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ON THE ANALYTICITY PROPERTIES OF FIVE-PARTICLE AMPLITUDES
IN QUANTUM FIELD THEORY

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A B S T R A C T

In a five-particle process $a+b \rightarrow 1+2+3$, where the invariant mass of 2 and 3 is such that this pair can scatter only elastically in the final state, the conventional production amplitude T is related easily to a new amplitude, T_{23}^- , which has simpler analyticity properties than T in relativistic quantum field theory. Concerning the physical interpretation of T_{23}^- , its (ℓ, m) partial wave can be consistently regarded as the inelastic scattering amplitude for $a+b \rightarrow 1 + \text{[one composite particle with angular momentum } (\ell, m)]$. It is shown that T_{23}^- and its absorptive part have analyticity properties in three angular variables which generalize those of four-particle amplitudes inside the Lehmann ellipses. The problem of deriving dispersion relations for T_{23}^- is also investigated.

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1. INTRODUCTION

Rather little is known about the analyticity properties of production amplitudes (two-particles $\longrightarrow$ three or more particles) in the framework of relativistic quantum field theory, in spite of their importance for the development of dispersion theory. Concerning properties in angles, the only results seem to be those of Ascoli and Minguzzi ^{1), 2)}, which are a generalization of the analyticity properties of four-particle amplitudes inside the small Lehmann ellipse ³⁾. On the other hand, although several attempts have been made to conjecture or to derive dispersion relations in the total energy for production processes ⁴⁾, well-established results are scarce: they have been proved only for the double Compton effect, $\gamma p \longrightarrow \gamma \gamma p$, under the condition that the squared total invariant mass of the two outgoing photons be smaller than $4\mu^2$ (μ = pion mass), that is, when they cannot interact strongly in the final state ⁵⁾. Concerning physically more interesting cases like $\pi N \longrightarrow \pi \pi N$, perturbation theory has provided counter-examples which make the validity of those dispersion relations in the total energy which are natural generalizations of fixed momentum transfer dispersion relations for four-particle processes very doubtful. In fact, triangle diagrams for production amplitudes produce complex cuts both in the total energy and in partial subenergies ⁶⁾. However, in the framework of perturbation theory, dispersion relations of the above kind may be valid for $\gamma N \longrightarrow \gamma \pi N$, $\pi N \longrightarrow \gamma \gamma N$ and $\pi N \longrightarrow \gamma \pi N$ under certain restrictions on the fixed variables ⁷⁾ and other dispersion relations in the total energy which are, somehow, a generalization of fixed-angle dispersion relations for four-particle amplitudes are indeed valid in physically interesting cases like $\pi N \longrightarrow \pi \pi N$ ⁸⁾. The latter fact is a direct consequence of some interesting studies on the majorization properties of Feynman diagrams ⁹⁾.

This paper contains an attempt to understand the intricate analytic properties of the production amplitudes for two particles $\longrightarrow$ three particles in relativistic quantum field theory, along a line started by Ishida ⁴⁾, who pointed out that the conventional production amplitude could be expressed in terms of new production amplitudes whose absorptive parts are real and which may be more convenient for studying analyticity properties, and pursued later by Mohan ⁴⁾. A brief description of Mohan's paper in comparison with the present work will be given later. Since many problems remain open in the works of those authors, it seemed quite convenient to investigate further Ishida's proposal and to see in detail what analyticity properties both in angles and in the total energy can be obtained for some of Ishida's amplitudes (actually, not all the amplitudes put forward by him have simple analyticity properties) in relativistic quantum field theory. This we do on the basis of some appropriate sets of variables and reference frames which have proved useful in studying production processes in recent years. We argue that the new amplitudes can be regarded as the superposition of all inelastic scattering amplitudes for a certain four-particle process where one particle is composite and has any angular momentum ℓ , m .

Our main result is that the new amplitudes and their absorptive parts have analyticity properties in three variables related to angles, in suitable helicity frames, which generalize the ones of four-particle processes found by Lehmann³⁾ and are related to the ones established by Ascoli and Minguzzi^{1),2)}. We also investigate the analyticity properties of the new amplitudes in the total energy using the variables of Bali, Chew and Pignotti¹⁰⁾. We establish the off-shell dispersion relation and we study the analytic continuation of the absorptive part in the mass of the off-shell particle, concluding that it makes sense for all energies, except for a finite low-energy interval. We end up with the conjecture that the new amplitudes may be analytic in the complex plane of the total energy except for the left and right cuts and a finite and bounded low energy region, where all complex cuts should be contained (at least, when suitable stability conditions are fulfilled).

This paper is organized as follows.

Section 2 is devoted to kinematics. It collects and summarizes several sets of variables to be used afterwards and it presents a restricted graphical representation of the physical regions for five-particle processes which hopefully will help to visualize their structure. It is a straightforward generalization of the Mandelstam diagram for four-particle processes.

Section 3 introduces the new production amplitudes, which seem to possess some simpler properties than the conventional amplitude. Subsection 3.A studies a new set of state vectors in the Hilbert subspaces of two-particle subsystems. Subsection 3.B contains the definition of the new amplitudes using the above new states and the relation between them and the conventional amplitude.

Section 4 deals with the analyticity properties of the new amplitudes and their absorptive parts in angles, as well as with their physical interpretation. Some analyticity domains in three angles found here with the aid of the Jost-Lehmann-Dyson representation and the fact that the absorptive parts of the new amplitudes coincide with their imaginary parts justify a posteriori their introduction.

Section 5 is concerned with the analyticity properties of the new amplitudes in the total energy. The off-shell dispersion and crossing relations and the asymptotic behaviour are discussed in Subsection 5.A. It is found, moreover, that the interesting crossing relation is simpler for the new amplitudes than for the conventional one. The analytic continuation in the mass, the problems posed by it, its relation to the findings of perturbation and S matrix theories and the above-mentioned conjecture are contained in Subsection 5.B.

Throughout this paper we consider spinless stable bosons of arbitrary masses.

2. KINEMATICS FOR FIVE-PARTICLE PROCESSES

Here we shall consider the kinematics for a five-particle process $a+b \rightarrow 1+2+3$. The four-momentum and mass of particle i are $p_i = (p_i^0, \vec{p}_i)$ and $m_i (p_i^2 = m_i^2)$, $i = a, b, 1, 2, 3$. The invariant mass of particles i, j (both in the initial or in the final state) is $s_{ij} = (p_i + p_j)^2$. The momentum transfer between particles i, j (one in the initial and the other in the final state) is $t_{ij} = (p_i - p_j)^2$. Using four-momentum conservation and the mass-shell condition, one realizes easily that only five kinematic invariants are independent. If, for instance, we choose the ones shown in Fig. 1 as independent, then the remaining five invariants are given by ($i, j, k = 1, 2, 3, i \neq j \neq k, i \neq k$)

$$s_{ij} = s_{ab} - s_{ik} - s_{jk} + m_i^2 + m_j^2 + m_k^2 \quad (2.1)$$

$$t_{aj} = s_{ik} - s_{ab} - t_{bj} + m_a^2 + m_b^2 + m_j^2 \quad (2.2)$$

$$t_{ak} = t_{bj} - t_{ai} - s_{ik} + m_a^2 + m_i^2 + m_k^2 \quad (2.3)$$

$$t_{bi} = s_{jk} - s_{ab} - t_{ai} + m_a^2 + m_b^2 + m_i^2 \quad (2.4)$$

$$t_{bk} = t_{ai} - t_{bj} - s_{jk} + m_b^2 + m_j^2 + m_k^2 \quad (2.5)$$

Afterwards, we shall be using several different variables and reference frames, so that it seems convenient to collect them all in this section for further reference, together with their relations to the above kinematic invariants.

