we may also define the reduced waves:

$$\psi_\ell^{(+)}(t) \equiv \frac{\pi M}{N_\ell Q_t^\ell q_t^\ell} k_\ell^{(+)}(t),$$

$$N_\ell \equiv \sum_{n=0}^{\ell+1} \binom{\ell+1}{n} (-1)^n \frac{2}{2n+1} > 0,$$
}

(2.14)

remarking that, due to (2.7),

$$\psi_{\ell}^{(+)}(4\mu^2) = \frac{8\pi^2 M (\ell+1)}{N_{\ell} (M^2 - \mu^2)(2\ell+3)} \left\{ \alpha_{\ell}^{(+)} - \frac{2\mu^2}{M} \left(\frac{\ell}{\ell+1}\right)^{1/2} \alpha_{\ell}^{(-)} \right\} \quad (2.15)$$

Similar reduced waves could be defined for the h, μ , but we shall not do so here.

Next, we introduce two more bits of notation. By putting a bar over a quantity we shall understand that we have subtracted from that quantity the contribution of the nucleon pole. We just give two explicit expressions, referring to 14), 15) for a more complete list:

$$\begin{aligned} \bar{k}_{\ell}^{(+)}(z) &\equiv k_{\ell}^{(+)}(z) \\ &+ \frac{2\mu^2 - z}{2Q_z q_z} \left\{ \left[\frac{M^2 + q_z^2 + Q_z^2}{2q_z Q_z} \right]^2 - 1 \right\} G^2 Q_{\ell}^{(1,1)} \left(\frac{M^2 + q_z^2 + Q_z^2}{2q_z Q_z} \right), \end{aligned} \quad (2.16a)$$

$$\bar{\psi}_{\ell}^{(+)}(z) \equiv \frac{\pi M}{N_{\ell} q_z^{\ell} Q_z^{\ell}} \bar{k}_{\ell}^{(+)}(z). \quad (2.16b)$$

Here G^2 is the renormalized πN coupling constant, and $Q_{\ell}^{(1,1)}$ the Jacobi function of the second kind ²²⁾. Also, we shall denote by F^{σ} to the symmetrized (in $\cos \theta_t$) part of $F^{(+)}$:

$$F^{\sigma} = \frac{1}{2} \{ F^{(+)}(\cos \theta_t, z) + F^{(+)}(-\cos \theta_t, z) \}. \quad (2.17)$$

To finish this section, we quote the Froissart-Gribov ²³⁾ representations of the waves we have introduced ^{14), 15)}. These formulae hold for $\ell = \text{even} \geq 2$:

$$\bar{h}_{\ell}^{(+)}(z) = \frac{1}{2\pi q_z Q_z} \int_{(M+\mu)^2}^{\infty} ds \{ F_s^{(+)}(z, z) + F_u^{(+)}(s, z) \} Q_{\ell} \left(\frac{s + q_z^2 + Q_z^2}{2q_z Q_z} \right), \quad (2.18)$$

where Q_{ℓ} is the Legendre function of the second kind ²²⁾,

$$\bar{\mu}_{\ell}(z) = \frac{1}{\pi} \int_{z_0(z)}^{\infty} dz \{ F_s^{(+)}(z, z) + F_u^{(+)}(z, z) \} z^{-\ell-2} \quad (2.19)$$

where we have changed variables:

$$z_0(t) = |(t - 4\mu M)/4q_t Q_t|, \quad z = (1 + q_t^2 + Q_t^2)/2|q_t Q_t|,$$

and, finally,

$$\bar{k}_\ell^{(+)}(t) = \frac{1}{\pi q_t Q_t} \int_{(M+\mu)^2}^{\infty} ds \{ F_s^{(+)}(s, t) + F_u^{(+)}(s, t) \}$$

$$\left\{ \left(\frac{1 + q_t^2 + Q_t^2}{2q_t Q_t} \right)^2 - 1 \right\} Q_\ell^{(1,1)} \left(\frac{1 + q_t^2 + Q_t^2}{2q_t Q_t} \right).$$

(2.20)

We consistently denote by F_x the absorptive part ^{*} of F in channel x .

Let us remark again that $F^{(+)}$ has Martin's positivity ²⁴⁾ in both the s and u channels, and, therefore $F_s^{(+)} + F_u^{(+)}$ satisfies

$$\frac{\partial^n}{\partial t^n} \{ F_s^{(+)}(s, t) + F_u^{(+)}(s, t) \} \geq 0,$$

$$n=0, 1, 2, \dots; \quad 0 \leq t \leq 4\mu^2.$$

(2.21)

^{*}) The reader should be warned that there is a slight change of notation from Refs. 14), 15), in which F_x was the whole discontinuity, i.e., $F_x^{\text{old}} = 2i F_x^{\text{now}}$.

3. $\bar{N}N \rightarrow \pi\pi$ PARTIAL WAVES IN THE UNPHYSICAL REGION, AND ERRORS

As stated in the previous section, we shall work explicitly only with the k (or, equivalently, the ψ). We start by defining the truncated (absorptive part of the) amplitude and difference functions by the analogues of Eqs. (1.3), (1.4),

$$\begin{aligned} & F_s^{(+)}(s, t; L, s_0) + F_u^{(+)}(s, t; L, s_0) \\ &= \frac{4\pi\sqrt{s}}{M q_s} \left\{ \sum_0^L \left[(1+K_+(s, t)) P'_{\ell+1}(\cos\theta_s) - (1+K_-(s, t)) P'_\ell(\cos\theta_s) \right] f_{\ell+}^{(+)}(s) \right. \\ & \quad \left. + \sum_1^L \left[(1+K_-(s, t)) P'_\ell(\cos\theta_s) - (1+K_+(s, t)) P'_{\ell-1}(\cos\theta_s) \right] f_{\ell-}^{(+)}(s) \right\}, \end{aligned} \quad (3.1a)$$

for $(M+\mu)^2 \leq s \leq s_0$,

$$= \frac{2\sqrt{s} q_s}{M} \sigma_{tot}^{(+)}(s), \quad (3.1b)$$

for $s > s_0$, where $\sigma_{tot}^{(+)} = (1/3) \sigma_{tot}^{(1/2)} + (2/3) \sigma_{tot}^{(3/2)}$, $\sigma_{tot}^{(I)}$ being the total cross-section with isospin I in the s channel, and we have taken advantage of the optical theorem:

$$F_s^{(+)}(s, t=0) = F_u^{(+)}(s, t=0) = (q_s \sqrt{s}/M) \sigma_{tot}^{(+)}$$

We shall, for simplicity and to keep this article within a finite size, work as if we were using (3.1b). However, it is not difficult to construct a much better estimate. To do so, we begin by defining the integer $I(s)$ as the maximum integer such that

$$\frac{4\pi}{q_s} \left\{ \sum_0^{I(s)} (\ell+1) + \sum_1^{I(s)} \ell \right\} \leq q_s \sigma_{tot}^{(+)}(s), \quad (3.1c)$$

and we let $\eta_n^{\pm}(s)$ be given by

$$\left. \begin{aligned} \eta_n^{\pm}(s) &= 1, 1 < n \leq I(s), \quad \eta_0^+(s) = 1/2, \quad \eta_1^-(s) = 1/2 \\ \eta_n^{\pm}(s) &= \left\{ q_s^2 \sigma_{tot}^{(+)}(s) - 4\pi \left[\sum_0^{I(s)} (\ell+1) + \sum_1^{I(s)} \ell \right] \right\} / 4\pi [2I(s)+3], \\ n &= I(s)+1. \end{aligned} \right\} \quad (3.1d)$$

Then, one should substitute (3.1b) by the definition

$$\begin{aligned}
 & F_s^{(+)}(s, t; L, s_0) + F_u^{(+)}(s, t; L, s_0) \\
 &= \frac{4\pi\sqrt{s}}{M_{q_s}} \left\{ \sum_0^{I(s)+1} \left[(1+K_+(s, t)) P'_{\ell+1}(\cos \theta_s) - (1+K_-(s, t)) P'_\ell(\cos \theta_s) \right] \eta_\ell^+(s) \right. \\
 & \quad \left. + \sum_1^{I(s)+1} \left[(1+K_-(s, t)) P'_\ell(\cos \theta_s) - (1+K_+(s, t)) P'_{\ell-1}(\cos \theta_s) \right] \eta_\ell^-(s) \right\},
 \end{aligned} \tag{3.1e}$$

for $s > s_0$.

All the results may be generalized to the case in which (3.1b) is substituted by the more accurate form (3.1e), although the proofs (especially that of theorem 3.1) will become more cumbersome. The reason is that, with $F(., L, s_0)$ given by (3.1a) and (3.1e), one still has the positivity properties

$$\left. \begin{aligned}
 & \frac{\partial^n}{\partial t^n} \left\{ F_s^{(+)}(s, t) + F_u^{(+)}(s, t) \right. \\
 & \quad \left. - F_s^{(+)}(s, t; L, s_0) - F_u^{(+)}(s, t; L, s_0) \right\} > 0, \\
 & 0 \leq t \leq 4\mu^2.
 \end{aligned} \right\} \tag{3.2}$$

Proof: For the spinless case this has been shown by A. Martin [Phys.Rev. 129, 1432 (1963)]. We recall that such proof just uses unitarity, in the sense of requiring $0 \leq \text{Im} f_\ell \leq 1$, and the fact that $(2\ell+1)P_\ell(\cos \theta_s)$ is an increasing function of ℓ for fixed $\cos \theta_s > 1$. Now, the first property also holds for us, and the second is replaced by the analogue property for the combinations $(1+K_+)P'_{\ell+1} - (1+K_-)P'_\ell$ and $(1+K_-)P'_\ell - (1+K_+)P'_{\ell-1}$. To check that such combinations are still increasing functions of ℓ is straightforward, although tedious; we shall leave the details to the reader, and just note that if one is content with an approximative proof, it is sufficient to realize that, in the region of interest for us, one can set, to a negligible error, $K_\pm(s, t) = 1$, so you are left with combinations of the type $P'_n - P'_{n-1}$ which are well known to be increasing functions of n . Finally, we note that, whichever form we use, one has that (recall also the discussion in the Introduction), $F_s^{(+)}(s, t=0; L, s_0) + F_u^{(+)}(s, t=0; L, s_0)$ equals the exact absorptive part at $t=0$, up to the forward contribution of the "tail" of $L+1$ to ∞ left-over waves, that we shall neglect.

The error committed can be split into two parts, $\Delta_L F^{(+)}$ and $\Delta_{s_0} F^{(+)}$ given by the exact form of Eqs. (1.4), the writing of which we leave to the reader.

Similarly, we define truncated waves, and the corresponding errors; we shall do it directly for the $\bar{\Psi}_\ell^{(+)}$ because this quantity does not have any more the kinematical threshold zero of $F_\ell^{(+)}$:

$$\bar{\Psi}_\ell^{(+)}(t; L, s_0) = M \int_{(M+\mu)^2}^{\infty} ds \{ F_s^{(+)}(s, t; L, s_0) + F_u^{(+)}(s, t; L, s_0) \} R_\ell^{(1,1)}(s, t), \quad (3.3a)$$

where we have defined, to save notation,

$$R_\ell^{(1,1)}(s, t) = \frac{[(s + q_t^2 + Q_t^2)/2q_t Q_t]^2 - 1}{N_\ell Q_t^{2l+1} q_t^{2l+1}} Q_\ell^{(1,1)}\left(\frac{s + q_t^2 + Q_t^2}{2q_t Q_t}\right). \quad (3.3b)$$

$$\Delta_L \bar{\Psi}_\ell^{(+)}(t) = M \int_{(M+\mu)^2}^{s_0} ds (\Delta_L F^{(+)}(s, t)) R_\ell^{(1,1)}(s, t), \quad (3.4)$$

$$\Delta_{s_0} \bar{\Psi}_\ell^{(+)}(t) = M \int_{s_0}^{\infty} ds (\Delta_{s_0} F^{(+)}(s, t)) R_\ell^{(1,1)}(s, t). \quad (3.5)$$

At this point it may be worth halting for a moment to discuss the lower range of the integrals in (3.3), (3.4) for $t \approx 0$. The reason is that, as may be seen from (3.3b) and the properties of the $Q_\ell^{(1,1)}$, $R_\ell^{(1,1)}(s, t)$ has a logarithmic singularity for $t=0$, $s=(M+\mu)^2$. Now, near this point, the $F_x^{(+)}(s, t)$ e.g., behaves like

$$F_x^{(+)}(s, t) \simeq q_s \frac{M+\mu}{M} \sigma_{t+t}^{(+)}(s=(M+\mu)^2) + \pm C,$$

where C is related to a combination of all the scattering lengths for πN except the S wave one. Therefore, the singularity mentioned is overkilled by the behaviour of the F , and it does not cause any difficulty. This is well known.

Let us look now more closely to the errors. With respect to them we shall prove the following three results:

Theorem 3.1: Both $\Delta_L \bar{\Psi}_2^{(+)}(t)$ and $\Delta_{s_0} \bar{\Psi}_2^{(+)}(t)$ are increasing functions of t .

Theorem 3.2: $\Delta_{s_0} \bar{\Psi}_\ell^{(+)}(t)$ is extremely small for $\ell=4, 6, \dots$

Theorem 3.3: $\Delta_L \bar{\Psi}_\ell^{(+)}(t)$ is an increasing function of t for $\ell < L+2$.

We start with the proof of Theorem 3.1. Let us recall that, because of the positivity properties of our function $F^{(+)}$, one can write ²⁴⁾

$$\begin{aligned} \Delta_L F^{(+)}(s, t) &\equiv \Delta_L F_s^{(+)}(s, t) + \Delta_L F_u^{(+)}(s, t) \\ &= \sum_{n=0}^{\infty} T_n(\cos \theta_s) \varphi_n(s) \end{aligned} \quad (3.6)$$

with $\varphi_n(s) > 0$ and where the T_n are the Tchebycheff (Chebyshev) polynomials ²²⁾.

Instead of proving Theorem 3.1 for the $\Delta_L \psi_\ell$, we shall prove it for the quantities

$$\Delta'_L \bar{\psi}_\ell^{(+)}(t) \equiv M \int d\mathbf{j} \{ \Delta_L F^{(+)}(\mathbf{j}, t) - \Delta_L F^{(+)}(\mathbf{j}, t=0) \} R_\ell^{(1,1)}(\mathbf{j}, t).$$

For the practical point of view, this is irrelevant as the difference between $\Delta'_L \bar{\psi}_\ell$ and $\Delta_L \bar{\psi}_\ell$ is a quantity which, using the properties [essentially the exact analogue of Eq. (24) of Ref. 14] of the $R_\ell^{(1,1)}$ can easily be shown to be less than twice the value of the difference between $\bar{\psi}_\ell^{(+)}(t=0)$ and $\bar{\psi}_\ell^{(+)}(t=0; L, s_0)$, which we have assumed to be negligible.

Using (3.6), we then get, with obvious notation,

$$\Delta'_L F^{(+)}(\mathbf{j}, t) = \sum_{n=1}^{\infty} T_n(\cos \theta_s) \varphi_n(\mathbf{j}). \quad (3.7)$$

We shall consider separately two pieces of the above sum:

$$\Delta'^{(1)}_L F^{(+)}(\mathbf{j}, t) \equiv (\cos \theta_s) \varphi_1(\mathbf{j}), \quad (3.8a)$$

$$\Delta'^{(2)}_L F^{(+)}(\mathbf{j}, t) \equiv \sum_{n=2}^{\infty} T_n(\cos \theta_s) \varphi_n(\mathbf{j}), \quad (3.8b)$$

and their contributions to $\Delta'_L \bar{\psi}_2^{(+)}$,

$$\Delta'^{(1)}_L \bar{\psi}_2^{(+)}(t) = M \int d\mathbf{j} (\Delta'^{(1)}_L F^{(+)}(\mathbf{j}, t)) R_2^{(1,1)}, \quad (3.9a)$$

$$\Delta'^{(2)}_L \bar{\psi}_2^{(+)}(t) = M \int d\mathbf{j} (\Delta'^{(2)}_L F^{(+)}(\mathbf{j}, t)) R_2^{(1,1)}. \quad (3.9b)$$

The first can be treated explicitly: take its derivative with respect to t , and check that it is positive. So we turn to the second. Because of (3.8b), it is the integral of sums of positive quantities (the φ) times terms of the type

$$T_n(\cos \theta_s) R_2^{(1,1)}(\mathbf{j}, t), \quad n \geq 2. \quad (3.10)$$

Although the result can, with some effort, be shown to hold exactly, we shall prove it only in the s, t region such that $\cos \theta_t \gg 1$. This fills most of the range $0 \leq t \leq 4\mu^2$, $s \geq (M + \mu)^2$.

Under such condition, we can use the asymptotic form of the R [Ref. 14), Eq. (27)]; therefore, we are led to show that

$$(\partial/\partial t) T_n(\cos \theta_s) \frac{1}{(s + q_t^2 + Q_t^2)^3} \equiv x(\mathbf{j}, t) > 0.$$

To this end, we shall use the formula ²²⁾

$$T_n(z) = \frac{z}{n} T_n'(z) - \frac{n}{n-1} T_{n-1}'(z) < \frac{z}{n} T_n'(z),$$

obtaining then

$$\chi(s, t) > \frac{T_n'(z)}{2(s+q_t^2+Q_t^2)^{\frac{4}{3}} q_t^2} \left\{ s+q_t^2+Q_t^2 - \frac{3}{2} \left(q_t^2 + \frac{t}{2} \right) \right\}, \quad z \equiv \cos \theta_t,$$

which is obviously positive.

The proof that also $\Delta_{s_0} \bar{\psi}_2^{(+)}(t)$ is increasing is exactly similar, and so is the proof of Theorem 3.3, where, however, one has to be slightly more careful in the majorizations. Finally, we pass to the proof, and explanation of the exact meaning of Theorem 3.2. From the results of Ref. 14), Eq. (24), it follows that

$$\Delta_{s_0} \bar{\psi}_\ell^{(+)}(t) < \frac{R_\ell^{(1,1)}(s_0, t)}{R_2^{(1,1)}(s_0, t)} \Delta_{s_0} \bar{\psi}_2^{(+)}(t).$$

Since s_0 is fairly large, $\cos \theta_t (s=s_0) \gg 1$, and we can use the asymptotic form of the R_ℓ ; hence

$$\Delta_{s_0} \bar{\psi}_\ell^{(+)}(t) < \frac{1}{2^{\ell-2} (s_0 + Q_t^2 + q_t^2)^{\ell-2}} \Delta_{s_0} \bar{\psi}_2^{(+)}(t). \quad (3.11a)$$

Putting numbers, recalling that good phase shift analysis exist up to $s_0 = 7M^2$, we get that

$$\Delta_{s_0} \bar{\psi}_\ell^{(+)}(t) < \frac{1}{[12M^2]^{\ell-2}} \Delta_{s_0} \bar{\psi}_2^{(+)}(t). \quad (3.11b)$$

This is to be compared with the expected decrease of the $\bar{\psi}_\ell^{(+)}$ themselves: if we consider that the bulk of the contribution in Eq. (3.3) comes from the region of the N_{33}^* , we shall have that, roughly,

$$\begin{aligned} \bar{\psi}_\ell^{(+)}(t; L, s_0) &\simeq \frac{1}{2^{\ell-2}} \cdot \frac{1}{(s(N_{33}^*) + Q_t^2 + q_t^2)^{\ell-2}} \bar{\psi}_2^{(+)}(t; L, s_0) \\ &\simeq \frac{1}{[2M^2]^{\ell-2}} \bar{\psi}_2^{(+)}(t; L, s_0), \end{aligned} \quad (3.12)$$

where we have again used Eq. (24) of Ref. 14), and taken the asymptotic value of the $R_l^{(1,1)}$, because also $\cos \theta_t(s=s_{N*}) \gg 1$. Comparing (3.12) with (3.11b), we see that the relative importance of the $\Delta_{s_0}^{33}$ error decreases by a factor $(2M^2/12M^2)^2 = 1/36$ for each consecutive wave. This is the meaning of Theorem 3.2.

Putting together the conclusions of all three theorems, it follows that we only have to calculate the errors for $t = 4\mu^2$, because this is the maximum value they can reach.

Let us now assume that, by some means, we have been able to calculate the errors for $l=2$, $t=4\mu^2$:

$$\Delta_L \bar{\Psi}_2^{(t)}(4\mu^2) \equiv \epsilon_2, \quad \Delta_{s_0} \bar{\Psi}_2^{(t)}(4\mu^2) \equiv \eta_2. \quad (3.13)$$

Then, using Eqs. (19), (28), (29) of Ref. 14), and Theorem 3.2, it will follow that

$$(\Delta_L + \Delta_{s_0}) \bar{\Psi}_\ell^{(t)}(t=4\mu^2) < \frac{1}{[2\mu(M+\mu)]^{\ell-2}} \Delta_L \bar{\Psi}_2^{(t)}(t=4\mu^2), \quad (3.14)$$

so that, actually, it is only necessary to know ϵ_2 and η_2 of (3.13).

4. THE FULL AMPLITUDE INSIDE THE MANDELSTAM REGION

Let us now turn to the extrapolation of the full amplitude to the Mandelstam region, $0 \leq t \leq 4\mu^2$, $(M - \mu)^2 \leq s \leq (M + \mu)^2$. There are two different approaches: one can either try to reconstruct the amplitude from its partial waves, as determined in the previous section, or use directly fixed t dispersion relations giving the whole of the amplitude at once. Clearly, both methods should be used thereby reducing errors and cross-checking the estimates.

We shall begin by discussing the reconstruction of the amplitude from its partial waves. Again here we have two possibilities: we can sum naively the partial wave series, or take advantage of the results of Ref. 14) to reduce the waves to solutions of what is called a Hausdorff moment problem ^{12),13),14),17)} for which the powerful summation methods of Padé approximants can be used ¹³⁾. We shall describe both methods (and not only the second which is doubtlessly the more useful one) for the sake of completeness and also to show explicitly the mechanism at work.

We define, exactly as before, truncated amplitudes,

$$\bar{F}^\sigma(s, t; L, s_0) \equiv \sum_{\ell=even} \rho_\ell^{-1} P_\ell^{(1,1)}(\cos \theta_t) \bar{k}_\ell^{(+)}(t; L, s_0) \quad (4.1)$$

and the errors

$$\Delta_L \bar{F}^\sigma(s, t) \equiv \sum_{\ell=even} \rho_\ell^{-1} P_\ell^{(1,1)}(\cos \theta_t) \Delta_L \bar{k}_\ell^{(+)}(t), \quad (4.2)$$

$$\Delta_{s_0} \bar{F}^\sigma(s, t) \equiv \sum_{\ell=even} \rho_\ell^{-1} P_\ell^{(1,1)}(\cos \theta_t) \Delta_{s_0} \bar{k}_\ell^{(+)}(t). \quad (4.3)$$

Note that it is better here to work directly with the k_ℓ rather than using the ψ_ℓ . In the region of interest,

$$P_\ell^{(1,1)}(\cos \theta_t) \geq 1, \quad (4.4)$$

so that the waves, and the errors just add up, and no instability may arise. Note also that only the symmetric part of $\bar{F}^{(+)}$, viz., $\bar{F}^\sigma$ [cf., Eq. (2.17)] can be obtained by this method. Also, the S wave being out of the scope of our work, we are forced to consider, rather than $\bar{F}^\sigma$ itself, a subtracted form of it

$$\begin{aligned} \bar{F}^{\sigma \text{ sub}}(s, t) &\equiv \bar{F}^\sigma(s, t) - \bar{F}^\sigma(s_1, t), \\ (M - \mu)^2 &\leq s_1 \leq (M + \mu)^2, \end{aligned} \quad \left. \vphantom{\begin{aligned} \bar{F}^{\sigma \text{ sub}}(s, t) &\equiv \bar{F}^\sigma(s, t) - \bar{F}^\sigma(s_1, t), \\ (M - \mu)^2 &\leq s_1 \leq (M + \mu)^2, \end{aligned}} \right\} \quad (4.5)$$

or, equivalently,

$$\bar{F}^{\sigma \text{red}}(s, t) \equiv \bar{F}^{\sigma}(s, t) - \frac{4}{3} \bar{k}_0^{(+)}(t). \quad (4.6)$$

Essentially, this amounts to starting the sums in (4.1) to (4.3) at $\ell=2$. Of course, $\bar{F}^{\sigma}(s, t)$ or $\bar{k}_0^{(+)}(t)$ have to be taken from an independent source. To be definite, let us work now with $\bar{F}^{\sigma \text{red}}$. One has [cf., Eq. (24) of Ref. 14]

$$\Delta_L \bar{k}_\ell^{(+)}(t) < [Q_\ell^{(1,1)}(z_{th})/Q_2^{(1,1)}(z_{th})] \Delta_L \bar{k}_2^{(+)}(t), \quad (4.7a)$$

$$\Delta_{s_0} \bar{k}_\ell^{(+)}(t) < [Q_\ell^{(1,1)}(z_0)/Q_2^{(1,1)}(z_0)] \Delta_{s_0} \bar{k}_2^{(+)}(t), \quad (4.7b)$$

$$z_{th} \equiv (2M\mu + t/2)/2q_t Q_t, \quad z_0 \equiv (s_0 + q_t^2 + Q_t^2)/2q_t Q_t.$$

Plugging this into (4.2), (4.3) and using (B.1) of Appendix B gives

$$\begin{aligned} & \Delta_L \bar{F}^{\sigma \text{red}}(s, t) \\ & < \left\{ \left[\frac{z_{th}(s^2-1)}{2(z_{th}^2-1)(z_{th}^2-s^2)} - \frac{4}{3} Q_0^{(1,1)}(z_{th}) \right] / Q_2^{(1,1)}(z_{th}) \right\} \Delta_L \bar{k}_2^{(+)}(t), \end{aligned} \quad (4.8a)$$

with $s = \cos \theta_t$, $z_{th} > s$, and

$$\begin{aligned} & \Delta_{s_0} \bar{F}^{\sigma \text{red}}(s, t) \\ & < \left\{ \left[\frac{z_0(s^2-1)}{2(z_0^2-1)(z_0^2-s^2)} - \frac{4}{3} Q_0^{(1,1)}(z_0) \right] / Q_2^{(1,1)}(z_0) \right\} \Delta_{s_0} \bar{k}_2^{(+)}(t). \end{aligned} \quad (4.8b)$$

The limitation $z > s$ means simply that the above inequalities only hold inside the Mandelstam region, as it should obviously be the case.

With this, we have reduced the problem to the knowledge of $\Delta_L \bar{k}_2^{(+)}(t)$, $\Delta_{s_0} \bar{k}_2^{(+)}(t)$ which, in turn, using the results of Section 3, can be bounded from knowledge of $\Delta_L \bar{\psi}_2^{(+)}(4\mu^2)$, $\Delta_{s_0} \bar{\psi}_2^{(+)}(4\mu^2)$.