2.A THE FRAME O_{ab}^{xy}

Let us consider the centre-of-mass frame O_{ab}^{xy} ($\vec{p}_a + \vec{p}_b = 0$), the axis being chosen, so that the final particles 1, 2 and 3 lie in the xy plane and the x axis bisects the angle $2x_{ij}$ between $\vec{p}_i$ and $\vec{p}_j$, $i, j = 1, 2, 3, i \neq j$ (see Fig. 2). Some calculations lead to express the different magnitudes shown in Fig. 2 and the modulus of the particle three-momenta in O_{ab}^{xy} in terms of the invariants $[\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)]$

$$X_a = \frac{\cos \theta_{ai} + \cos \theta_{aj}}{2 \cos X_{ij}} \quad Y_a = \frac{\cos \theta_{ai} - \cos \theta_{aj}}{2 \sin X_{ij}} \quad (2.A.1)$$

$$\cos 2\chi_{ij} = \frac{(s_{ab} + m_i^2 - s_{jk})(s_{ab} + m_j^2 - s_{ik}) + 2s_{ab}(m_i^2 + m_j^2 - s_{ij})}{\sqrt{\lambda(s_{ab}, m_i^2, s_{jk}) \cdot \lambda(s_{ab}, m_j^2, s_{ik})}} \quad \begin{matrix} i, j, k = 1, 2, 3 \\ i \neq j \neq k, \quad i \neq k \end{matrix} \quad (2.A.2)$$

$$\cos \theta_{ah} = \frac{(s_{ab} + m_a^2 - m_b^2)(s_{ab} + m_h^2 - s_{lm}) + 2s_{ab}(t_{ah} - m_a^2 - m_b^2)}{\sqrt{\lambda(s_{ab}, m_a^2, m_b^2) \cdot \lambda(s_{ab}, m_h^2, s_{lm})}} \quad \begin{matrix} h = i, j \\ l, m = i, j, k \\ l \neq m \neq h, \quad l \neq h \end{matrix} \quad (2.A.3)$$

$$\bar{p}_a^2 = \bar{p}_b^2 = \frac{\lambda(s_{ab}, m_a^2, m_b^2)}{4s_{ab}} \quad \bar{p}_i^2 = \frac{\lambda(s_{ab}, m_i^2, s_{jk})}{4s_{ab}} \quad (2.A.4)$$

Notice that if we take s_{ab} , s_{jk} , s_{ik} , t_{ai} , t_{bj} as basic variables in this situation, the above formulae and Eqs. (2.1)-(2.5) allow to express all variables in terms of them.

2.B THE FRAMES O_{ab}^i AND O_{jk}^i

We shall consider the following two reference frames: the over-all centre-of-mass system O_{ab}^i ($\bar{p}_a + \bar{p}_b = 0$) and the rest system of particles j and k ($\bar{p}_j + \bar{p}_k = 0$), $i, j, k = 1, 2, 3$, $i \neq j \neq k$, $i \neq k$ in such a way that the three-momentum of particle i lies along the negative z axis, and the x and y axis are parallel in both frames. In O_{ab}^i , $\bar{p}_a$ is taken in the z - x plane. Let us define the four-vector $q_i = \frac{1}{2}(\bar{p}_j - \bar{p}_k)$ for further reference. The following formulae give several quantities (some of them being defined in Fig. 3) in both frames in terms of the invariants

$$\text{in } O_{ab}^i \quad P_i = \begin{pmatrix} \frac{s_{ab} + m_i^2 - s_{jk}}{2\sqrt{s_{ab}}} \\ 0 \\ 0 \\ -\sqrt{\frac{\lambda(s_{ab}, m_i^2, s_{jk})}{4s_{ab}}} \end{pmatrix} \quad (2.B.1)$$

$$\text{in } O_{jk}^i \quad P_i = \begin{pmatrix} \frac{s_{ab} - m_i^2 - s_{jk}}{2\sqrt{s_{jk}}} \\ 0 \\ 0 \\ -\sqrt{\frac{\lambda(s_{ab}, m_i^2, s_{jk})}{4s_{jk}}} \end{pmatrix}, \quad \bar{P}_j^2 = \bar{P}_k^2 = \frac{\lambda(s_{jk}, m_j^2, m_k^2)}{4s_{jk}}$$

$$x_i = \cos \Psi_i \cdot \sqrt{1 - z_i^2} \quad (2.B.2)$$

$$z_i = - \frac{(s_{jk} + m_j^2 - m_k^2)(s_{ab} - m_i^2 - s_{jk}) + 2s_{jk}(m_i^2 + m_j^2 - s_{ij})}{\sqrt{\lambda(s_{jk}, m_j^2, m_k^2)} \lambda(s_{ab}, m_i^2, s_{jk})}$$

$$\bar{P}_a^2 = \frac{\lambda(s_{jk}, m_a^2, t_{bi})}{4s_{jk}}, \quad \bar{P}_b^2 = \frac{\lambda(s_{jk}, m_b^2, t_{ai})}{4s_{jk}} \quad (2.B.3)$$

In O_{jk}^i , x_i has a somewhat complicated expression

$$x_i = \frac{(\bar{P}_a \bar{P}_j) \bar{P}_i^2 - (\bar{P}_a \bar{P}_i)(\bar{P}_j \bar{P}_i)}{|\bar{P}_i| \cdot |\bar{P}_j| \cdot \sqrt{\bar{P}_i^2 \bar{P}_a^2 - (\bar{P}_a \bar{P}_i)^2}}, \quad \bar{P}_h \bar{P}_\ell = \frac{t_{h\ell} - m_h^2 - m_\ell^2 + 2P_h^0 P_\ell^0}{2} \quad \begin{matrix} h=a,b \\ \ell=i,j \end{matrix} \quad (2.B.4)$$

$$\bar{P}_i \bar{P}_j = \frac{m_i^2 + m_j^2 + 2P_i^0 P_j^0 - s_{ij}}{2}, \quad \bar{P}_a \bar{P}_b = \frac{m_a^2 + m_b^2 + 2P_a^0 P_b^0 - s_{ab}}{2} \quad (2.B.5)$$

the energies P_h^0 , $h = a, b, i, j$ being trivially obtained from $|\bar{P}_h|$. We remark that the angles Θ_{ai} in Figs. 2 and 3 are the same. Consequently, $z = \cos \Theta_{ai}$ is given by Eq. (2.A.3) for $h = i$.

We stress that our choice of axis in Fig. 3 is not arbitrary. If, for a moment, we consider the following particular reaction: $N(a) + \pi(b) \rightarrow N(1) + \pi(2) + \pi(3)$, we see that the frame O_{23}^1 is precisely the standard helicity frame for the outgoing di-pion system. See also Wick's analysis of three-particle states ¹¹⁾.

Here we may take s_{ab} , s_{jk} , z , z_i and $\cos \Psi_i$ as basic variables. Notice that for fixed given values of s_{ab} and s_{jk} , z and t_{ai} are interrelated by Eq. (2.A.2) while z_i and s_{ik} determine each other reciprocally by virtue of Eqs. (2.B.3) and (2.1). On the other hand, by using the above expression for $\bar{p}_a^2$ in O_{jk}^i , Eq. (2.4), the above equation for $\cos \Psi_i$, the above correspondence between z_i and s_{ik} , Eqs. (2.B.4)-(2.B.5) and Eq. (2.2), one gets the relation between $\cos \Psi_i$ and t_{bj} . So, one concludes that the set s_{ab} , s_{jk} , z , z_i and $\cos \Psi_i$ is equivalent to the set s_{ab} , s_{jk} , t_{ai} , s_{ik} , t_{bj} , although to pass from one set to the other, implies the use of some lengthy expressions.

2.C THE BALI-CHEW-PIGNOTTI (BCP) VARIABLES

Let us associate particle a to i and particle b to j . We shall consider a certain frame, O^{BCP} , where the different particle four-momenta have the following expressions, by definition [the consistency of which having been shown in Ref. 10)]

$$P_b = \begin{pmatrix} \sqrt{\frac{\lambda(t_{bj}, m_b^2, m_j^2)}{-4t_{bj}}} \\ 0 \\ 0 \\ \frac{t_{bj} + m_b^2 - m_j^2}{\sqrt{-4t_{bj}}} \end{pmatrix}, \quad P_j = \begin{pmatrix} \sqrt{\frac{\lambda(t_{bj}, m_b^2, m_j^2)}{-4t_{bj}}} \\ 0 \\ 0 \\ \frac{-t_{bj} + m_b^2 - m_j^2}{\sqrt{-4t_{bj}}} \end{pmatrix} \quad (2.C.1)$$

$$P_k = L_1(L^{jk}) \begin{pmatrix} \sqrt{\frac{\lambda(t_{ai}, t_{bj}, m_k^2)}{-4t_{bj}}} \\ 0 \\ 0 \\ \frac{t_{bj} - t_{ai} + m_k^2}{\sqrt{-4t_{bj}}} \end{pmatrix}, \quad L_1(L) = \begin{pmatrix} \cosh(2\mathcal{E}) & \sinh(2\mathcal{E}) & 0 & 0 \\ \sinh(2\mathcal{E}) & \cosh(2\mathcal{E}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.C.2)$$

$$P_a = L_1(\xi^{jk}) L_2(\omega) L_3(q) L_4(\xi^{ik}) \begin{pmatrix} \sqrt{\frac{\lambda(t_{ai}, m_a^2, m_i^2)}{-4t_{ai}}} \\ 0 \\ 0 \\ \frac{-t_{ai} + m_i^2 - m_a^2}{\sqrt{-4t_{ai}}} \end{pmatrix} \quad (2.0.3)$$

$$P_i = L_1(\xi^{jk}) L_2(\omega) L_3(q) L_4(\xi^{ik}) \begin{pmatrix} \sqrt{\frac{\lambda(t_{ai}, m_a^2, m_i^2)}{-4t_{ai}}} \\ 0 \\ 0 \\ \frac{t_{ai} + m_i^2 - m_a^2}{\sqrt{-4t_{ai}}} \end{pmatrix} \quad (2.0.4)$$

$$L_2(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\omega & -\sin 2\omega & 0 \\ 0 & \sin 2\omega & \cos 2\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, L_3(q) = \begin{pmatrix} \cosh 2q & 0 & 0 & \sinh 2q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh 2q & 0 & 0 & \cosh 2q \end{pmatrix} \quad (2.0.5)$$

$$\cosh 2q = \frac{m_k^2 - t_{ai} - t_{bj}}{\sqrt{4 t_{ai} t_{bj}}} \quad (2.0.6)$$