There remains a last problem: although in principle we can compute as many $\bar{k}_\ell^{(+)}(t; L, s_0)$ as we want, for practical reasons it is convenient to have an estimate of the error that is being committed when we cut the sum in (4.8) at some value, N . For this, one writes again

$$\bar{k}_l^{(+)}(t; L, s_0) < \frac{Q_l(z_{th})}{Q_N(z_{th})} \bar{k}_N^{(+)}(t; L, s_0),$$

and uses now Eq. (B.4), to sum up all these bounds and obtain that the tail in the summation (4.1) satisfies the estimate

$$\begin{aligned} & \sum_{\substack{l=N+2 \\ l=\text{even}}}^{\infty} P_l^{-1} P_l^{(1,1)}(\cos \theta_z) \bar{k}_l^{(+)}(t; L, s_0) \\ & < \bar{k}_N^{(+)}(t; L, s_0) \frac{1}{Q_N^{(1,1)}(z_{th})} \\ & \times \left\{ \frac{z_{th} (z^2 - 1)}{2(z_{th}^2 - 1)(z_{th}^2 - z^2)} \right. \\ & \left. - \frac{N+3}{P_{N+2}(N+4)} \cdot \frac{P_{N+3}^{(1,1)}(z) Q_{N+2}^{(1,1)}(z_{th}) z - P_{N+2}^{(1,1)}(z) Q_{N+3}^{(1,1)}(z_{th}) z_{th}}{z_{th}^2 - z^2} \right\}. \quad (4.9) \end{aligned}$$

This shows that we can obtain explicit summation formulae and explicit bounds. These methods work best away from the border of the Mandelstam region; for example, they will be very tight for $M^2 + \mu^2 - M\mu \leq s \leq M^2 + \mu^2 + M\mu$.

After having so demonstrated the possibility of explicit reconstruction of F , let us proceed to the sophisticated procedure we spoke about at the beginning. The point is that due to the positivity of $F_s^{(+)}$, $F_u^{(+)}$, the exact $\bar{k}_l^{(+)}(t)$ must satisfy the constraints ¹⁴⁾

$$\sum_{m=0}^n \binom{n}{m} (-1)^m [\chi_0(t)]^{-2(n-m)} \bar{k}_{l+2m}^{(+)}(t) \geq 0, \quad (4.10a)$$

$l \geq 2$; here

$$\chi_0(t) \equiv z_{th} + (z_{th}^2 - 1)^{1/2}. \quad (4.10b)$$

Then, as shown in Refs. 17), 25), this in turn implies that one can write

$$H_n(t) \equiv \bar{k}_{2n+2}^{(+)}(t) = \int_0^{1/\chi_0(t)} d\xi \cdot \xi^n B(\xi, t), \quad (4.11a)$$

$$B(\xi, t) \geq 0. \quad (4.11b)$$

Using (4.11) and (2.10), it follows that, by straightforward extension of the analysis of Ref. 12), one can write, inside the Mandelstam's region,

$$\bar{F}^{\sigma nd}(s, t) = \frac{1}{8} \int_0^{1/x_0(t)} d\xi \Psi_2^+(\cos \theta_t, \xi) B(\xi, t), \quad (4.12)$$

where Ψ_2^+ is the function defined in Appendix C, [Eq. (C.4)]. Now, to solve (4.12) given (4.11) is what the mathematicians know as the Markoff problem, after the person who first posed and solved it. The solutions are through what are popularly known among physicists as Padé approximants; we pass to describe it, following more or less the analysis of Common¹³⁾. He states that, if a function $[\Psi_2^+(\cdot, \xi)]$ in our case] has the properties that there exists the number $a > 1/x_0(t)$ [in our case, $a = [\cos \theta_t + \sqrt{\cos^2 \theta_t - 1}]^{-1}$, which is obviously larger than $1/x_0(t)$] such that i) $\Psi_2^+(\cdot, \xi)$ is infinitely many times differentiable in the interval $(0, a)$, which holds in one case as Ψ_2^+ is even analytic there; ii) $(\partial^n / \partial \xi^n) \Psi_2^+(\cdot, \xi) \geq 0$, on $(0, a)$, which is again true for us using (C.4a), and iii) $\Psi_2^+(\cdot, \xi) \rightarrow \infty$ as $\xi \rightarrow a$, which is still evident from the representation (C.4a) then one has the bounds *)

$$\sum_{i=0}^{N/2} w_i \Psi_2^+(\cos \theta_t, \xi_i) > \bar{F}^{\sigma nd}(s, t) > \sum_{i=0}^{N/2} W_i \Psi_2^+(\cos \theta_t, \Xi_i), \quad (4.13)$$

where we have assumed that we are given the first $N+1$ waves, and that N is even.

[Similar bounds can be written if N is odd; we send to Refs. 17) and 13) for details.]

The weights w_i , W_i , and the points ξ_i, Ξ_i , are defined as follows: one constructs the so-called quasi-orthogonal polynomials $q(\xi)$, $Q(\xi)$

$$q(\xi) \equiv \det \begin{pmatrix} 1 & \xi & \xi^2 & \dots & \xi^{N/2+1} \\ 1 & x_0^{-1} & x_0^{-2} & \dots & x_0^{-N/2+1} \\ H_0 & H_1 & H_2 & \dots & H_{N/2+1} \\ H_1 & H_2 & H_3 & \dots & H_{N/2+2} \\ & & & \dots & \\ H_{N/2-1} & \dots & & & H_N \end{pmatrix}, \quad (4.14a)$$

*) Other, equivalent bounds, have been given by Baker²⁴⁾.

$$Q(\xi) \equiv \det \begin{pmatrix} 1 & \xi & \xi^2 & \dots & \xi^{N/2+1} \\ 1 & 0 & 0 & \dots & 0 \\ H_0 & H_1 & H_2 & \dots & H_{N/2+1} \\ H_1 & H_2 & H_3 & \dots & H_{N/2+2} \\ & & & \dots & \\ H_{N/2-1} & \dots & & & H_N \end{pmatrix} \quad (4.14b)$$

It may then be shown that the zeros of $q(\xi)$, are real different, and so are those of $Q(\xi)$. These zeros are, respectively,

$$\left. \begin{aligned} 0 < \xi_1 < \xi_2 < \dots < \xi_{N/2} < \xi_{N/2+1} = [x_0(t)]^{-1}, \\ \xi_0 = 0 < \xi_1 < \xi_2 < \dots < \xi_{N/2}. \end{aligned} \right\} \quad (4.15)$$

Denoting by a prime to the derivative of q or Q , we may use the notation

$$\left. \begin{aligned} \frac{q(\xi)}{(\xi - \xi_i) q'(\xi_i)} &\equiv \sum_{n=0}^{N/2} a_{in} \xi^n, \\ \frac{Q(\xi)}{(\xi - \xi_i) Q'(\xi_i)} &\equiv \sum_{n=0}^{N/2} A_{in} \xi^n, \end{aligned} \right\} \quad (4.16)$$

Then,

$$w_i = \sum_{n=0}^{N/2} a_{in} H_n, \quad W_i = \sum_{n=0}^{N/2} A_{in} H_n. \quad (4.17)$$

Therefore, if, by the method of the previous section we know the first waves, $\bar{k}_2^+(t), \bar{k}_4^+(t), \dots, \bar{k}_{2+2N}^+(t)$, $N = \text{even}$, we can reconstruct, within explicit bounds, the full amplitude inside the Mandelstam region. We finish this discussion with two remarks: firstly, the upper and lower approximants of Eq. (4.13) will be recognized as Padé approximants. It can be proved that, in the present situation, they converge exponentially towards $\bar{F}^{\sigma \text{red}}$. This is due to the positivity of B in (4.11). Second, from the mathematical experiments of the French Padé school [cf., Ref. 25) for a review and bibliography] we may expect them to be extremely stable against perturbations of the $\bar{k}_l^+(t)$. Actually, in our case, and due again to positivity, we shall be able to present explicit estimates of the stability of the Padés.

The method just described can be compared with the calculations of, e.g., Höhler et al.,¹⁶⁾. These authors use directly a fixed t dispersion relation.

$$\begin{aligned} \bar{F}^{\sigma}(s, t) - \bar{F}^{\sigma}(-q_t^2 - Q_t^2, t) = \\ = \frac{[s + q_t^2 + Q_t^2]^2}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \frac{2s' - (s+u)}{(s'-s)(s'-u)(s'-u')^2} \{F_s^{\sigma}(s', t) + F_u^{\sigma}(s', t)\}. \end{aligned} \quad (4.18)$$

[Eq. (4.18) is actually different from the relation of Ref. 29), although they can be easily related.] Then, using the known πN phase shifts plus an assumption for the high energy behaviour, they are able to compute the left-hand side of (4.18) [that we shall denote by $\bar{F}^{\sigma \text{sub}}$, as it is just a particular case of (4.5)] in the range $(M - \mu)^2 \leq s \leq (M + \mu)^2$, and for $-2\mu^2 \leq t \leq 2\mu^2$. Here, of course, the problem is that it is again impossible to give reliable estimates of the errors.

However, (4.18) has still positivity properties, and, therefore, we can apply our methods to it. This will give again an estimate of $\bar{F}^{\sigma \text{sub}}$, with rigorous errors. This method is in many respects simpler than that introduced at the beginning of this section, and provides a cross-check of the other; therefore, one should use both methods, which has the further advantage of reducing the errors.

Because the analysis is totally similar to the ones we have been making up to now, we shall run quickly over the formulae without halting to give detailed proofs. As before, we start by defining the truncated amplitude, $\bar{F}^{\sigma \text{sub}}(s, t; L, s_0)$, and the corresponding errors, $\Delta_L \bar{F}^{\sigma \text{sub}}$, $\Delta_{s_0} \bar{F}^{\sigma \text{sub}}$. These are expressed by substituting, in (4.18), $\bar{F}_s^{(+)} + \bar{F}_u^{(+)}$ by $\bar{F}_s^{(+)}(s, t; L, s_0) + \bar{F}_u^{(+)}(s, t; L, s_0)$ and by $\Delta_L \bar{F}_s^{(+)} + \Delta_L \bar{F}_u^{(+)}$, $\Delta_{s_0} \bar{F}_s^{(+)} + \Delta_{s_0} \bar{F}_u^{(+)}$, respectively. As these equations are totally similar to those for $\Delta_L \bar{\Psi}^{(+)}(t)$, $\Delta_{s_0} \bar{\Psi}^{(+)}(t)$ (Section 3), with the simplification of replacing $R_{\ell}^{(1,1)}(s, t)$ by $[2s' - (s+u)] / (s'-s)(s'-u)(s'-u')^2$, which has still positivity for $(M - \mu)^2 \leq s \leq (M + \mu)^2$, we can repeat the proof of Theorem 3.1, obtaining the result

$$\begin{aligned} 0 < [s + q_t^2 + Q_t^2]^{-2} \{(\Delta_L + \Delta_{s_0}) \bar{F}^{\sigma \text{sub}}(s, t)\} \\ < [s + \mu^2 - M^2]^{-2} \{(\Delta_L + \Delta_{s_0}) \bar{F}^{\sigma \text{sub}}(s, t = 4\mu^2)\}, \end{aligned} \quad (4.19)$$

so that all we need is an estimate of $\bar{F}^{\sigma \text{sub}}(s, t = 4\mu^2)$ to obtain bounds, valid in the whole region, to the error committed when replacing $\bar{F}^{\sigma \text{sub}}(s, t)$ by its truncated value, $\bar{F}^{\sigma \text{sub}}(s, t; L, s_0)$.

Such estimate could be obtained with the methods described at the beginning of this section; at the precise point $t=4\mu^2$ it is, however, easier to give a direct estimate. To get it, one considers the expansion (2.10). First of all, it is clear that, since $s = -q_t^2 - Q_t^2$ implies $\cos \theta_t = 0$, and because of the threshold behaviour of the $\bar{k}_\ell^{(+)}$ waves, Eq. (2.14), it follows that, as $t \rightarrow 4\mu^2$,

$$\bar{F}^\sigma(-q_t^2 - Q_t^2, t=4\mu^2) = \beta_0^{-1} \bar{k}_0^{(+)}(4\mu^2),$$

so that

$$\bar{F}^{\sigma \text{ sub}}(s, t \rightarrow 4\mu^2) = \sum_{\substack{\ell=2 \\ \ell=\text{even}}}^{\infty} \beta_\ell^{-1} P_\ell^{(1,1)}(\cos \theta_{4\mu^2}) \bar{k}_\ell^{(+)}(4\mu^2).$$

(4.20)

Next, one has the formula ²²⁾

$$P_n^{(1,1)}(x) = \frac{2^{n+2} \Gamma(n+3/2)}{\sqrt{\pi} (n+2) n!} x^n + \sum_{\nu \leq n-2} C_\nu x^\nu,$$

and thus, using again the threshold behaviour of the k , we see that (4.20) reduces to

$$\bar{F}^{\sigma \text{ sub}}(s, t=4\mu^2) = \frac{1}{\pi^{3/2} M} \sum_{\substack{\ell=2 \\ \ell=\text{even}}}^{\infty} \frac{\Gamma(\ell+5/2) N_\ell}{(\ell+1)!} (s+\mu^2-M^2)^\ell \bar{\psi}_\ell^{(+)}(4\mu^2).$$

Now, we use that N_ℓ is clearly a decreasing function of ℓ , and the relations ¹⁴⁾

$$\bar{\psi}_\ell^{(+)}(4\mu^2) < \frac{1}{[2\mu(M+\mu)]^{\ell-2}} \bar{\psi}_2^{(+)}(4\mu^2),$$

(4.21)

to write

$$\begin{aligned} \bar{F}^{\sigma \text{ sub}}(s, t=4\mu^2) &< \frac{16 (s+\mu^2-M^2)^2}{35 \pi^{3/2} M} \bar{\psi}_2^{(+)}(4\mu^2) \\ &\times \sum_{\ell=2}^{\infty} \frac{\Gamma(\ell+5/2)}{(\ell+1)!} \left[\frac{s+\mu^2-M^2}{2\mu(M+\mu)} \right]^{\ell-2}. \end{aligned}$$

(4.22)

As pointed out in the first half of this section, this type of calculations will only be useful in the region $M^2 + \mu^2 - \mu M \leq s \leq M + \mu^2 + \mu M$. With this limitation, it is easy to check that only the terms with $\ell=2$ and $\ell=4$ have to be taken into account in the sum in (4.22). Thus (4.22) gives a very tight bound to $\bar{F}^{\sigma \text{ sub}}(s, t=4\mu^2)$ in terms only of the usual quantity, $\bar{\psi}_2^{(+)}(4\mu^2)$.