The variables ξ^{jk} and ξ^{ik} are related to s_{jk} , s_{ik} through the following equations

$$s_{jk} - t_{ai} - m_b^2 + \frac{(m_j^2 - m_b^2 - t_{ai})(m_k^2 - t_{ai} - t_{bj})}{2t_{bj}} = \frac{\sqrt{\lambda(t_{ai}, t_{bj}, m_k^2)} \lambda(t_{bj}, m_b^2, m_j^2)}{-2t_{bj}} \cosh 2\xi^{jk}$$

(2.0.7)

$$s_{ik} - t_{bj} - m_a^2 + \frac{(m_i^2 - m_a^2 - t_{bj})(m_k^2 - t_{ai} - t_{bj})}{2t_{ai}} = \frac{\sqrt{\lambda(t_{ai}, t_{bj}, m_k^2)} \lambda(t_{ai}, m_a^2, m_i^2)}{-2t_{ai}} \cosh 2\zeta^{ik}$$

(2.C.8)

Thus, Eqs. (2.C.1)-(2.C.8) parametrize all four-momenta in terms of either ζ^{jk} , ζ^{ik} , ω , t_{ai} , t_{bj} or s_{jk} , s_{ik} , ω , t_{ai} , t_{bj} . We remark that the above matrices $\mathcal{L}_h$, $h = 1, 2, 3$ represent suitable Lorentz transformations and that the angle ω (the Toller angle) is the angle between the two planes containing $\vec{p}_a$, $\vec{p}_i$ and $\vec{p}_b$, $\vec{p}_j$ respectively in the rest system of particle k . The equation relating s_{ab} to s_{jk} , s_{ik} , ω , t_{ai} and t_{bj} is the following

$$\begin{aligned} s_{ab} = m_a^2 + m_b^2 + \frac{\sqrt{\lambda(t_{ai}, t_{bj}, m_k^2)}}{4t_{ai}t_{bj}} (m_i^2 - m_a^2 - t_{ai})(m_j^2 - m_b^2 - t_{bj}) + \frac{m_i^2 - m_a^2 - t_{ai}}{-2t_{ai}} \\ \cdot \left[s_{jk} - t_{ai} - m_b^2 + \frac{(m_j^2 - m_b^2 - t_{ai})(m_k^2 - t_{ai} - t_{bj})}{2t_{bj}} \right] + \frac{m_j^2 - m_b^2 - t_{bj}}{-2t_{bj}} \left[s_{ik} - t_{bj} - m_a^2 + \right. \\ \left. + \frac{(m_i^2 - m_a^2 - t_{bj})(m_k^2 - t_{ai} - t_{bj})}{2t_{ai}} \right] + \frac{m_k^2 - t_{ai} - t_{bj}}{\lambda(t_{ai}, t_{bj}, m_k^2)} \left[s_{jk} - t_{ai} - m_b^2 + \frac{(m_j^2 - m_b^2 - t_{ai})(m_k^2 - t_{ai} - t_{bj})}{2t_{bj}} \right] \\ \left[s_{ik} - t_{bj} - m_a^2 + \frac{(m_i^2 - m_a^2 - t_{bj})(m_k^2 - t_{ai} - t_{bj})}{2t_{ai}} \right] + \cos 2\omega \cdot \frac{2\sqrt{t_{ai}t_{bj}}}{\lambda(t_{ai}, t_{bj}, m_k^2)} \\ \cdot \sqrt{\left[s_{jk} - t_{ai} - m_b^2 + \frac{(m_j^2 - m_b^2 - t_{ai})(m_k^2 - t_{ai} - t_{bj})}{2t_{bj}} \right]^2 - \frac{\lambda(t_{bj}, m_j^2, m_b^2) \cdot \lambda(t_{ai}, t_{bj}, m_k^2)}{(-2t_{bj})^2}} \\ \cdot \sqrt{\left[s_{ik} - t_{bj} - m_a^2 + \frac{(m_i^2 - m_a^2 - t_{bj})(m_k^2 - t_{ai} - t_{bj})}{2t_{ai}} \right]^2 - \frac{\lambda(t_{ai}, m_a^2, m_i^2) \cdot \lambda(t_{ai}, t_{bj}, m_k^2)}{(-2t_{ai})^2}} \end{aligned}$$

(2.C.9)

and it enables one to take s_{ab} , s_{jk} , t_{ai} , t_{bj} and ω as independent variables, if desired [when using Eq. (2.C.9) to express s_{ik} in terms of s_{ab} , the possible ambiguity due to the square root is easily solved by taking the limit $s_{ab} \rightarrow \infty$, $s_{ik} \rightarrow \infty$ for fixed t_{ai} , t_{bj} , s_{jk}]. Notice that Eq. (2.C.9) indicates that for fixed t_{ai} , t_{bj} , s_{jk} and s_{ik} the range of variation of s_{ab} as ω varies in $0 \leq \omega \leq 2\pi$ is finite.

The relation between the set s_{ab} , s_{jk} , z , z_i and $\cos \Psi_i$ introduced in Subsection 2.B and the present set s_{ab} , s_{jk} , t_{ai} , t_{bj} , ω is not difficult to obtain: in Subsection 2.B we indicated how to go from s_{ab} , s_{jk} , z , z_i and $\cos \Psi_i$ to s_{ab} , s_{jk} , t_{ai} , s_{ik} , t_{bj} and, on the other hand, Eq. (2.C.9) also gives all one needs to go from the latter set to s_{ab} , s_{jk} , t_{ai} , t_{bj} and ω .

2.D LONGITUDINAL AND TRANSVERSE MOMENTA

In the over-all centre-of-mass frame (either O_{ab}^{xy} or O_{ab}^i), it is convenient to decompose each outgoing three-momentum $\vec{p}_i$ $i = 1, 2, 3$ into longitudinal ($p_{i,L}$) and two-dimensional transverse ($\vec{r}_i$) components¹²⁾

$$\vec{p}_i = p_{i,L} \frac{\vec{p}_a}{|\vec{p}_a|} + \vec{r}_i, \quad \vec{p}_a \vec{r}_i = 0 \quad (2.D.1)$$

$$p_{i,L} = \frac{2s_{ab}(m_a^2 + m_b^2 - t_{ai}) + (s_{ab} + m_a^2 - m_b^2)(s_{ab} + m_i^2 - s_{jk})}{2\sqrt{s_{ab} \cdot \lambda(s_{ab}, m_a^2, m_b^2)}} \quad \begin{matrix} i \neq j \neq k \\ i \neq k \end{matrix} \quad (2.D.2)$$

$$\vec{r}_i^2 = \vec{p}_i^2 - p_{i,L}^2 \quad (2.D.3)$$

$\vec{p}_i^2$ being given in Eq. (2.A.4). The above equations will be useful when discussing a graphical representation of the physical regions for the five-particle process, in Subsection 2.E.

2.E GRAPHICAL REPRESENTATION FOR PHYSICAL REGIONS

It may be quite convenient to dispose of graphical representations of the physical regions of the kinematic invariants in five-particle reactions. Even though there is no hope to represent the ranges of variation of five independent variables in a plane we will introduce a restricted diagram which may help to visualize things.

We remark that when all particle three-momenta are parallel or antiparallel in the centre-of-mass frame, then four-momentum conservation and the mass shell condition imply that there are only two independent variables. Moreover, such configurations constitute the boundary of the physical region. Consequently, if, for instance, we consider the following equation which comes trivially from Eqs. (2.2) and (2.3) for $i = 1, j = 2, k = 3$

$$s_{ab} + t_{a1} + t_{a2} + t_{a3} = 2m_a^2 + m_b^2 + m_1^2 + m_2^2 + m_3^2 \equiv \sum \quad (2.E.1)$$

we see that it leads directly to draw the following plot for the boundary of the physical regions of the five-particle process, which is a straightforward generalization of the Mandelstam diagram for four-particle reactions.

First, we draw a regular tetrahedron of the height $\sum$ in three-dimensional space. The four planes which constitute the faces of the tetrahedron, together with its vertex and sides, divide the whole three-dimensional space in 14 different regions (see Fig. 4): four V regions, determined by vertex, four F regions, determined by faces and six S regions, determined by sides. The tetrahedron determines a system of tetrahedral co-ordinates for the four invariants $s_{ab}, t_{a1}, t_{a2}, t_{a3}$: each point in the three-dimensional space has four co-ordinates, namely, the four invariants related by Eq. (2.E.1). Those co-ordinates are measured along the normal from the point under consideration down to the four faces. On each face, side and vertex, one, two and three invariants vanish respectively. The signs of the co-ordinates are determined by the fact that the only non-vanishing co-ordinate of each vertex is $+\sum$.

Clearly, the points corresponding to the physical values of the invariants when all particle three-momenta are parallel or antiparallel in the centre-of-mass should be contained in the V region $V(s_{ab})$. Since only two variables are independent, there is one further equation relating s_{ab}, t_{ai} $i = 1, 2, 3$ besides Eq. (2.E.1). The new relation comes from eliminating $s_{12}, s_{23}, s_{13}, t_{b1}, t_{b2}$ and t_{b3} among Eqs. (2.1), (2.4), (2.5) and (2.2) [or (2.3)] and the three equations $\vec{r}_i^2 = 0$, $\vec{r}_i^2$ being given in Eq. (2.D.3). Of course, there are several choices for the two independent variables: s_{ab} and t_{a1}, t_{a1} and t_{a2} , etc.

Consequently, the corresponding points lie on some surface, as shown in Fig. 5. Assuming all particles to be stable and $(m_a + m_b)^2 < (m_1 + m_2 + m_3)^2$, the co-ordinates of the threshold are $s_{ij} = (m_i + m_j)^2$, $i, j = 1, 2, 3$ $i \neq j$, $s_{ab} = (m_1 + m_2 + m_3)^2$ and t_{ai} is given by the vanishing of the numerator in Eq. (2.D.2). For non-vanishing transverse momenta, we may take $\vec{r}_1^2, \vec{r}_2^2$ and $\vec{r}_3^2$ as the remaining three independent variables, the corresponding points lying inside the surface $\Pi(s_{ab})$. Clearly, we cannot represent the whole variation of five parameters in our three-dimensional plot. However, since the $\vec{r}_i^2$ appear in the energy conservation law as effective masses to be added to the particle masses, we see trivially that

for each set of fixed transverse momenta the variation of the other two variables describes a surface more or less like (and contained within) $\Gamma(s_{ab})$.

We shall point out other nice features of the preceding plot, namely that other V and S regions of it can be made to correspond to the different processes which can be obtained from $a+b \rightarrow l+2+3$ by crossing. In fact, the three crossed processes $a+\bar{l} \rightarrow \bar{b}+2+3$, $a+\bar{2} \rightarrow l+\bar{b}+3$ and $a+\bar{3} \rightarrow l+2+\bar{b}$ can be represented by corresponding regions in the V regions $V(t_{a1})$, $V(t_{a2})$, $V(t_{a3})$ as t_{a1} , t_{a2} and t_{a3} become positive respectively and s_{ab} becomes negative. The surfaces $\Gamma(t_{a1})$, the thresholds, etc., are similar to the ones for $a+b \rightarrow l+2+3$. Moreover, if we consider in particular the S region $S(t_{a1}, s_{ab})$, we see that inside of it we have $s_{ab} > 0$, $t_{a1} > 0$, $t_{a3} < 0$, $t_{a2} < 0$, which is precisely what happens for $a+b+\bar{l} \rightarrow 2+3$. Thus, after some further verifications we conclude that the physical regions for the six crossed processes $a+b+\bar{l} \rightarrow 2+3$, $a+b+\bar{2} \rightarrow l+3$, $a+b+\bar{3} \rightarrow l+2$, $a+\bar{l}+\bar{2} \rightarrow \bar{b}+3$, $a+\bar{l}+\bar{3} \rightarrow \bar{b}+2$, $a+\bar{2}+\bar{3} \rightarrow \bar{b}+l$ can be represented inside the six S regions $S(t_{a1}, s_{ab})$, $S(t_{a2}, s_{ab})$, $S(t_{a3}, s_{ab})$, $S(t_{a1}, t_{a2})$, $S(t_{a1}, t_{a3})$, $S(t_{a2}, t_{a3})$, (see Fig. 6).

Since we are dealing with stable particles, the F regions, where three invariants are positive [for instance, in $F(t_{a1}, s_{ab}, t_{a3})$ we have $t_{a1} > 0$, $s_{ab} > 0$, $t_{a3} > 0$], cannot correspond to any physical process. Briefly, the unphysical regions are the F regions, the tetrahedron and those parts of the V and S regions not contained inside the physical regions.

The equations for the surfaces $\Omega(t_{a1}, s_{ab})$, etc., which constitute the boundaries of the physical region for the processes $a+b+\bar{l} \rightarrow 2+3$, etc., can be found similarly by considering the corresponding transverse momenta for those channels.

The following property of the variables s_{ab} (or s_{13}), s_{23} , t_{a1} , t_{b2} , and ω summarized in Subsection 2.C is worth noticing. Let us give a set of values to them so that the corresponding point lies in the physical region for $a+b \rightarrow l+2+3$. Then, leaving s_{23} , t_{a1} , t_{b2} and ω fixed at those physical values, let us consider the curve C obtained when s_{ab} (or s_{13}) varies from $s_{ab} = +\infty$ down to $s_{ab} = -\infty$. Clearly, from $s_{ab} = +\infty$ down to some $s_{ab}^{(1)}$, the curve C stays inside the physical region for $a+b \rightarrow l+2+3$, that is, inside the V region $V(s_{ab})$ bounded by $\Gamma(s_{ab})$. Also, from $s_{ab} = -\infty$ up to some $s_{ab}^{(2)}$, C will be contained inside the S region $S(t_{a2}, t_{a3})$ bounded by $\Omega(t_{a2}, t_{a3})$, since the latter physical region is also the physical region for $\bar{l}+b \rightarrow \bar{a}+2+3$. The values $s_{ab}^{(1)}$ and $s_{ab}^{(2)}$ will be the simultaneous solution of Eq. (2.C.9) for $i=1$, $j=2$, $k=3$ and the equations giving the boundary $\Gamma(s_{ab})$ ($\Omega(t_{a2}, t_{a3})$), for the fixed values of t_{a1} , t_{b2} , ω and s_{23} under consideration. Since, in general, $t_{a1} < 0$, $t_{b2} < 0$, $s_{23} > (m_2+m_3)^2$, $0 \leq \omega < 2\pi$, it is clear that for $s_{ab}^{(1)} > s_{ab} > s_{ab}^{(2)}$ the curve C cannot be contained inside the physical region of any other channel.

Several remarks are worth to be made.

- a) By virtue of the empirical law of small transverse momenta ¹²⁾ at high energies the most probable kinematical configurations are those corresponding to points close to the boundaries of the physical regions: $\Gamma(s_{ab})$, $\Omega(t_{al}, s_{ab})$, etc.
- b) The above plot is a natural generalization of the Dalitz plot idea. In particular, the section of the physical region for $a+b \rightarrow 1+2+3$ with the planes $s_{ab} = \text{constant}$ is the t plot of Muirhead ¹³⁾.
- c) Of course, other plots like the one just discussed but based on other combinations of Eqs. (2.1)-(2.5) different from Eq. (2.E.1) can be studied. In some of those new cases, the above correspondence between V and S regions and physical regions may be altered as the physical regions corresponding to two different channels can be in the same S region, for instance. In particular, the section of the plot for Eq. (2.1) with the planes $s_{ab} = \text{constant}$ is the standard Dalitz plot.
- d) The inside of the tetrahedron in Figs. 4 and 7 is the obvious counterpart of the Mandelstam triangle in the four-particle diagram. We shall come later on to that tetrahedron, as it plays an important role in connection with the analyticity properties of the production amplitudes in the total energy.