5. πN WAVES IN THE UNPHYSICAL REGION

Whatever the method is, we have shown in the preceding section that one can write

$$\bar{F}_1^\sigma(s, t) \leq \bar{F}^\sigma(s, t) \leq \bar{F}_2^\sigma(s, t) \quad (5.1a)$$

where $\bar{F}_1^\sigma$, $\bar{F}_2^\sigma$ are known functions and (5.1) holds inside the Mandelstam region. Let us define Δ :

$$\Delta(s, t) \equiv \bar{F}_2^\sigma(s, t) - \bar{F}_1^\sigma(s, t) \geq 0. \quad (5.1b)$$

To project onto partial waves in the unphysical region of the s channel, $(M - \mu)^2 \leq s \leq (M + \mu)^2$, one uses (2.3) and writes another expansion

$$\bar{F}^{(+)}(s, t) = \frac{4\pi\sqrt{s}}{Mq_s} \sum_{\ell=0}^{\infty} P_{\ell}^{-1} P_{\ell}^{(1,1)}(\cos \theta_s) f_{\ell}^{(+)}(s), \quad (5.2)$$

so that

$$\begin{aligned} \bar{f}_{\ell}^{(+)}(s) = & \frac{1}{2} \left\{ P_{\ell}(\ell+4) \left[f_{\ell+}^{(+)}(s) + f_{\ell+1,-}^{(+)}(s) \right] \right. \\ & \left. - P_{\ell+1}(\ell+4) \left[f_{\ell+1,+}^{(+)}(s) + f_{\ell+2,-}^{(+)}(s) \right] \right\}. \end{aligned} \quad (5.3)$$

Then, one has

$$\begin{aligned} \bar{f}_{\ell}^{(+)}(s) = & \\ = & \frac{Mq_s}{4\pi\sqrt{s}} \int_{-1}^1 d\cos \theta_s (1 - \cos^2 \theta_s) P_{\ell}^{(1,1)}(\cos \theta_s) \bar{F}^{(+)}(s, t) \\ = & \frac{M}{8\pi q_s \sqrt{s}} \int_0^{-4q_s^2} dt (1 - \cos^2 \theta_s) P_{\ell}^{(1,1)}(\cos \theta_s) \bar{F}^{(+)}(s, t). \end{aligned} \quad (5.4)$$

As it is well known, when s is in the unphysical range one has $0 \leq -4q_s^2 \leq 4\mu^2$, so that the whole range of integration in (5.4) lies within the Mandelstam region. Further, as the methods described in Section 4 give good bounds only for, say, $M^2 + \mu^2 - M\mu \lesssim s \lesssim M^2 + \mu^2 + M\mu$, we do not have to care about the threshold behaviour of the $\bar{f}_{\ell}^{(+)}(s)$, as thresholds will not be reached. Symmetrizing with respect to $\cos \theta_t$ just brings in the u channel expansion, so that, if we write

$$\bar{F}^\sigma(s, t) = \frac{4\pi\sqrt{s}}{Mq_s} \sum_0^\infty p_\ell^{-1} p_\ell^{(1,1)}(\cos\theta_s) \bar{f}_\ell^\sigma(s), \quad (5.5a)$$

then, remarking that $\cos\theta_t \rightarrow -\cos\theta_t$ is equivalent to $s \rightarrow u$, we see that

$$\begin{aligned} \bar{f}_\ell^\sigma(s) = & \frac{p_\ell}{2} \left\{ \frac{\ell(\ell+2)}{2\ell+1} [k_+(s) \bar{f}_{\ell-1,+}^{(+)}(s) + k_-(s) \bar{f}_{\ell,-}^{(+)}(s)] \right. \\ & + \frac{(\ell+2)(\ell+3)}{2\ell+5} [k_+(s) \bar{f}_{\ell+1,+}^{(+)}(s) + k_-(s) \bar{f}_{\ell+2,-}^{(+)}(s)] \\ & - \frac{\ell(\ell+2)}{2\ell+1} [k_-(s) \bar{f}_{\ell,+}^{(+)}(s) + k_+(s) \bar{f}_{\ell+1,-}^{(+)}(s)] \\ & \left. - \frac{(\ell+2)(\ell+3)}{2\ell+3} [k_-(s) \bar{f}_{\ell+2,+}^{(+)}(s) + k_+(s) \bar{f}_{\ell+3,-}^{(+)}(s)] \right\}, \end{aligned} \quad (5.5b)$$

with

$$k_\pm(s) = \frac{q_s^2}{(s^{1/2} \pm M)^2 - \mu^2}. \quad (5.5c)$$

Then, $\bar{f}_\ell^\sigma$ is given by Eq. (5.4) changing $\bar{F}^{(+)}$ by $\bar{F}^\sigma$. Using now (5.1), it follows that, if we write

$$\bar{f}_\ell^\sigma(s, i) = \frac{Mq_s}{4\pi\sqrt{s}} \int_{-1}^{+1} d\cos\theta_s p_\ell^{(1,1)}(\cos\theta_s) [1 - \cos^2\theta_s] \bar{F}_i^\sigma(s, t), \quad (5.6)$$

$i = 1, 2,$
then

$$\bar{f}_\ell^\sigma(s, 2) + \delta_\ell(s, 2) \geq \bar{f}_\ell^\sigma(s) \geq \bar{f}_\ell^\sigma(s, 1) - \delta_\ell(s, 1), \quad (5.7a)$$

where,

$$\begin{aligned}
 \delta_\ell(j;2) &\equiv \frac{1}{2} \left\{ \max_{0 \leq t \leq -t_{\frac{1}{2}}} \Delta(j,t) \right\} \int_{-1}^{+1} d\cos\theta_s (1-\cos^2\theta_s) [|P_\ell^{(1,1)}(\cos\theta_s)| - P_\ell^{(1,1)}(\cos\theta_s)], \\
 \delta_\ell(j;1) &\equiv \\
 &\equiv \frac{1}{2} \left\{ \max_{0 \leq t \leq -t_{\frac{1}{2}}} \Delta(j,t) \right\} \int_{-1}^{+1} d\cos\theta_s (1-\cos^2\theta_s) [|P_\ell^{(1,1)}(\cos\theta_s)| + P_\ell^{(1,1)}(\cos\theta_s)].
 \end{aligned}
 \tag{5.7b}$$

Therefore, we can still project, with known errors, into partial πN waves in the unphysical region.

6. SEPARATION OF WAVES AND AMPLITUDES WITHOUT POSITIVITY

6.1 Separation of waves

Our equations, say (2.13), (5.5) give only certain combinations of waves. We will show now how to recover individual waves from them.

To separate waves with different total angular momentum is easy enough. Concentrating in (2.13) for definiteness, we just remark that

$$\begin{aligned} & \frac{2\ell+3}{\ell+1} \bar{k}_\ell^{(+)}(t) + \frac{2\ell+7}{\ell+3} \bar{k}_{\ell+2}^{(+)}(t) + \dots + \frac{2\ell+3+2n}{\ell+1+2n} \bar{k}_{\ell+2n}^{(+)}(t) + \dots \\ &= 8\pi \frac{t^{1/2}}{q_t^{1/2} Q_t^{3/2}} \left\{ \bar{g}_\ell^{(+)}(t) - \frac{t^{1/2}}{2M} \bar{g}_\ell^{(-)}(t) \right\}, \end{aligned} \quad (6.1)$$

and the series can be truncated with little error using, with $z_{th}(t)$ being as usual,

$$\bar{k}_{\ell+2n}^{(+)}(t) < \frac{Q_{\ell+2n}^{(1,1)}(z_{th})}{Q_\ell^{(1,1)}(z_{th})} \bar{k}_\ell^{(+)}(t).$$

Then it remains to separate the $\bar{g}_\ell^{(\pm)}(t)$. This is not very important from the practical point of view, because in all models the contribution of $\bar{g}_\ell^{(-)}$ is negligible; this is clear in (6.1) due to the kinematical factor $t^{1/2}/2M < 13\%$. For the sake of completeness, however, we shall indicate how to solve the problem.

The point is to use another amplitude with positivity; for example, we can take Bell's amplitude ¹⁸⁾

$$F'^{(+)}(s, t) = A'^{(+)}(s, t) + \frac{M(s-u)}{4M^2 - t} B'^{(+)}(s, t).$$

If one then expands it in Jacobi polynomials, in the t channel

$$\bar{F}'^{(+)}(s, t) = \sum_0^\infty A_\ell^{-1} P_\ell^{(1,1)}(\cos \theta_t) \bar{k}_\ell'^{(+)}(t),$$

then everything that was due here for the $\bar{k}_\ell^{(+)}$ can be repeated without any change. So we can find the $\bar{k}_\ell'^{(+)}$, and from the knowledge of both $\bar{k}_\ell^{(+)}$, $\bar{k}_\ell'^{(+)}$ the $\bar{g}_\ell^{(\pm)}$ can be separated.

6.2 Amplitudes without positivity

It is clear that since our analysis is based on positivity, it will not be possible to apply it directly to isospin states different from $I=0$ in the t channel. However, it is not difficult to extend our methods to such cases. Unfortunately, as it will be obvious in a moment, we cannot put bounds on the errors for the very important t channel P wave so that the problem becomes less interesting, and we shall therefore run quickly over the discussion.

Let us start from the amplitude with isospin one in the t channel,

$$F^{(-)}(s, t) = \frac{1}{3} F^{(s; 1/2)} - \frac{2}{3} F^{(s; 3/2)} = F^{(t; 1)}, \quad (6.2a)$$

$$F^{(-)}(s, t) = A^{(-)}(s, t) + \frac{s - M^2 - m^2}{2M} B^{(-)}(s, t). \quad (6.2b)$$

Its s , u and t channel expansions are exactly equal to those of $F^{(+)}$ with the replacements e.g., of $f_{\ell\pm}^{(+)}$ by $f_{\ell\pm}^{(-)}$ of $k_{\ell}^{(+)}$ by $k_{\ell}^{(-)}$ and of $g_J^{(\pm)}$ which correspond to isospin $I_t=0$ by what may be called $\gamma_J^{(\pm)}$, i.e., waves with $I_t=1$. Also the sum replacing (2.5) will run over odd values of J now.

The Froissart-Gribov representation can still be written. When $\ell = \text{odd}$, it reads

$$\begin{aligned} \bar{k}_{\ell}^{(-)}(t) &= \frac{1}{\pi q_t Q_t} \int_{(M+\mu)^2}^{\infty} ds \{ F_s^{(-)}(s, t) - F_u^{(-)}(s, t) \} \\ &\times \left\{ \left(\frac{s + q_t^2 + Q_t^2}{2q_t Q_t} \right)^2 - 1 \right\} Q_{\ell}^{(1,1)} \left(\frac{s + q_t^2 + Q_t^2}{2q_t Q_t} \right). \end{aligned} \quad (6.3)$$

Just as before, we can define truncated amplitudes, and waves ^{*}):

$$F^{(-)}(s, t; L, s_0) = \begin{cases} \frac{4\pi\sqrt{s}}{Mq_s} \left\{ \sum_0^L [P'_{\ell+1} - P'_{\ell}] f_{\ell+}^{(-)} + \sum_1^L [P'_{\ell} - P'_{\ell-1}] f_{\ell-}^{(-)} \right\}, & s \leq s_0, \\ \frac{2\sqrt{s} q_s}{M} \sigma_{tot}^{(-)}(s), & s > s_0, \end{cases} \quad (6.4a)$$

^{*}) Actually one should replace (6.4a) by the equivalent to (3.1e).

$$\begin{aligned} \bar{k}_l^{(-)}(t; L, s_0) &= \frac{1}{\pi q_t Q_t} \int_{(M+\mu)^2}^{\infty} ds \{ F_s^{(-)}(s, t; L, s_0) - F_u^{(-)}(s, t; L, s_0) \} \\ &\times \left\{ \left(\frac{s + q_t^2 + Q_t^2}{2q_t Q_t} \right)^2 - 1 \right\} Q_l^{(1,1)} \left(\frac{s + q_t^2 + Q_t^2}{2q_t Q_t} \right). \end{aligned} \quad (6.4b)$$

Similarly, one would define $\bar{\psi}_l^{(-)}(t)$, $\bar{\psi}_l^{(-)}(t; L, s_0)$, etc. Therefore, we can calculate all odd waves.

Now, the only thing one can do if one wants to put rigorous errors is to remark the obvious fact that, again using positivity and the results of Ref. 14)

$$\begin{aligned} &|\bar{k}_l^{(-)}(t) - \bar{k}_l^{(-)}(t; L, s_0)| \\ &< \frac{1}{\pi q_t Q_t} \int_{(M+\mu)^2}^{\infty} ds \{ (\Delta_L + \Delta_{s_0}) F^{(+)}(s, t) \} \\ &\times \left\{ \left(\frac{s + q_t^2 + Q_t^2}{2q_t Q_t} \right)^2 - 1 \right\} Q_l^{(1,1)} \left(\frac{s + q_t^2 + Q_t^2}{2q_t Q_t} \right) \\ &< \frac{Q_l^{(1,1)} \left(\frac{2M\mu + t/2}{2q_t Q_t} \right)}{Q_{l-1}^{(1,1)} \left(\frac{2M\mu + t/2}{2q_t Q_t} \right)} \{ (\Delta_L + \Delta_{s_0}) \bar{k}_{l-1}^{(+)}(t) \}. \end{aligned} \quad (6.5)$$

(6.5)

This relates the bounds over odd l isospin one waves to the already familiar errors on the isospin zero, even l ones. It is clear that the method gives nothing for $l=1$, as already the first majorization diverges.