3. DEFINITION OF THE PRODUCTION AMPLITUDES T_{jk}

We shall assume throughout that all particles are spinless and stable bosons, their masses being otherwise arbitrary. The standard assumptions of relativistic quantum field theory as formulated by Lehmann, Symanzik and Zimmermann ¹⁴⁾ are taken as our starting point. We write the S matrix and the invariant production amplitude T , as

$$S(a+b \rightarrow 1+2+3) = \frac{i(2\pi)^4 \delta^{(4)}(P_1+P_2+P_3-P_a-P_b)}{[(2\pi)^{3/2}]^5 (2P_1^0 2P_2^0 2P_3^0 2P_a^0 2P_b^0)^{1/2}} T \quad (3.1)$$

We shall assume that there is an interval $(m_j+m_k)^2 \leq s_{jk} < m_{jk,\text{in}}^2$ where the process $j+k \rightarrow j+k$ is completely elastic (no particle production occurs) and, moreover, we shall keep s_{jk} inside such an interval ($m_{jk,\text{in}}^2$ being, of course, the first inelastic threshold for $j+k \rightarrow j+k$). Moreover, throughout this and the following sections, we shall be always inside the physical region for $a+b \rightarrow 1+2+3$, [that is, inside $\Gamma(s_{ab})$ in Figs. 5 and 7] all particles being on-shell. In what follows, we shall introduce new state vectors and production amplitudes, T_{jk} , study some of their interesting properties and the precise connection between them and the conventional production amplitude, T .

3.A THE "STANDING WAVES" $|p_j p_k^- \rangle$

We shall introduce the following new states in the Hilbert subspace of the two-particle subsystem j, k

$$|p_j p_k^- \rangle \equiv \frac{1}{2i} \left[|p_j p_k^{out} \rangle - |p_j p_k^{in} \rangle \right] \quad (3.A.1)$$

The main properties of the new states (a sort of "standing waves") are:

- a) they are eigenstates of PT (P = parity, T = time reversal)

$$PT |p_j p_k^- \rangle = |p_j p_k^- \rangle \quad (3.A.2)$$

- b) they are not a complete set in the Hilbert subspace as (I being the identity operator inside of it)

$$I - \sum_{p_j p_k} |p_j p_k^- \rangle \langle p_j p_k^-| \neq 0 \quad (3.A.3)$$

If we define new "standing waves" by

$$|p_j p_k^+ \rangle \equiv \frac{1}{2} \left[|p_j p_k^{out} \rangle + |p_j p_k^{in} \rangle \right]$$

then the right-hand side of Eq. (3.A.3) becomes precisely $\sum_{p_j p_k} |p_j p_k^+ \rangle \langle p_j p_k^+|$. However, the new states $|p_j p_k^+ \rangle$ will be no longer needed. Both states

$|p_j p_k^+ \rangle$ and $|p_j p_k^- \rangle$ were introduced by Ishida ⁴⁾.

- c) Let S, K be the S and K matrices for the elastic process $j+k \rightarrow j+k$
 [we recall that $K(S+I)+i(S-I) = 0$] and let us define the F operator by either of the equivalent integral equations

$$F(S-I) + 2iS = 0 \quad F = -2iS(S-I)^{-1} \quad (3.A.4)$$

or formally

$$(F+iI)K + I = 0 \quad F = -(iI + K^{-1}) \quad (3.A.5)$$

Then we can obtain readily the following relations between the in, out and - states $\sqrt{T}_{el}(p_j+p_k \rightarrow p'_j+p'_k)$ being the two-particle elastic scattering amplitude

$$\langle p_j p_k \text{ out} | = \langle p_j p_k - | F = \sum_{p'_j p'_k} \langle p_j p_k \text{ in} | F | p'_j p'_k \text{ in} \rangle \langle p'_j p'_k - | \quad (3.A.6)$$

$$\langle p_j p_k - | = \sum_{p'_j p'_k} \frac{(-) \delta^{(4)}(p'_j + p'_k - p_j - p_k)}{2(2\pi)^2 (2p'_j \cdot 2p'_k \cdot 2p_j \cdot 2p_k)^{1/2}} T_{el}(p_j + p_k \rightarrow p'_j + p'_k)^* \langle p'_j p'_k \text{ out} | \quad (3.A.7)$$

$$| p_j p_k \text{ in} \rangle = F | p_j p_k - \rangle \quad (3.A.8)$$

In the centre-of-mass frame of the j,k pair, let $\bar{n}$ and $\bar{n}'$ be unit vectors directed along the three-momenta of particle j in the states $| p_j p_k \text{ in} \rangle$ and $| p'_j p'_k \text{ in} \rangle$, respectively, and let $\delta_\ell(s_{jk})$ be the elastic (and real) phase-shift for the process $j+k \rightarrow j+k$ at angular momentum ℓ . Then, starting from Eq. (3.A.4), we get

$$\langle p_j p_k \text{ in} | F | p'_j p'_k \text{ in} \rangle = \delta^{(4)}(p_j + p_k - p'_j - p'_k) \frac{\sqrt{s_{jk}}}{|\bar{p}_j| p_j \cdot p_k} F_{red} \quad (3.A.9)$$

$|\bar{p}_j|$ being given in Eq. (2.B.2) and F_{red} being the reduced F matrix

$$F_{red} = F_{red}(\bar{n}, \bar{n}') = \sum_{\ell, m} Y_\ell^m(\bar{n}) Y_\ell^m(\bar{n}')^* F_\ell(s_{jk}) \quad (3.A.10)$$

$$F_\ell(s_{jk}) = - \left[i + \cot \delta_\ell(s_{jk}) \right]$$

the Y_ℓ^m 's being, of course, spherical harmonics.

Equation (3.A.7) implies clearly that the states $| p_j p_k - \rangle$ do not constitute an orthonormal set. Nevertheless, neither this fact nor Eq. (3.A.3) prevent in principle to expand any state vector lying in the Hilbert subspace of two-particle states in terms of the states $| p_j p_k - \rangle$, by virtue of Eqs. (3.A.6) and (3.A.8) (in this connection see the comments on the F matrix in Subsection 4.D).

3.B THE AMPLITUDES T_{jk}^-

Keeping always the restriction $(m_j + m_k)^2 \leq s_{jk} < m_{jk, in}^2$, we shall write the standard production amplitudes with particles a, i ($i = 1, 2, 3$) reduced as

$$T = i f^{ai} \int d^4x \exp\left[\frac{i}{2}(p_a + p_i)x\right] \langle p_j p_k^{out} | \left\{ (\square_{x_1} + m_i^2)(\square_{x_2} + m_a^2) \cdot \theta(x_1 - x_2) [\phi_i(x_1), \phi_a(x_2)] \right\}_{x_1 = \frac{x}{2}, x_2 = -\frac{x}{2}} | p_b \rangle \quad (3.B.1)$$

$$f^{ai} = \left[\frac{(2\pi)^9 2p_j^0 2p_k^0 2p_b^0}{z_a z_i} \right]^{1/2} \quad (3.B.2)$$

z_a and z_b being renormalization constants, which play no role in what follows, and $\phi_i(x_1)$, $\phi_a(x_2)$ being local interpolating fields for particles i, a . Then, we shall define the announced new amplitudes T_{jk}^- as

$$T_{jk}^- = i f^{ai} \int d^4x \exp\left[\frac{i}{2}(p_a + p_i)x\right] \langle p_j p_k^- | \left\{ (\square_{x_1} + m_i^2)(\square_{x_2} + m_a^2) \cdot \theta(x_1 - x_2) [\phi_i(x_1), \phi_a(x_2)] \right\}_{x_1 = \frac{x}{2}, x_2 = -\frac{x}{2}} | p_b \rangle \quad (3.B.3)$$

Upon performing the indicated differentiations and discarding equal time-commutator contributions, Eq. (3.B.3) can be written as

$$T_{jk}^- = i f^{ai} \int d^4x \exp\left[\frac{i}{2}(p_a + p_i)x\right] \theta(x^0) \langle p_j p_k^- | \left[J_i\left(\frac{x}{2}\right), J_a\left(-\frac{x}{2}\right) \right] | p_b \rangle \quad (3.B.4)$$

$J_h(x) = (\square_{x+m_h^2})\phi_h(x)$, $h = a, i$ being the current for particle h . The neglect of such equal time contributions is permissible only when analyticity properties in p_a, p_i are concerned.