From (6.4), (6.5) it is easy to repeat the analysis of Section 4, and, therefore, obtain estimates, and absolute bounds, on the antisymmetric part $2\bar{F}^{\sigma} = \bar{F}^{(-)}(\cos \Theta_t, t) - \bar{F}^{(-)}(-\cos \Theta_t, t)$ of $\bar{F}^{(-)}$ inside the Mandelstam region; the errors committed will be bounded by the errors of $\bar{F}^{\sigma}$. Two remarks only have to be made: as just noted, the method gives the approximate P wave, but not the error. Hence, such errors have to be taken from an independent source [in this sense the situation is better than for $I_t=0$ where not even $\bar{k}_0^{(+)}(t; L, s_0)$ could be computed]. Then, of the three reconstruction methods of Section 4, only the first and last will

work: the second will blow up as the quantity corresponding to $B(\xi)$ will not be positive any more. To be precise, one can still write Padé approximants for $F^{(-)}$, but, although such approximants can be expected to converge very fast²⁵⁾, we will not be able to obtain exact bound estimates like those of Eq. (4.13).

PART II: STABLE EXTRAPOLATION TO THE CUTS

7. t CHANNEL POSITIVITY PROPERTIES

7.1 Positivity of partial waves

Up to now, we have been using the positivity properties of the amplitude in the s and u channels. We will now show that one also has positivity in some regions of the t channel. In this subsection it will be shown that, under reasonable assumptions, one has positivity in a t interval which, rigorously, extends from $4\mu^2$ to $16\mu^2$ and most likely reaches up to $t \sim 2M^2$. This positivity had been obtained, in an unpublished work of the author (communicated at the 1971 Meeting of the Spanish Physical Society) for processes without spin; we present here the discussion with the spin of the nucleon taken into account. Actually, the proof works for any process of the type $\bar{A}A \rightarrow \pi\pi$, A with any spin, but we shall not bother to give such a general proof here.

The starting point is the remark, due to Mandelstam²⁶⁾, that generalized unitarity implies that

$$ph \hat{g}_\ell^{(\pm)}(t) = ph f_\ell^{\pi\pi}(t) + i n_\pm \pi, \quad (7.1)$$

where $f_\ell^{\pi\pi}$ is the $\pi\pi$ ℓ^{th} wave with isospin zero, $n_\pm$ may equal 0 or 1, $ph z$ is the phase of z , and

$$\hat{g}_\ell^{(\pm)}(t) \equiv t^{1/2} q_\pm^{-1/2} Q_\pm^{-3/2} g_\ell^{(\pm)}(t).$$

If we now consider the combination

$$\alpha_\ell(t) = \hat{g}_\ell^{(+)}(t) - \frac{t^{1/2}}{2M\sqrt{t+1}} \hat{g}_\ell^{(-)}(t),$$

then, by virtue of (7.1) we shall be able to write

$$ph \alpha_\ell(t) = ph f_\ell^{\pi\pi}(t) + i n \pi, \quad n = 0 \text{ or } 1. \quad (7.2)$$

Now, we assume that n has a constant value, i.e., $n=0$ always or $n=1$ always. This will be true if, for example, $\text{Im} \alpha_\ell$ and $\text{Im} f_\ell^{\pi\pi}$ do not vanish, which seems to be the case. This will be discussed further on.

Since because of unitarity, $\text{Im} f_\ell^{\pi\pi}(t) > 0$ for $t \geq 4\mu^2$, using (7.2) it follows that, in the region in which (7.1) holds,

$$(-1)^n \text{Im} \alpha_\ell(t) > 0. \quad (7.3)$$

Because of the constancy of n , it is sufficient to find it at one point, which we take to be $t = 4\mu^2$. But, here, due to the threshold behaviour of the $g_\ell^{(\pm)}$, Eq. (2.7), it follows that (apart from irrelevant numerical factors) $\alpha_\ell(t) \simeq k_\ell^{(+)}(t)$. Still, as $t \rightarrow 4\mu^2$, $k_\ell^{(+)}(t)$ becomes purely real: hence, we have the alternative

$$0 < \text{ph } k_\ell^{(+)}(t) < \delta, \quad t \gtrsim 4\mu^2 \quad \text{if } n=0 \quad (7.4a)$$

$$\pi < \text{ph } k_\ell^{(+)}(t) < \pi + \delta, \quad t \gtrsim 4\mu^2, \quad \text{if } n=1, \quad (7.4b)$$

δ small, and conversely. Therefore, we shall have that $n=0$, if, and only if, $\text{Re } k_\ell^{(+)}(t \gtrsim 4\mu^2) > 0$, and $n=1$, if, and only if, $\text{Re } k_\ell^{(+)}(t \gtrsim 4\mu^2) < 0$. Since $\text{Re } k_\ell^{(+)}$ is continuous across the point $t = 4\mu^2$, we may just check this for $t \lesssim 4\mu^2$. But, in this region we can use (2.16a), (2.20), for $\ell = \text{even} \geq 2$, together with the asymptotic behaviour of the $Q_\ell^{(1,1)}$,

$$Q_\ell^{(1,1)}(z) \simeq \frac{N_\ell}{2^{\ell+1} (z^2 - 1) z^{\ell+1}}, \quad z \rightarrow \infty,$$

to get that

$$\text{Re } k_\ell^{(+)}(t) \simeq N_\ell \left[\frac{a_\ell}{\pi M} + G^2 \mu^{-\ell} \right] (\mu^2 - M^2)^{\ell/2} [q_t^2]^{\ell/2}, \quad (7.5)$$

where

$$a_\ell = \int_{(M+\mu)^2}^{\infty} ds \left\{ F_s^{(+)}(s, 4\mu^2) + F_u^{(+)}(s, 4\mu^2) \right\} / (s + \mu^2 - M^2)^{\ell+1},$$

which is positive due to the s and u channel positivity properties of $F^{(+)}$. Hence, everything in (7.5) is positive except the factors $(\mu^2 - M^2)^{\ell/2} (q_t^2)^{\ell/2}$: so that the sign of $\text{Re } k_\ell^{(+)}(t \lesssim 4\mu^2)$ will be (+) if $\ell/2 = \text{even}$, (-) if $\ell/2 = \text{odd}$. Climbing back the ladder, we have therefore proved: in the region in which (7.1) holds one has $(-1)^{\ell/2} \text{Im } \alpha_\ell(t) > 0$, and, hence, using the definition of α_ℓ and (2.10),

$$(-1)^{\ell/2} \text{Im } k_\ell^{(+)}(t) = (-1)^{\ell/2} \text{Im } \bar{k}_\ell^{(+)}(t) > 0. \quad (7.6)$$

Equation (7.1) is expected to hold far ahead of the region in which it holds rigorously ($4\mu^2 \leq t \leq 16\mu^2$). Since, to invalidate (7.6), what one needs is inelastic contributions larger than elastic ones, it is reasonable to expect (7.6) to hold inside the interval $(4\mu^2, \sim 2M^2)$.

Let us now turn back to the discussion of our assumption, $n = \text{constant}$. If we assume validity of the Mandelstam representation, and a reasonable behaviour for the Pomeranchuk trajectory, we can write, up to irrelevant kinematical factors,

$$\text{Im } \bar{k}_\ell^{(+)}(t) = \int ds \, \rho(s, t) Q_\ell^{(1,1)}(z(s, t)), \quad (7.7)$$

even when $t > 4\mu^2$. Now, Mahoux and Martin²⁷⁾ have proved that $\rho(s, t) > 0$ in the double elastic region^{*}). Since this region covers the lower part of the integral in (7.7) it is clear that, for large ℓ , this region will dominate and, therefore, $\text{Im } \bar{k}_\ell^{(+)}$ will keep a constant sign: (7.6) is proved, for large ℓ , without needing to assume $n = \text{constant}$. For any ℓ we shall have a constant sign of $\text{Im } \bar{k}_\ell^{(+)}$ if the inelastic contributions to (7.7) are not large enough to overpower the elastic ones. Thus, we may substitute the assumption $n = \text{constant}$ by the (more physical?) assumption of "elastic dominance".

Needless to say, the arguments work only for $\ell = \text{even}$ and isospin $I_t = 0$ so that the zero which seems to exist in the $\text{Im } k_1^{(-)}(t)$ $I_t = 1$ wave has nothing to do with our situation.

7.2 Positivity of the amplitude

It is evident that neither of the above methods is going to work for $\ell = 0$. Therefore, we shall consider the amplitude $F^{\sigma \text{red}}(s, t)$. In the region $4\mu^2 \leq t \leq 2M^2$, say, one can write

$$\bar{F}_t^{\sigma \text{red}}(s, t) = \sum_{\substack{\ell=2 \\ \ell=\text{even}}}^{\infty} \rho_\ell^{-1} P_\ell^{(1,1)}(\cos \theta_t) \text{Im } \bar{k}_\ell^{(+)}(t).$$

By relating the $P_\ell^{(1,1)}$ to the so-called Gegenbauer polynomials²²⁾ one gets the representation

$$P_\ell^{(1,1)}(z) = \frac{\ell+1}{2} \int_0^\pi d\xi \, (\sin^2 \xi) [z + (\cos \xi)(z^2 - 1)^{1/2}]^\ell. \quad (7.9)$$

In the above described region of t , $\cos \theta_t$ is pure imaginary if $s \geq (M - \mu)^2$; writing then $\cos \theta_t = ir$, $r = \text{real}$, and substituting into (7.9) it follows immediately that
($\ell = \text{even}$)

^{*}) Actually, Mahoux and Martin's proof was for $\pi\pi$ scattering, but it can easily be arranged to suit our case. We are grateful to A. Martin for suggesting the alternate approach that is now being described.

$$(-1)^{\ell/2} P_{\ell}^{(1,1)}(\cos \theta_t) > 0: \quad (7.10)$$

hence, the amplitude has the positivity properties

$$\overline{F}_{\ell}^{\sigma \text{ sed}}(s, t) > 0; \quad 4\mu^2 \leq t \lesssim 2M^2, \quad (M-\mu)^2 \leq s \leq (M+\mu)^2. \quad (7.11)$$

7.3 t channel representation of waves and amplitudes

We shall use three representations: dispersion relations from the $\bar{k}_{\ell}^{(+)}$, and for $\bar{F}^{\sigma}(s, t)$, and Froissart-Gribov for the f_{ℓ}^{σ} . The most interesting ones are the first and last, that we now discuss briefly. One has

$$\begin{aligned} \bar{\Psi}_{\ell}^{(+)}(t) &= \frac{1}{\pi} \int_{-\infty}^0 dt' \frac{\text{Im } \bar{\Psi}_{\ell}^{(+)}(t')}{t' - t} \\ &+ \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\text{Im } \bar{\Psi}_{\ell}^{(+)}(t')}{t' - t}. \end{aligned} \quad (7.12)$$

Now, we have shown that $\text{Im } \bar{\Psi}_{\ell}^{(+)}(t') > 0$ in the range $4\mu^2 < t' \lesssim 2M^2$, but we know nothing about the range $2M^2$ to $4M^2$. This suggests that we use the combination

$$\left[Q_t^{\ell/2} / (\mu^2 - M^2)^{\ell/2} \right] \bar{\Psi}^{(+)}(t) \equiv \bar{\Psi}_{\ell}^{(+)}(t), \quad (7.13)$$

instead of the $\bar{\Psi}_{\ell}^{(+)}$ itself so as to depress the $2M^2$ to $4M^2$ region. Then, we define

$$\begin{aligned} \bar{\Phi}_{\ell}^{(+)}(t) \\ \equiv \frac{1}{\pi} \int_{4\mu^2}^{2M^2} dt' \frac{\text{Im } \bar{\Psi}_{\ell}^{(+)}(t')}{t' - t}, \end{aligned} \quad (7.14a)$$

whose relation to ψ (Ψ) may be conveniently written as

$$\begin{aligned}
 \bar{\varphi}_\ell^{(+)}(t) = & \bar{\Psi}_\ell^{(+)}(t) \\
 & - \left\{ \frac{1}{\pi} \int_{-\infty}^{-28\mu^2} dt' \frac{\text{Im } \bar{\Psi}_\ell^{(+)}(t')}{t' - t} \right. \\
 & + \frac{1}{\pi} \int_{2M^2}^{4M^2} dt' \frac{\text{Im } \bar{\Psi}_\ell^{(+)}(t')}{t' - t} \\
 & + \frac{1}{\pi} \int_{-28\mu^2}^0 dt' \frac{\text{Im } \bar{\Psi}_\ell^{(+)}(t')}{t' - t} \\
 & \left. + \frac{1}{\pi} \int_{4M^2}^{\infty} dt' \frac{\text{Im } \bar{\Psi}_\ell^{(+)}(t')}{t' - t} \right\}.
 \end{aligned} \tag{7.14b}$$

We will discuss this in some detail. First, the two first integrals in the right-hand side of (7.14b). We cannot say much about them. In a first analysis one would altogether neglect their contribution; this may be justified because $\bar{\Psi}_\ell^{(+)}(t') \sim Q_{t'}^{\ell/2}$, for $t' \rightarrow -\infty$, and $\bar{\Psi}_\ell^{(+)}(t') \sim Q_{t'}^{\ell/2}$, for $t' \rightarrow 4M^2$, respectively.

The third integral can be computed using πN scattering data ³⁾. The last contributes a quantity that satisfies the bound, when $t < 4\mu^2$,

$$\left| \frac{1}{\pi} \int_{4M^2}^{\infty} dt' \frac{\text{Im } \bar{\Psi}_\ell^{(+)}(t')}{t' - t} \right| < \frac{128 \pi^2}{\sqrt{6} N_\ell M^{2\ell+1/2}} \int_0^\infty dx \frac{1}{1+x^4}, \tag{7.15}$$

where we have used (2.13), (2.14) and the unitarity bound, (the $1/\sqrt{6}$ comes from isospin crossing), $|g_\ell^{(\pm)}| < 1/\sqrt{6}$, and neglected μ^2/M^2 as compared to unity.

Thus, the contribution of such term will just add a known quantity to the errors with which $\bar{\Psi}_\ell^{(+)}$ was known. Hence, we may work with (7.14a), assuming $\bar{\varphi}_\ell^{(+)}$ to be known within certain errors, inside the region $t < 4\mu^2$.

Finally, we turn to the Froissart-Gribov representation of the $\bar{F}_\ell^\sigma(s)$. This reads

$$\begin{aligned} \bar{F}_\ell^\sigma(s) = & \frac{1}{\pi q_s^2} \int_{4\mu^2}^{\infty} dt \left\{ \left(\frac{t+2q_s^2}{2q_s^2} \right)^2 - 1 \right\} Q_\ell^{(1,1)} \left(\frac{t+2q_s^2}{2q_s^2} \right) \bar{F}_t^\sigma(s,t) \\ & + (-1)^\ell \frac{1}{\pi q_s^2} \int_{(M+\mu)^2}^{\infty} du \left\{ \left(\frac{u+s-2M^2-2\mu^2-2q_s^2}{2q_s^2} \right)^2 - 1 \right\} \\ & \times Q_\ell^{(1,1)} \left(\frac{u+s-2M^2-2\mu^2-2q_s^2}{2q_s^2} \right) \bar{F}_u^\sigma(u,t). \end{aligned} \quad (7.16)$$

Here, the second integral in the right-hand side can be computed because the $\bar{F}_u^\sigma$ is expressible in terms of the physical πN phase shifts. It then remains the piece

$$\int_{2M^2}^{\infty} dt \left\{ \left(\frac{t+2q_s^2}{2q_s^2} \right)^2 - 1 \right\} Q_\ell^{(1,1)} \bar{F}_t^\sigma(s,t),$$

with which we cannot do anything yet, and, therefore, just assume to be negligible in a first analysis. Thus, we are left with the quantity

$$\begin{aligned} \bar{a}_\ell^\sigma(s) \equiv & \frac{1}{\pi q_s^2} \int_{4\mu^2}^{2M^2} dt \left\{ \left(\frac{t+2q_s^2}{2q_s^2} \right)^2 - 1 \right\} \\ & \times Q_\ell^{(1,1)} \left(\frac{t+2q_s^2}{2q_s^2} \right) \bar{F}_t^\sigma(s,t), \end{aligned} \quad (7.17)$$

with $\bar{F}_t^\sigma(s,t) \geq 0$.

8. STABLE EXTRAPOLATION TO THE CUTS USING PADE APPROXIMANTS ^{*)}

From the analysis of the previous sections we have been led to study equations of the type

$$\mu_n = \int_0^1 dx \cdot x^n \rho(x), \quad \rho(x) \geq 0, \quad (8.1)$$

or of the type

$$\varphi(z) = \int_0^1 dx \frac{\rho(x)}{z-x}, \quad \rho(x) \geq 0, \quad (8.2)$$

and we are supposed to know $\mu_0, \mu_1, \dots, \mu_N$, respectively $\varphi(z_0), \varphi(z_1) \dots \varphi(z_N)$ ($z_1 > 1$) with some prescribed errors.

To treat such problems we usually apply the various extrapolation techniques available in the literature ⁹⁾. However, in the present situation, and due to the positivity of ρ , much more powerful methods can be devised, based on the technique of Padé approximants. Since these methods are of interest in themselves, and have applications to problems other than the ones with which we are concerned here, we shall discuss (8.1) and (8.2) in their own, specializing later.

First of all, it is obvious that we shall not get pointwise convergence. So, the proper way to formulate the problem is to demand a reconstruction not of ρ itself, but of averages of the type

$$\rho_I(x, x_0) \equiv \int dx' I(x-x') \rho(x'), \quad (8.3a)$$

where $I(x)$ is peaked around x_0 , and

$$\int dx I(x) = 1, \quad I(x) \geq 0. \quad (8.3b)$$

The better our extrapolation, the sharper we can take I and, in the limit of exact knowledge of all the μ_n [respectively, $\varphi(z_n)$], we could let $I(x) \rightarrow \delta(x-x_0)$.

Finally, we shall use Bernstein polynomials and Bernstein functions for I ; however, other types of functions can be used provided that 1) for (8.1), they be polynomials of degree $\leq N$; 2) for (8.2) they be of the form (polynomial of degree $< N$) / $\prod_{i=0}^N (1-x/z_i)$. The Bernstein choice has some advantages, and that is why we use them.

^{*)} A complete and mathematically rigorous treatment of this problem will be presented separately ²⁸⁾.

8.1 The Hausdorff problem

Equation (8.1) is what is usually referred to in the mathematical literature ¹⁷⁾ as the Hausdorff problem. As is known, sufficient and necessary conditions for $\rho > 0$ to exist are

$$\delta^k_{\mu_n} \equiv \sum_{j=0}^k \binom{k}{j} (-1)^j \mu_{n+j} > 0, \quad k+n \leq N. \quad (8.4)$$

The first thing that is clear is that (8.4) can be used to improve the error estimations. In fact, if we started with an error band

$$\mu'_n \leq \mu_n \leq \mu''_n, \quad n=0, 1, \dots, N, \quad (8.5)$$

we must have, in order to have reachable errors, that

$$\left. \begin{aligned} \sum \binom{k}{j} (-1)^j \mu''_{n+j} &> 0, \\ \sum \binom{k}{j} (-1)^j \mu'_{n+j} &> 0. \end{aligned} \right\} \quad (8.6)$$

This will reduce the range of allowed variation (see the Figure for an example of this). We shall henceforth assume that the range (8.5) is an allowed range. Then, from the values of the first $N+1$ moments, one can deduce an error band within which the higher moments, $n > N$, have to lie. This band is obtained by application of the techniques of Refs. 17), 26), getting

$$\max_{\mu'' \geq \mu \geq \mu'} \sum_{i=0}^{N/2} w_i \xi_i^\nu > \mu_\nu > \min_{\mu'' \geq \mu \geq \mu'} \left| \sum_{i=0}^{N/2} w_i \xi_i^\nu \right|, \quad (8.7)$$

where, just as in Section 4,

$$\left. \begin{aligned} 0 < \xi_1 < \dots < \xi_{N/2} < \xi_{N/2+1} = 1, \\ \xi_0 = 0 < \xi_1 < \dots < \xi_{N/2}, \end{aligned} \right\} \quad (8.8a)$$

are, respectively, the roots of the polynomials

$$q(\xi) = \det \begin{pmatrix} 1 & \xi & \xi^2 & \dots & \xi^{N/2+1} \\ 1 & 1 & 1 & \dots & 1 \\ \mu_0 & \mu_1 & \mu_2 & \dots & \mu_{N/2+1} \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{N/2-1} & \mu_{N/2} & \dots & \mu_N \end{pmatrix}, \quad (8.8b)$$

$$Q(\xi) = \det \begin{pmatrix} 1 & \xi & \dots & \xi^{N/2+1} \\ 1 & 0 & \dots & 0 \\ \mu_1 & \mu_2 & \dots & \mu_{N/2+1} \\ \dots & \dots & \dots & \dots \\ \mu_{N/2-1} & \dots & \dots & \mu_N \end{pmatrix}, \quad (8.8c)$$

and, if we define

$$\frac{q(\xi)}{(\xi - \xi_i) q'(\xi_i)} \equiv \sum_{n=0}^{N/2} a_{in} \xi^n, \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad (8.8d)$$

$$\frac{Q(\xi)}{(\xi - \xi_i) Q'(\xi_i)} \equiv \sum_{n=0}^{N/2} A_{in} \xi^n,$$

then,

$$w_i = \sum_0^{N/2} a_{in} \mu_n, \quad W_i = \sum_0^{N/2} A_{in} \mu_n. \quad (8.8e)$$

Furthermore, the max and min in Eq. (8.7) are to be taken for $\mu'' \equiv \mu_n''$, $\mu \equiv \mu_n$, $\mu' \equiv \mu_n'$ and $n \leq N$. Therefore, we can, out of the knowledge of the first few μ 's deduce an allowed band for all of them. We shall see that this may, in some cases, be useful in reducing errors.

Next, we shall construct the Padé approximants to the function

$$\varphi(z) \equiv \frac{1}{\pi} \int_0^1 dx \frac{\rho(x)}{x-z}.$$

Clearly, $\text{Im } \varphi = \rho$. These approximants are similar to those considered in Section 4 of this paper [see also Ref. 26]; what we use here is denoted by $[\bar{N}, \bar{N}]$ or $[\bar{N}+1, \bar{N}]$ in this reference] and we shall not repeat the construction. We will just denote them by

$$\varphi_N^\mu(z), \quad \text{Im } \varphi_N^\mu(x) \equiv \rho_N^\mu(x), \quad (8.9a)$$

and recall their main property:

$$\mu_n = \int_0^1 dx \cdot x^n \rho_N^\mu(x). \quad (8.9b)$$

Instead of considering general averages, we shall take what may be called Bernstein-Stieltjes polynomials:

$$\left. \begin{aligned} B_{n,k}^{(N)}(x) &\equiv b_{n,k} x^n (1-x)^k, \quad n+k \leq N, \\ b_{n,k}^{-1} &\equiv \int_0^1 dx (1-x)^k x^n. \end{aligned} \right\} \quad (8.10)$$

As is known ¹⁷⁾, (A) $B_{n,k}^{(N)}$ is peaked around $x = n/(n+k)$, (B) any positive functions can be written as accurately as wishes by sums with positive coefficients of Bernstein polynomials. (This, in particular, extends our proof for any function) and (C) as $N \rightarrow \infty$, letting $n/(n+k) \rightarrow x_0$, one has $B_{n,k}^{(N)}(x) \rightarrow \delta(x - x_0)$: we recover local convergence in the limit.

Now, obviously, (8.9), (8.10) imply that

$$\int_0^1 dx B_{n,k}^{(N)}(x) \rho(x) = \int_0^1 dx B_{n,k}^{(N)}(x) \rho_N^\mu(x),$$

and, hence, taking now into account the possible errors,

$$\begin{aligned} \delta_{\max}^k \mu_n &\geq \int_0^1 dx B_{n,k}^{(N)}(x) \rho(x) \\ &\geq \max \{ \delta_{\min}^k \mu_n, 0 \}, \end{aligned} \quad (8.11a)$$

where

$$\left. \begin{aligned} \delta_{\max}^k \mu_n &\equiv \sum_{\substack{j=0 \\ j=\text{even} \leq k}} \binom{k}{j} \mu_{n+j}'' - \sum_{\substack{j=1 \\ j=\text{odd} \leq k}} \binom{k}{j} \mu_{n+j}' , \\ \delta_{\min}^k \mu_n &\equiv \sum_{\substack{j=0 \\ j=\text{even} \leq k}} \binom{k}{j} \mu_{n+j}' - \sum_{\substack{j=1 \\ j=\text{odd} \leq k}} \binom{k}{j} \mu_{n+j}'' . \end{aligned} \right\} \quad (8.11b)$$

Note that the over-all error permitted by (8.11) is less than

$$\delta_{\max}^k \mu_n - \delta_{\min}^k \mu_n = \sum_{j=0}^k \binom{k}{j} (\mu_{n+j}'' - \mu_{n+j}') . \quad (8.12)$$

Hence, if we say, for example, that $\mu_{n+j}'' - \mu_{n+j}' < \epsilon$, then

$$\delta_{\max}^k \mu_n - \delta_{\min}^k \mu_n < 2^k \epsilon : \quad (8.13)$$

This gives the maximum allowed error for the average extrapolation. It also gives the power of resolution of the extrapolation, viz., the errors have to be such that, in the above example, $2^k \epsilon \ll \delta_{\max}^k \mu_{N-k}$. In general, we define the resolution power, χ , as

$$\chi_N \equiv \max_{0 \leq k \leq N} \frac{\delta_{\max}^k \mu_{N-k} - \max \{ \delta_{\min}^k \mu_{N-k}, 0 \}}{\delta_{\max}^k \mu_{N-k}} , \quad (8.14)$$

and we shall be able to obtain resolution with B's provided that $\chi_N \ll 1$.

8.2 The Hilbert moment problem

For lack of a better name we shall call (8.2) a Hilbert moment problem. We shall assume that the z_i are such that the series $\sum 1/z_i$ is convergent; this is always the case in applications, and, moreover, one could, just as in Section 8.1, extrapolate stably the $\varphi(z)$ to points satisfying such condition, if it were not the case. Here, we also have an error band

$$\varphi''(z_i) \geq \varphi(z_i) \geq \varphi'(z_i), \quad i=0, 1, \dots, N, \quad (8.15)$$

that we may assume to be allowed. For the sake of definiteness, we shall also assume that N = even.