By using Eqs. (3.A.6), (3.A.9) and expliciting the dependence on p_j, p_k , we obtain the relation between the standard and the new amplitudes

$$\begin{aligned} T(p_j, p_k) &= \sum_{p'_j, p'_k} \langle p_j, p_k \text{ in} | F | p'_j, p'_k \text{ in} \rangle T(p'_j, p'_k)_{jk-} = \\ &= T(\bar{n}) = \int d\Omega(\bar{n}') F_{red}(\bar{n}, \bar{n}') T(\bar{n}')_{jk-} \end{aligned} \quad (3.B.5)$$

$d\Omega(\bar{n}')$ being the solid angle about $\bar{n}'$. In writing the last line, we have taken the j, k pair in their centre-of-mass frame, but we specified nothing concerning the remaining particles. For what follows, the best is to consider both T and T_{jk-} as defined both in the pair of frames O_{ab}^i, O_{jk}^i of Subsection 2.B and to regard Eq. (3.B.5) as a relation between T and T_{jk-} in O_{jk}^i so that both $\bar{n}$ and $\bar{n}'$ are vectors of the type $(\bar{q}_i / |\bar{q}_i|)$ in Fig. 3. Then, if we perform a partial-wave analysis of T_{jk-} in O_{jk}^i

$$T(\bar{n}')_{jk-} = \sum_{\ell, m} Y_{\ell}^m(\bar{n}') T_{jk-, \ell m} \quad (3.B.6)$$

Equation (3.B.5) yields after some manipulations

$$T(\bar{n}) = \sum_{\ell, m} Y_{\ell}^m(\bar{n}) F_{\ell}(s_{jk}) T_{jk-, \ell m} = \sum_{\ell, m} Y_{\ell}^m(\bar{n}) T_{\ell m} \quad (3.B.7)$$

$T_{\ell m}$ being the corresponding partial wave for the standard production amplitude.

We remark that the two angles defining $\bar{n}, \bar{n}'$ are nothing but the angles Ψ_i and $\arccos z_i$ in Fig. 3.

By using Eq. (3.A.7) or the above equations, one may express T_{jk-} in terms of T quite easily

$$T(\bar{n})_{jk-} = - \frac{|\bar{p}_j|}{32\pi^2 \sqrt{s_{jk}}} \int d\Omega(\bar{n}') T_{\ell}(\bar{n}', \bar{n})^* T(\bar{n}') \quad (3.B.8)$$

Notice that Eqs. (3.B.5) and (3.B.8) imply

$$-\frac{|\bar{p}_j|}{32\pi^2\sqrt{s_{jk}}}\int d\Omega(\bar{n}')T_d(\bar{n},\bar{n}')^*F_{nd}(\bar{n}',\bar{n}'')=\delta^{(2)}(\bar{n}-\bar{n}'') \quad (3.B.9)$$

We remark that the developments presented here can be easily generalized when the pair j,k may go to other two-particle coupled channels (besides undergoing elastic scattering), provided that no three-particle channel be coupled to it. One just needs to generalize the F matrix. And so on when spin particles are involved.

Needless to say, inelastic unitarity for $a+b \rightarrow a+b$ below the four-particle threshold imposes restrictions on the amplitudes T_{jk}^- , if one takes the elastic scattering amplitude for $a+b \rightarrow a+b$ as input data.

4. ANALYTIC PROPERTIES IN ANGLES AND PHYSICAL INTERPRETATION OF T_{jk}^-

From the definition of $|p_j p_k^- \rangle$, and using the asymptotic condition for particle k in order to reduce it, it is not difficult to obtain the following useful formula from Eq. (3.B.3)

$$T_{jk}^- = -\frac{i f^{ai}}{2[(2\pi)^3 2p_k^0 z_k]^{1/2}} \int d^4x d^4y \exp\left\{i\left[\frac{1}{2}(p+p_i)x + p_k y\right]\right\} \cdot$$

$$\langle p_j | J_k(\gamma) \left\{ (\square_{x_1} + m_i^2)(\square_{x_2} + m_a^2) \theta(x_1 - x_2) [\Phi_i(x_1), \Phi_a(x_2)] \right\}_{x_1 = \frac{x}{2}, x_2 = -\frac{x}{2}} | p_b \rangle \quad (4.1)$$

Some properties of T_{jk}^- , which justify its definition, are the following.

4.A T_{jk-} HAS ANALYTICITY PROPERTIES IN TWO ANGLES

In fact, upon inserting a complete set of states in Eq. (4.1), one has

$$T_{jk-} = - \frac{1}{2 [(2\pi)^3 2p_k^0 \bar{e}_k]^{1/2}} (2\pi)^4 \sum_{\alpha} \delta^{(4)}(p_j + p_k - p_{\alpha}) \langle p_j | J_k(0) | \alpha_{out} \rangle \cdot \left\{ i f^{ai} \int d^4x \exp\left[\frac{i}{2}(p_a + p_i)x\right] \langle \alpha_{out} | \left[(\square_{x_1} + m_i^2)(\square_{x_2} + m_a^2) \Theta(x_1 - x_2) \cdot [\phi_i(x_1), \phi_a(x_2)] \right]_{x_1 = \frac{x}{2}, x_2 = -\frac{x}{2}} | p_b \rangle \right\} \quad (4.A.1)$$

Clearly, the state $|\alpha, out\rangle$ has at most two particles in it. On the other hand, it cannot be the vacuum and it is not an essential restriction to assume that it is necessarily a two-particle state. In fact, the above assumption implies the absence of one-particle states decaying into two-particle states, which is well inside the spirit of considering only interactions between stable particles. First, we realize that the term inside the curly bracket in Eq. (4.A.1) is again a standard amplitude for $a+b \rightarrow i+\alpha$. It is easy to see that one may undo the reduction of particle i in it and reduce, in turn, particle b in order to have a retarded commutator for particles a and b . Consequently, Eq. (4.A.1) becomes

$$T_{jk-} = - \frac{(2\pi)^4}{2 [(2\pi)^3 2p_k^0 \bar{e}_k]^{1/2}} \sum_{\alpha} \delta^{(4)}(p_j + p_k - p_{\alpha}) \langle p_j | J_k(0) | \alpha_{out} \rangle \cdot \left\{ (-i f^{ab}) \int d^4x \exp\left[\frac{i}{2}(p_a - p_b)x\right] \left[(\square_{x_1} + m_i^2)(\square_{x_2} + m^2) \Theta(x_1 - x_2) \cdot \langle p_i, \alpha_{out} | [\phi(x_1), \phi(x_2)] | 0 \rangle \right]_{x_1 = \frac{x}{2}, x_2 = -\frac{x}{2}} \right\} \quad (4.A.2)$$

Upon applying the Jost-Lehmann-Dyson (JLD) representation to the above retarded commutator¹⁵⁾ and interchanging orders of integration and summation, we get

$$T_{jk-} = \int \frac{d^4u dx^2 \varphi(u, x^2, p_j, p_k, p_i)}{\left[\frac{1}{2}(p_a - p_b) - u\right]^2 - x^2} \quad (4.A.3)$$

$$\Phi(u, x^2, p_i, p_j, p_k) = - \frac{(2\pi)^4}{[2(2\pi)^3 2p_k^0 z_k]^{1/2}} \sum_{\alpha} \delta^{(4)}(p_j + p_k - p_i).$$

$$\cdot \langle p_j | J_k(0) | \alpha_{out} \rangle \Phi_{\alpha}(u, x^2, p_i) \quad (4.A.4)$$