Sufficient and necessary conditions for the $\varphi(z_i)$ to satisfy (8.2) can be found in the literature; we shall, however, derive a new set that will be useful to provide the desired extrapolation. To do so, we start by constructing the so-called second kind Padé approximants ^{24), 25)} to $\varphi(z)$,

$$\psi_N(z) \equiv \frac{a_0/z}{1 + \frac{(1/z_0 - 1/z) a_1}{1 + \frac{(1/z_1 - 1/z) a_2}{1 + \dots \frac{(1/z_N - 1/z) a_{N+1}}{1}}}}, \quad (8.16a)$$

where the a 's are defined by the following recursion formula: we set

$$\begin{aligned} \gamma_0(z) &\equiv \frac{1}{z} \varphi(z), \\ \gamma_i(z) &\equiv \frac{\gamma_{i-1}(z_{i-1})/z_{i-1} - \gamma_{i-1}(z)/z}{(1/z_{i-1} - 1/z) \gamma_{i-1}(z)/z}; \end{aligned} \quad \left. \vphantom{\begin{aligned} \gamma_0(z) &\equiv \frac{1}{z} \varphi(z), \\ \gamma_i(z) &\equiv \frac{\gamma_{i-1}(z_{i-1})/z_{i-1} - \gamma_{i-1}(z)/z}{(1/z_{i-1} - 1/z) \gamma_{i-1}(z)/z}; \end{aligned}} \right\} \quad (8.16b)$$

then,

$$a_i = \gamma_i(z_i). \quad (8.16c)$$

Due to their very construction, these approximants satisfy the conditions

$$\psi_N(z_i) = \varphi(z_i), \quad i=0, 1, \dots, N.$$

Then, we introduce what may be called the Bernstein fractions,

$$\beta_{n,k}^{(N)}(x) = B_{n,k}^{(N)}(x) / c_{n,k} \prod_{i=0}^N (1 - x/z_i), \quad n+k+1 \leq N, \quad (8.17a)$$

where

$$c_{n,k} \equiv \int_0^1 dx \frac{B_{n,k}^{(N)}(x)}{\prod_{i=0}^N (1 - x/z_i)}. \quad (8.17b)$$

Note that, due to a well-known theorem of algebra, one can still write the β in terms of simple fractions,

$$\begin{aligned}
 \beta_{n,k}^{(N)}(x) &= \sum_{i=0}^N \frac{\lambda_i}{z_i - x} \\
 &= \sum_{i=\text{even}} \frac{|\lambda_i|}{z_i - x} - \sum_{i=\text{odd}} \frac{|\lambda_i|}{z_i - x}, \\
 \lambda_j &= (-1)^j |\lambda_j| = \\
 &= \frac{z_j^n (1 - z_j)^k b_{n,k}}{c_{n,k} \prod_{j \neq i} (1 - z_j/z_i)}.
 \end{aligned}
 \tag{8.18}$$

It is easy to check that the Bernstein fractions have the following properties:

(A) $\int_0^1 dx \beta_{n,k}^{(N)}(x) = 1$;

(B) $\beta_{n,k}^{(N)}(x) \geq 0$, for $0 \leq x \leq 1$;

(C) $\beta_{n,k}^{(N)}(x)$

is peaked around a point near $x = n/(n+k)$ (the exact location of the maximum depends, of course, on the z_i , and can, in general, be found only numerically);

(D) letting $N \rightarrow \infty$ and, given x_0 , $n/(n+k) \rightarrow x_0$, $\beta_{n,k}^{(N)}(x)$ tends to $\delta(x - x_0)$.

Thus, in particular, a necessary and sufficient condition for (8.2) to hold is that

$$\delta^{k,n} \varphi \equiv \sum_{i=0}^N \lambda_i \varphi(z_i) > 0.
 \tag{8.19}$$

We shall, just as before, define

$$\delta_{\max}^{k,n} \varphi \equiv \sum_{i=\text{even}} |\lambda_i| \varphi''(z_i) - \sum_{i=\text{odd}} |\lambda_i| \varphi'(z_i),
 \tag{8.20a}$$

$$\delta_{\min}^{k,n} \varphi \equiv \sum_{i=\text{even}} |\lambda_i| \varphi'(z_i) - \sum_{i=\text{odd}} |\lambda_i| \varphi''(z_i).
 \tag{8.20b}$$

Hence, reasoning just as in Section 8.1, we shall have

$$\begin{aligned} \delta_{\max}^{k,n} \varphi &\geq \int_0^1 dx \beta_{n,k}^{(N)}(x) \rho(x) \\ &\geq \max \{ \delta_{\min}^{k,n} \varphi, 0 \}, \end{aligned} \quad (8.21)$$

that is to say, we have performed the desired stable extrapolation from the φ to the ρ . Note that, although we have used the Bernstein fractions, due to property (D) above we might from this obtain convergence for any average. This same property ensures that, if the errors tend to zero, and we have $N \rightarrow \infty$, the convergence becomes point-like. Finally, we may define the focusing power to be

$$\begin{aligned} \chi_N &= \max_{0 \leq k \leq N-1} \frac{\delta_{\max}^{k,N-k-1} \varphi - \max \{ \delta_{\min}^{k,N-k-1} \varphi, 0 \}}{\delta_{\max}^{k,N-k-1} \varphi}, \end{aligned} \quad (8.22)$$

so that the extrapolation will be meaningful as long as $\chi_N \leq 1$.

8.3. Reconstruction of ρ

In the general case, the analysis of the two previous sections is the best one can do. If, however, we assume some boundedness for ρ , then there is some interest in constructing approximations to ρ itself rather than to its averages. Thus, we set ourselves the problem of reconstructing ρ from the knowledge of its averages. We shall treat the second case (Section 8.2); the other is totally similar.

One begins by defining normalized Bernstein functions:

$$\tilde{\beta}_{N-k-l,k}(x) = \frac{\beta_{N-k-l,k}(x)}{\left\{ \int_0^1 dx [\beta_{N-k-l,k}(x')]^2 \right\}^{1/2}} \equiv h_{Nk} \beta_{N-k-l,k}(x). \quad (8.23a)$$

These functions are "almost orthogonal" in the sense that, if we denote by $\delta_{k,j}$ to the usual Kronecker delta

$$\lim_{N \rightarrow \infty} \int_0^1 dx \tilde{\beta}_{N-k-l,k}(x) \tilde{\beta}_{N-j-l,j}(x) = \delta_{k,j}. \quad (8.23b)$$

Then, one has the bounds

$$\begin{aligned} \sum_{k=0}^{N-1} h_{Nk} \tilde{\beta}_{N-k-l, k}(x) \delta_{\max}^{k, N-k-l} \varphi &\geq \rho(x; N) \\ &\geq \sum_{k=0}^{N-1} h_{Nk} \tilde{\beta}_{N-k-l, k}(x) \max \left\{ \delta_{\min}^{k, N-k-l} \varphi, 0 \right\}, \end{aligned} \quad (8.24a)$$

where

$$\rho(x; N) \equiv \sum_{k=0}^{N-1} \tilde{\beta}_{N-k-l, k}(x) \int_0^1 dx' \tilde{\beta}_{N-k-l, k}(x') \rho(x'). \quad (8.24b)$$

It is not difficult to show that $\rho(x; N) \rightarrow \rho(x)$, in the mean for the general case, pointwise if $\rho(x)$ was continuous.

8.4 The real part of $\varphi(z)$

The Padé approximants to $\varphi(z)$, as given by Eq. (8.2) are known to converge to it everywhere outside the cut $[0, 1]$. We have just shown that the imaginary part in the cut is also given by these approximants. It remains to consider the real part, $\text{Re } \varphi(x)$ in the cut $[0, 1]$. We have two possible approaches to this problem.

(A) One may define averages of $\text{Re } \varphi(x)$ around given points. For example, consider

$$\text{Re } \varphi_I(x) \equiv \int_0^1 dx' \text{Re } \varphi(x') I(x-x')$$

where I is peaked around a given point, x_0 , and I is continuous with $\int_0^1 dx I(x) = 1$.

This can be written as

$$\left. \begin{aligned} \text{Re } \varphi_I(x) &= \int_0^1 dx \rho(x) J(x), \\ J(x) &= \frac{1}{\pi} \int dx' \frac{I(x')}{x' - x}. \end{aligned} \right\} \quad (8.25)$$

$J(x)$ is a continuous function, so we may write

$$\begin{aligned} \sum_{k=0}^{N-1} \alpha_{Nk} \tilde{\beta}_{N-k-l, k}(x) + \frac{\epsilon(N)}{x_0 - x} &\geq J(x) \\ &\geq \sum_{k=0}^{N-1} \alpha_{Nk} \tilde{\beta}_{N-k-l, k}(x) - \frac{\epsilon(N)}{x_0 - x}, \end{aligned} \quad (8.26a)$$

with

$$\alpha_{Nk} \equiv \int_0^1 dx \tilde{\beta}_{N-k-1,k}(x) J(x), \quad (8.26b)$$

and where $\epsilon(N)$ can be computed explicitly, as $J(x)$ is known, and satisfies

$$\lim_{N \rightarrow \infty} \epsilon(N) = 0.$$

Plugging then (8.26a) into (8.25) gives

$$\begin{aligned} & \sum_{k=0}^{N-1} \alpha_{Nk} \int_0^1 dx \rho(x) \tilde{\beta}_{N-k-1,k}(x) + \pi \epsilon(N) \varphi(z_0) \\ & \geq \operatorname{Re} \varphi_I(x) \\ & \geq \sum_{k=0}^{N-1} \alpha_{Nk} \int_0^1 dx \rho(x) \tilde{\beta}_{N-k-1,k}(x) - \pi \epsilon(N) \varphi(z_0). \end{aligned} \quad (8.27)$$

Therefore, using (8.21), upper and lower bounds to $\operatorname{Re} \varphi_I(x)$ can be obtained immediately.

(B) Rather than averages, we may approximate $\operatorname{Re} \varphi(x_0)$ by $\operatorname{Re} \varphi(x_0 + i\epsilon)$ because, if $\rho(x)$ is reasonable, $\lim_{\epsilon \rightarrow 0} \operatorname{Re} \varphi(x_0 + i\epsilon) = \operatorname{Re} \varphi(x_0)$. Then,

$$\operatorname{Re} \varphi(x_0 + i\epsilon) = \frac{1}{\pi} \int_0^1 dx \rho(x) \frac{x - x_0}{(x - x_0)^2 + \epsilon^2},$$

so, the situation is just as before with $J(x)$ substituted by $(x - x_0)/[(x - x_0)^2 + \epsilon^2]$, which is also a continuous function.

8.5 Relation to the partial waves for the π -N system

Considering Eq. (7.17), and using exactly the same arguments as in Ref. 14), Section 3.2, we obtain that the $\bar{a}_l^\sigma$ obey the constraints

$$\delta^n \bar{a}_l^\sigma(s) \equiv \sum_{m=0}^n \binom{n}{m} (-1)^m [\Lambda(s)]^{-n+m} \bar{a}_{l+m}^\sigma(s) \geq 0,$$

with

$$\Lambda(s) = \frac{2\mu^2 + q_s^2 + 2\mu(q_s^2 + \mu^2)^{1/2}}{q_s^2}.$$

Hence, it follows ¹²⁾ that the a satisfy the representation

$$\bar{a}_l^\sigma(s) = \int_{\Lambda(s)}^\infty dz z^{-l-1} B(z), \quad B(z) > 0.$$

Changing variables, $x = \Lambda(s)/z$, one can rewrite this still as

$$\left. \begin{aligned} \bar{a}_\ell^\sigma(s) [\Lambda(s)]^\ell &= \int_0^1 dx \cdot x^\ell \rho(x), \\ \rho(x) &= \frac{1}{x} B(\Lambda(s)/x) \geq 0, \end{aligned} \right\} \quad (8.28)$$

so it suffices to identify μ_ℓ and $\bar{a}_\ell^\sigma \Lambda^\ell$ to apply directly the analysis of Sections 8.1, ff.

The methods developed in Sections 8.2 ff., for solving the Hilbert moment problem have obvious applications for dispersion relations on any function with positivity properties; for example, for the $F(s,t)$, $f_\ell(s)$ or the $\bar{k}_\ell^{(+)}(t)$. We shall, to exemplify the method, discuss the last two. This maintains consistency of the paper and keeps it within finite bounds. What one does is just write Eqs. (7.14), so that, by changing variables, $t' = 4\mu^2/x$, one gets

$$t \bar{\alpha}_\ell^{(+)}(t) = \int_0^1 dx \frac{\rho(x)}{(4\mu^2/t) - x}, \quad (8.29a)$$

where

$$\rho(x) = \begin{cases} \text{Im } \bar{\Psi}_\ell^{(+)}(4\mu^2/x) x^{-1}, & 4\mu^2/2M^2 < x < 1, \\ 0, & 0 \leq x \leq 4\mu^2/2M^2. \end{cases} \quad (8.29b)$$

Such equation is obviously identical with (8.2) with $z = 4\mu^2/t$.

The analysis for the $\bar{f}_\ell(s)$ is totally similar. Here the role of ρ would be played by the difference between $\bar{f}_\ell(s)$ and the contribution of the left-hand cut, the circle, and the low energy part of the right-hand cut, and the method would provide the extrapolation for $\text{Im } \bar{f}_\ell(s)$ when $s > s_0$, say.

8.6 Amplitudes without positivity

Much of what has been said in the preceding subsections applies also to the case in which the ρ of Eqs. (8.1) and (8.2), say, is not positive. There are, however, a number of important alterations that we pass to list. We will specialize to the Hausdorff moment problem, and leave the other case to the reader.

First of all, it is clear that Eqs. (8.4) will not hold any longer, and, therefore, there is no chance to improve the error estimates. Second, (8.7) will not hold, either, so it will be impossible to use more than the initial moments. Third, (8.11a) will change to

$$\delta_{\max}^k \mu_n \geq \int_0^1 dx B_{n,k}^{(N)}(x) \rho(x) \geq \delta_{\min}^k \mu_n,$$

thereby widening the error band. Lastly, and most important of all, error estimates, and, in particular, Eq. (8.14) do not have much meaning, as it is not difficult to construct oscillating ρ 's such that

$$\int_0^1 dx B_{n,k}^{(N)}(x) [\rho_1(x) - \rho_2(x)] = 0,$$

with $\rho_1 - \rho_2$ large at will. It should be clear, however, that this problem arises also in the usual extrapolation methods ⁹⁾.