$\Phi_{\alpha}(u, x^2, p_i)$ being the JLD weight function for the retarded commutator appearing in Eq. (4.A.2). Notice that in writing Eq. (4.A.3) we have not worried about possible regularizations in x^2 which could be necessary for the integral over x^2 to converge, as the regularization terms are polynomials in $\frac{1}{2}(p_a - p_b)$ and hence irrelevant as far as the analyticity properties of T_{jk}^- in $\frac{1}{2}(p_a - p_b)$ are concerned. The weight $\Phi_{\alpha}(u, x^2, p_i)$ is not vanishing only when: (i) $\frac{1}{2}(p_a + p_b) + u$ lies inside the forward light cone (L^+); (ii) $x \gg \text{Max} \left\{ 0, M_{i\alpha, ab} - \sqrt{\left[\frac{1}{2}(p_a + p_b) + u \right]^2}, M_{i\alpha, ab} - \sqrt{\left[\frac{1}{2}(p_a + p_b) - u \right]^2} \right\}$, $M_{i\alpha, ab}$ and $M_{i\alpha, ba}$ being the smallest invariant masses of any possible state γ such that $\langle p_i \alpha_{out} | J_a(0) | \gamma \rangle \cdot \langle \gamma | J_b(0) | 0 \rangle \neq 0$, $\langle p_i \alpha_{out} | J_b(0) | \gamma \rangle \cdot \langle \gamma | J_a(0) | 0 \rangle \neq 0$ respectively. Consequently $\Phi(u, x^2, p_j, p_k, p_i)$ has similar support properties to (i) and (ii), with some masses $M_{i, ab}$ and $M_{i, ba}$ which are the minima of $M_{i\alpha, ab}$ and $M_{i\alpha, ba}$ for all possible $|\alpha\rangle$ respectively. The frame O_{ab}^{xy} in Fig. 2 (see Subsection 2.A) is very well suited to study the analyticity properties of T_{jk}^- in $\frac{1}{2}(p_a - p_b)$ or, better, in the angles θ_{ai} , θ_{aj} defining $\bar{p}_a$ with respect to $\bar{p}_i$ and $\bar{p}_j$ for fixed s_{ij} , $i, j = 1, 2, 3$, $i \neq j$. Upon parametrizing $\bar{u}$ in O_{ab}^{xy} as $\bar{u} = |\bar{u}| \cdot (\sin\beta \cos\alpha, \sin\beta \sin\alpha, \cos\beta)$, realizing that Φ depends only on the scalars x^2 , u^2 , $p_h u$, $p_h p_e$, $h, e = i, j, k$, $h \neq e$ and hence, only on $\sin\beta$ (but not on $\cos\beta$) and doing some manipulations we get

$$T_{jk}^-(x_a, y_a) = \int_{L_{min}}^{+\infty} dL \int_0^1 d(\cos\beta) \int_0^{2\pi} d\alpha.$$

$$\frac{2 \sin\beta (\cos\alpha x_a + \sin\alpha y_a) \hat{\Phi}(L, \alpha, \sin\beta, s_{he})}{[\sin\beta (\cos\alpha x_a + \sin\alpha y_a) - L]^2 - \cos^2\beta (1 - x_a^2 - y_a^2)} \quad (4.A.5)$$

$$L = \frac{\bar{u}^2 + x^2 + \bar{p}_a^2 - \left(u^0 - \frac{m_a^2 - m_b^2}{2(s_{ab})^{1/2}} \right)^2}{2|\bar{p}_a| \cdot |\bar{u}|} \quad (4.A.6)$$

where: (i) the new weight, $\tilde{\varphi}$, comes from integrating the former, φ , over $|\bar{u}|$ and u^0 adequately; (ii) $L_{\min}$ is a function already familiar from the case of the four-point amplitude ³⁾

$$L_{\min} = \left\{ 1 + \frac{(M_{i,ab}^2 - m_a^2)(M_{i,ba}^2 - m_b^2)}{\bar{p}_a^2 [s_{ab} - (M_{i,ab} - M_{i,ba})^2]} \right\}^{\frac{1}{2}} \quad (4.A.7)$$

The analyticity domain of T_{jk-} in x_a, y_a is the largest set of complex points $x_a = \text{Re } x_a + i \text{Im } x_a, y_a = \text{Re } y_a + i \text{Im } y_a$ such that the denominator of Eq. (4.A.5) never vanishes for any α, β, L inside their respective integration ranges. By using symmetry considerations as Ascoli did ²⁾, the following equation for the boundary of the analyticity domain can be obtained

$$\frac{(\text{Re } x_a)^2 + (\text{Re } y_a)^2}{L_{\min}^2} + \frac{(\text{Im } x_a)^2 + (\text{Im } y_a)^2}{L_{\min}^2 - 1} - \frac{(\text{Re } x_a \cdot \text{Im } y_a - \text{Im } x_a \cdot \text{Re } y_a)^2}{L_{\min}^2 (L_{\min}^2 - 1)} = 1 \quad (4.A.8)$$

Second, upon reducing particle j in $\langle p_j | J_k(0) | \alpha_{\text{out}} \rangle$, using the fact that $\langle 0 | J_k(0) a(p_j)_{\text{out}} | \alpha \rangle = 0$ [as the mass $(p_\alpha - p_j)^2 = m_k^2$ is smaller than the smallest mass of a state $|\alpha\rangle$ for which $\langle 0 | J_k(0) | \alpha \rangle \neq 0$] in order to obtain a retarded commutator for particles j, k , we may write Eq. (4.A.1) as

$$T_{jk-} = \frac{-i\pi}{[2p_k^0 2p_j^0 z_k z_j]^{\frac{1}{2}}} \sum_{\alpha} \delta^{(4)}(p_j + p_k - p_\alpha) \left\{ \int d^4 y \exp\left[\frac{i}{2}(p_j - p_k)y\right] \cdot (\square_y + m_j^2) \theta(y^0) \langle 0 | [\phi_j(\frac{y}{2}), J_k(-\frac{y}{2})] | \alpha_{\text{out}} \rangle \right\} \cdot \left\{ i f^{ai} \int d^4 x \exp\left[\frac{i}{2}(p_a + p_i)x\right] \cdot \langle \alpha_{\text{out}} | \left[(\square_{x_1} + m_i^2) (\square_{x_2} + m_a^2) \theta(x_1 - x_2) [\phi_i(x_1), \phi_a(x_2)] \right]_{x_1 = \frac{x}{2}} \right|_{x_2 = -\frac{x}{2}} | p_b \rangle \right\}$$

(4.A.9)

Upon applying the JLD representation to the first retarded commutator in the above Eq. (4.A.9) and interchanging orders of integration and summation, Eq. (4.A.9) becomes

$$T_{jk-} = \int \frac{d^4 u d^2 x^2 \varphi(u, x^2, p_j + p_k, p_a + p_i, p_b)}{[q_i - u]^2 - x^2} \quad (4.A.10)$$

$$q_i = \frac{1}{2} (p_j - p_k)$$

where the new weight function φ contains all the remaining and unnecessary debris and is non-vanishing in a region like the one described in connection with Eq. (4.A.4) for the weight function $\varphi(u, x^2, p_j, p_k, p_i)$, but now $\frac{1}{2}(p_a + p_b)$ should be replaced by $\frac{1}{2}(p_j + p_k)$ and $M_{i\alpha, ab}$ and $M_{i\alpha, ba}$ by M_{jk} and M_{kj} which are the respective minimal masses of $|\alpha\rangle$ such that $\langle 0 | J_j(0) | \alpha \rangle \cdot \langle \alpha | J_k(0) | \alpha \rangle \neq 0$ and $\langle 0 | J_k(0) | \alpha \rangle \cdot \langle \alpha | J_j(0) | \alpha \text{ out} \rangle \neq 0$ for any contributing two-particle state, $|\alpha \text{ out}\rangle$. Consequently, we are in a situation similar to the one considered for Eq. (4.A.3): Eq. (4.A.10) implies analyticity properties in q_i . Their study is facilitated by going to the centre-of-mass frame of particles j, k and choosing the xy plane so that the three-momenta of particles a, b and i lie in it. Moreover, we shall take the x axis so that it bisects the angle between $\bar{p}_a$ and $\bar{p}_i$. Notice that this new frame is rotated with respect to O_{jk}^i in Fig. 3. Consequently, Eq. (4.A.10) will become similar to Eq. (4.A.5) after parametrizing $\bar{u}$ as there, x_a and y_a being replaced by the x and y components of $\bar{q}_i$. Upon arguing as in connection with Eq. (4.A.5), we conclude that T_{jk-} is analytic in both variables x and y in a domain given by Eq. (4.A.8), $L_{\min}$ being given now by

$$L_{\min} = \left\{ 1 + \frac{(M_{jk}^2 - m_j^2)(M_{ki}^2 - m_k^2)}{|\bar{p}_j|^2 [s_{jk} - (M_{jk} - M_{kj})^2]} \right\}^{1/2} \quad (4.A.11)$$

The relations between the actual x, y and the invariants can be easily obtained by using equations similar to the ones in Subsection 2.A but adapted to the new frame, and taking into account Eqs. (2.B.2)-(2.B.5). Equation (2.B.2) gives $\bar{p}_j^2$. T_{jk-} has even more analyticity properties in other angles, but we prefer to discuss them after considering its absorptive part which deserves a study on its own right and to which we now turn.

Of course, partial wave expansions of T_{jk-} in the angles considered above converge not only on the physical region but also inside the analyticity domains.