8.7 The treatment of the inaccessible regions of Section 7.3

In Section 7.3 we dropped, a bit cavalierly, the contributions of the inaccessible regions,

$$t < -28\mu^2 \quad \text{for} \quad \bar{\Psi}_\ell^{(+)}(t)$$

[cf. (7.14)] and $t \geq 2M^2$ for $\bar{f}_\ell^\sigma(s)$, Eq. (7.16). Let us focus the discussion on the first, for definiteness. Since we are not interested in the values of $\bar{\Psi}_\ell^{(+)}(t)$ in the far away left hand cut, but only in the influence of this cut for $t > -28\mu^2$, to take that cut into account becomes much simplified. What we propose to use is an iteration procedure, based on the fact that, as it is well-known, from knowledge of the πN waves not only $\text{Im} \bar{\Psi}_\ell^{(+)}(t)$ is known in the range $-28\mu^2 < t < 0$, but also $\text{Re} \bar{\Psi}_\ell^{(+)}(t)$ can be obtained there. Then, one begins by neglecting altogether the cut $(-\infty, -28\mu^2)$, and computes, as in the previous subsections, $\text{Im} \bar{\Psi}_\ell^{(+)}(t; 0)$ in $(4\mu^2, \infty)$; with this, one forms a discrepancy function

$$\Delta(t; 0) \equiv \bar{\Psi}_\ell^{(+)}(t) - \frac{1}{\pi} \int_{-28\mu^2}^{\infty} dt' \frac{\text{Im} \bar{\Psi}_\ell^{(+)}(t')}{t' - t} - \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\text{Im} \bar{\Psi}_\ell^{(+)}(t'; 0)}{t' - t}.$$

(8.30)

This function is known on $(-28\mu^2, 0)$; continuing it analytically, we get its values on $(0, 4\mu^2)$. This gives a new starting $\varphi_\ell^{(+)}(t; 1)$ in $(0, 4\mu^2)$, from which one repeats the analysis of the previous subsections getting a second approximation, $\text{Im} \bar{\Psi}_\ell^{(+)}(t; 1)$ on $(4\mu^2, \infty)$. Then (8.30) is written again with $\text{Im} \bar{\Psi}_\ell^{(+)}(t; 1)$ in lieu of $\text{Im} \bar{\Psi}_\ell^{(+)}(t; 0)$, thus giving a new discrepancy, $\Delta(t; 1)$ etc. This method has the obvious advantage over the usual ones ¹⁰⁾ of utilizing, at every step, the positivity properties of $\bar{\Psi}_\ell^{(+)}$.

9. DISCUSSION AND COMMENTS

We hope to have shown along the preceding pages that the fact that scattering amplitudes have positivity properties may, and should be used to stabilize analytical extrapolations. After all, this should be no surprise: if $SS^+ = 1$, one cannot have too large errors without violating this equality. Thus, we are able, from the knowledge of reliable data alone, to give accurate extrapolation formulae keeping at the same time control of the errors that are being committed. This gives the values of partial waves in the unphysical regions; from these, we extrapolate to the values of the crossed amplitudes in the physical or pseudophysical regions. The method of extrapolation uses Padé approximants and is such that explicit values of the errors are given. The situation is much improved when we have also positivity of the crossed channel amplitudes, for then these error bands are totally meaningful.

Our main aim here was to show that it is possible to obtain absolute error bands for the analytical extrapolations. Numerical calculations are under way, but it should perhaps not be idle to remark that firstly, determinations of $\bar{k}_\ell^{(+)}$ using the methods of Section 3, have been performed for $\pi\pi$ scattering and yield very precise results; second, that the reconstruction of the amplitude, using methods similar to the ones described in Section 4 has been made, again showing high precision ¹²⁾⁻¹⁴⁾, and that, from the work of Baker ²⁴⁾ and Common ¹³⁾, and of the French Padé school ²⁵⁾ one knows that, numerically, Padé approximants converge fantastically fast.

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A P P E N D I X A

If $p_n(x)$ are polynomials, orthogonal with a weight w in the interval $[-1, +1]$, we define their associated functions by

$$f_n(z) = \frac{\lambda}{q(z)} \int_{-1}^{+1} dx \frac{q(x) p_n(x)}{z-x}, \quad (\text{A.1})$$

where $q(x)$ is a fixed polynomial, λ a constant.

In Refs. 29), 14), it was shown that $f_{n+k}(z)/f_n(z)$ is a decreasing function of z for $z > 1$, when $p_n = P_n^{(0,0)}$ or $P_n^{(1,1)}$ and $f_n = Q_n$ or $Q_n^{(1,1)}$ respectively. We shall now prove that this result holds for very general p_n, f_n ; this result is necessary to generalize the analysis of Refs. 14), 29), and of this article to arbitrary spins.

The proof uses the Christoffel-Darboux formula ³⁰⁾

$$\begin{aligned} \sum_{m=0}^n \frac{1}{h_m} p_m(x) p_m(y) &= \\ &= \frac{k_m}{h_m k_{m+1}} \cdot \frac{p_{m+1}(x) p_m(y) - p_m(x) p_{m+1}(y)}{x-y}, \end{aligned} \quad (\text{A.2})$$

where

$$h_m \equiv \int_{-1}^{+1} dx w(x) [p_m(x)]^2, \quad k_m \equiv \frac{1}{m!} \left. \frac{d^m}{dx^m} p_m(x) \right|_{x=0}.$$

Explicit tables for k_n, h_n can be found in Ref. 30). Then, from (A.2), we can just repeat, without alteration, the proof of Ref. 29) just changing $P_n \leftrightarrow p_n, f_n \leftrightarrow Q_n$, obtaining that

$$\begin{aligned} &\frac{k_m}{k_{m+1} h_m} [f_{m+1}(z_1) f_m(z_2) - f_{m+1}(z_2) f_m(z_1)] \\ &= (z_2 - z_1) \sum_{n=m}^{\infty} \frac{k_m}{h_n} f_n(z_1) f_n(z_2); \quad z_1, z_2 > 1. \end{aligned} \quad (\text{A.3})$$

Now, k_n, h_n are always positive ³⁰⁾; therefore, the right-hand side of (A.3) will be positive if the polynomial q , and the constant λ in (A.1) are so chosen that $f_n(z) > 0$ for $z > 1$. This is easily seen to be the case, e.g., if $p_n(x) = P_n^{(\alpha, \beta)}(x)$ and $f_n(z) = Q_n^{(\alpha, \beta)}(z)$, which is the case relevant for arbitrary spin ³¹⁾.

A P P E N D I X B

TWO GENERALIZATIONS OF CHRISTOFFEL-DARBOUX FORMULAE

Lemma B.1

One has the generalized Christoffel-Darboux formula

$$\sum_{\substack{l=2 \\ l=2v+1}}^{\infty} P_l^{-1} Q_l^{(1,1)}(z) P_l^{(1,1)}(\xi) = \frac{z(\xi^2-1)}{2(z^2-1)(z^2-\xi^2)} - \frac{4}{3} Q_0^{(1,1)}(z). \quad (\text{B.1})$$

Proof:

Because of the completeness of the $P_\ell^{(1,1)}$ one can write

$$\sum_{\ell=0}^{\infty} P_\ell^{-1} P_\ell^{(1,1)}(x) P_\ell^{(1,1)}(\xi) = \delta(x-\xi). \quad (\text{B.2a})$$

Multiplying by $(1-x^2)/z-x$, integrating dx between -1 and $+1$, and using the analogue of the Neumann formula

$$Q_\ell^{(1,1)}(z) = \frac{1}{2(z^2-1)} \int_{-1}^{+1} dx P_\ell^{(1,1)}(x) \frac{1-x^2}{z-x},$$

we get

$$\sum_{l=0}^{\infty} P_l^{-1} Q_l^{(1,1)}(z) P_l^{(1,1)}(\xi) = \frac{1-\xi^2}{2(z^2-1)(z-\xi)}. \quad (\text{B.2b})$$

Although (4.2a) was only valid for $|\xi| \leq 1$ (B.2b) can be continued to any ξ provided only $z > \xi$. Adding (B.2b) with the result of changing there ξ into $-\xi$, and recalling that $P_\ell^{(1,1)}(-\xi) = (-1)^\ell P_\ell^{(1,1)}(\xi)$, we arrive at

$$\sum_{\substack{l=0 \\ l=2v+1}}^{\infty} P_l^{-1} Q_l^{(1,1)}(z) P_l^{(1,1)}(\xi) = \frac{z(1-\xi^2)}{2(z^2-1)(z^2-\xi^2)}, \quad (\text{B.3})$$

from which (B.1) follows directly.

Lemma B.2

A second generalization of the Christoffel-Darboux formula, viz.,

$$\sum_{\substack{l=N+2 \\ l=2N+1}}^{\infty} P_l^{-1} P_l^{(1,1)}(\xi) Q_l^{(1,1)}(z) = \frac{z(\xi^2-1)}{2(z^2-1)(z^2-\xi^2)}$$

$$- \frac{N+1}{(N+2)\rho_N} \cdot \frac{P_{N+1}^{(1,1)}(\xi) Q_N^{(1,1)}(z) \xi - P_N^{(1,1)}(\xi) Q_{N+1}^{(1,1)}(z) z}{z^2 - \xi^2},$$

(B.4)

also holds.

Proof

We start with the usual Christoffel-Darboux formula ^{22),30)}

$$\sum_{l=0}^n P_l^{-1} P_l^{(1,1)}(x) P_l^{(1,1)}(y) =$$

$$= \frac{n+1}{(n+2)\rho_n} \cdot \frac{P_{n+1}^{(1,1)}(x) P_n^{(1,1)}(y) - P_n^{(1,1)}(x) P_{n+1}^{(1,1)}(y)}{x-y}.$$

Multiplying by $(1-x^2)/(z-x)$, integrating dx between -1 and $+1$, and dividing the result by $2(z^2-1)$, this gives

$$\sum_{l=0}^n P_l^{-1} P_l^{(1,1)}(y) Q_l^{(1,1)}(z)$$

$$= \frac{n+1}{(n+2)\rho_n} \cdot \frac{P_n^{(1,1)}(y) Q_{n+1}^{(1,1)}(z) - P_{n+1}^{(1,1)}(y) Q_n^{(1,1)}(z)}{z-y}.$$

Adding up this formula for $y=\xi$ and $y=-\xi$ just kills the odd terms in the summation. Subtracting then the result with $n=\infty$ [which is just Eq. (B.3)] from the same with $n=N$ yields Eq. (B.4).

A P P E N D I X C

GENERATING FUNCTION FOR THE $P_l^{(1,1)}$

We have to sum explicitly the series

$$\sum_{\substack{l=2 \\ l=\text{even}}}^{\infty} \frac{(l+2)(2l+3)}{l+1} P_l^{(1,1)}(x) \xi^l.$$

Since such a formula could not be found in the literature, I have had to invent it.

One starts from the well-known formula

$$\sum_{l=0}^{\infty} P_l(x) \xi^l = [\xi^2 + 1 - 2x\xi]^{-1/2}.$$

Differentiating it, with respect to x , and using the relation between the P_l' and $P_l^{(1,1)}$ this gives

$$\sum_{l=0}^{\infty} \frac{l+2}{2} P_l^{(1,1)}(x) \xi^l = [\xi^2 + 1 - 2x\xi]^{-3/2}.$$

Integrating over $d\xi$ between 0 and ξ ,

$$\sum_{l=0}^{\infty} \frac{l+2}{l+1} P_l^{(1,1)}(x) \xi^l = \frac{2}{(x^2-1)\xi} \left[\frac{x-\xi}{(\xi^2+1-2x\xi)^{1/2}} - x \right]. \quad (C.1)$$

Differentiating with respect to ξ ,

$$\begin{aligned} \sum_{l=1}^{\infty} \frac{l(l+2)}{l+1} P_l^{(1,1)}(x) \xi^{l-1} \\ = \frac{2}{(x^2-1)\xi^2} \{ (x^2-x)x + (x-\xi)^2\xi \}, \quad r = [\xi^2+1-2x\xi]^{1/2}. \end{aligned} \quad (C.2)$$

Subtract from (C.1) itself with $\xi = 0$; hence,

$$\sum_{l=1}^{\infty} \frac{(l+2)}{l+1} P_l^{(1,1)}(x) \xi^l = \frac{2}{(x^2-1)\xi^2} \left[(1-x)x - \xi - (x^2-1)\xi x \right]. \quad (C.3)$$

Adding three times (C.3) to twice (C.2), and adding again the total result with $x=z$ and $x=-z$ gives, finally, the ugly but explicit result

$$\sum_{\substack{l=2 \\ l=\text{even}}}^{\infty} \frac{(l+2)(2l+3)}{l+1} P_l^{(1,1)}(z) \xi^l = \Psi_2^+(z, \xi), \quad (\text{C.4a})$$

$$\begin{aligned} \Psi_2^+(z, \xi) = & \frac{2}{(z^2-1)\xi r_-^3} \left\{ (r_-^2 - r_-) z + (z - \xi)^2 \xi \right. \\ & \left. + \frac{3}{2} r_-^2 (1 - r_-) z - \frac{3}{2} r_-^2 \xi - \frac{3}{2} (z^2 - 1) \xi r_-^3 \right\} \\ & + \frac{2}{(z^2-1)\xi r_+^3} \left\{ - (r_+^2 - r_+) z + (z + \xi)^2 \xi \right. \\ & \left. - \frac{3}{2} r_+^2 (1 - r_+) z - \frac{3}{2} r_+^2 \xi - \frac{3}{2} (z^2 - 1) \xi r_+^3 \right\}, \end{aligned}$$

$$r_{\pm} = \left(\xi^2 + 1 \mp 2\xi z \right)^{1/2}. \quad (\text{C.4b})$$

It is not difficult, if one recalls the origin of Ψ_2^+ to see that it is analytic in the variables z, ξ with the exception of branch cuts which can be traced to branch cuts of $\sqrt{\xi^2 + 1 \pm 2\xi z}$. Similarly, Eq. (C.4a) is easily shown to be convergent in a region that, for $z = \cos \theta_t$, $0 < \xi < 1/x_0(t)$ [$x_0(t)$ as given by Eq. (4.10b) in the text], coincides with the (complex) Mandelstam region.

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FIGURE CAPTION

Reduction of errors by use of positivity.
Error bars: initial errors. Hatched area:
allowed error band. Shaded area: extended
allowed range of variation.

μ_n