4.B THE ABSORPTIVE PART OF T_{jk}^- IS REAL

If we consider the absorptive part of the conventional production amplitude $\text{Abs } T$ [which is obtained from Eq. (3.B.2) just by dropping the step function], it is easy to see that it is in general a complex quantity due to the fact that $|p_j p_k^{\text{out}}\rangle \neq |p_j p_k^{\text{in}}\rangle$. Hence, it cannot coincide with $\text{Im } T$. This rather unfamiliar feature is avoided by using the new amplitudes T_{jk}^- . In fact, the absorptive part of T_{jk}^- as computed from either Eq. (3.B.3) or Eq. (3.B.4) (no equal-time contributions being discarded, as they disappear exactly in both cases):

$$\text{Abs } T_{jk}^- = \frac{1}{2} f^{ai} \int d^4 x \exp \left[\frac{i}{2} (p_a + p_i) x \right] \langle p_j p_k | [J_i(\frac{x}{2}), J_a(-\frac{x}{2})] | p_b \rangle$$

(4.B.1)

is real and coincides with $\text{Im } T_{jk}^-$ in the physical region, as one can show easily through the use of PT invariance, property a) of the states $|p_j p_k^-\rangle$ and some manipulations.

4.C ANALYTICITY PROPERTIES OF $\text{Abs } T_{jk}^-$ IN ANGLES

Moreover, $\text{Abs } T_{jk}^-$ has analyticity properties in three variables related to angles. In fact, upon introducing a complete set of intermediate states of four-momentum p_α , we get from Eq. (4.B.1)

$$\begin{aligned} \int d^4 x \exp \left[\frac{i}{2} (p_a + p_i) x \right] \langle p_j p_k | J_a(-\frac{x}{2}) J_i(\frac{x}{2}) | p_b \rangle = \\ = (2\pi)^4 \sum_{\alpha} \delta^{(4)}(p_a + p_i - p_b) \langle p_j p_k | J_a(0) | \alpha \rangle \langle \alpha | J_i(0) | p_b \rangle \end{aligned}$$

(4.C.1)

and we see that the right-hand side of this equation vanishes whenever $t_{bi} = (p_b - p_i)^2 (\leq (m_b - m_i)^2)$ is smaller than the smallest invariant mass of a state $|\alpha\rangle$ such that $\langle p_j p_k | J_a(0) | \alpha \rangle \cdot \langle \alpha | J_k(0) | p_b \rangle \neq 0$. Since we are inside the physical region for $a+b \rightarrow l+2+3$ [inside $\Gamma(s_{ab})$ in Figs. 5 and 7] t_{bi} fulfils the above condition. Consequently, it is easy to obtain

$$\begin{aligned} \text{Abs } T_{jk}^- = - \frac{f^{ai}}{4[(2\pi)^2 2p_k^0 z_k]^{1/2}} \int d^4 x d^4 y \exp \left\{ i \left[\frac{1}{2} (p_a + p_i) x + p_k y \right] \right\} \\ \cdot \langle p_j | J_k(y) J_i(\frac{x}{2}) J_a(-\frac{x}{2}) | p_b \rangle \end{aligned}$$

(4.C.2)

Next, by (i) introducing two complete sets of suitable intermediate states $|\beta \text{ in} \rangle$ and $|\alpha \text{ out} \rangle$ in Eq. (4.C.2); (ii) doing the integration in co-ordinate space; (iii) reducing particle b in the matrix element, $\langle \alpha \text{ out} | J_a(0) | p_b \rangle$ using the fact that $\langle \alpha \text{ out} | a^+(p_b)_{\text{out}} J_a(0) | 0 \rangle = 0$ [as the mass $(p_a - p_b)^2 = m_a^2$ is smaller than the smallest mass of a state $|\alpha \rangle$ for which $\langle \alpha | J_a(0) | 0 \rangle \neq 0$], reducing particle j in $\langle p_j | J_k(0) | \beta \text{ in} \rangle$ as we did to arrive at Eq. (4.A.9) and manipulating, we obtain

$$\begin{aligned}
 \text{Abs } T_{jk-} &= \frac{(2\pi)^8}{4(\bar{z}_a \bar{z}_b \bar{z}_i \bar{z}_j \bar{z}_k)^{3/2}} \sum_{\alpha, \beta} \delta^{(4)}(p_j + p_k - p_\beta) \delta^{(4)}(p_a + p_b - p_\alpha) \\
 &\cdot \left\{ d^4 \gamma \exp\left[\frac{i}{2}(p_j - p_k) \gamma\right] (\not{\gamma} + m_j^2) \theta(-\gamma^0) \langle 0 | [\phi_j(\frac{\gamma}{2}), J_k(-\frac{\gamma}{2})] | \beta \text{ in} \rangle \right\} \\
 &\cdot \langle \beta \text{ in} | J_i(0) | \alpha \text{ out} \rangle \cdot \left\{ d^4 z \exp\left[\frac{i}{2}(p_a - p_b) z\right] (\not{z} + m_b^2) \theta(z^0) \right. \\
 &\quad \cdot \langle \alpha \text{ out} | [J_a(-\frac{z}{2}), \phi_b(\frac{z}{2})] | 0 \rangle \left. \right\}
 \end{aligned} \tag{4.C.3}$$

Notice that since $(m_j + m_k)^2 \leq s_{jk} < m_{jk, \text{in}}^2$, the state $|\beta \rangle$ has necessarily two particles in it.

Upon applying the JLD representation to both retarded commutators without regularizing polynomials in $\frac{1}{2}(p_j - p_k)$ and $\frac{1}{2}(p_a - p_b)$ (as they do not alter the analyticity properties in them) and interchanging orders of summation over α, β and integration, we arrive at

$$\begin{aligned}
 \text{Abs } T_{jk-} &= \int \frac{d^4 u_1 d x_1^2 d^4 u_2 d x_2^2 \varphi(u_1, x_1^2, u_2, x_2^2, p_j + p_k, p_a + p_b)}{[(q_i - u_1)^2 - x_1^2] \left\{ \left[\frac{1}{2}(p_a - p_b) - u_2 \right]^2 - x_2^2 \right\}} \\
 q_i &= \frac{1}{2}(p_j - p_k)
 \end{aligned} \tag{4.C.4}$$

$$\begin{aligned}
 \varphi &= \frac{(2\pi)^8}{4(\bar{z}_a \bar{z}_b \bar{z}_i \bar{z}_j \bar{z}_k)^{3/2}} \sum_{\alpha, \beta} \delta^{(4)}(p_j + p_k - p_\beta) \delta^{(4)}(p_a + p_b - p_\alpha) \varphi_\beta(u_1, x_1^2) \\
 &\cdot \langle \beta \text{ in} | J_i(0) | \alpha \text{ out} \rangle \varphi_\alpha(u_2, x_2^2)
 \end{aligned} \tag{4.C.5}$$

Φ_β and Φ_α being the individual weight functions corresponding to the two retarded commutators in Eq. (4.C.3) respectively, whose supports are respectively:

$$i) \quad \frac{1}{2}(p_j + p_k) \pm u_1 \in L^+ \\ x_1 \geq \text{Max} \left\{ 0, M_{\beta, jk} - \sqrt{\left[\frac{1}{2}(p_j + p_k) + u_1 \right]^2}, M_{\beta, kj} - \sqrt{\left[\frac{1}{2}(p_j + p_k) - u_1 \right]^2} \right\}$$

$$ii) \quad \frac{1}{2}(p_a + p_b) \pm u_2 \in L^+ \\ x_2 \geq \text{Max} \left\{ 0, M_{\alpha, ab} - \sqrt{\left[\frac{1}{2}(p_a + p_b) + u_2 \right]^2}, M_{\alpha, ba} - \sqrt{\left[\frac{1}{2}(p_a + p_b) - u_2 \right]^2} \right\}$$

$M_{\beta, jk}$, $M_{\beta, kj}$, $M_{\alpha, ab}$ and $M_{\alpha, ba}$ are the smallest masses of any state $|\sigma\rangle$ such that $\langle 0 | J_j(0) | \sigma \rangle \cdot \langle \sigma | J_k(0) | \beta \text{ in} \rangle \neq 0$, $\langle 0 | J_k(0) | \sigma \rangle \cdot \langle \sigma | J_j(0) | \beta \text{ in} \rangle \neq 0$, $\langle \alpha \text{ out} | J_a(0) | \sigma \rangle \cdot \langle \sigma | J_b(0) | 0 \rangle \neq 0$, $\langle \alpha \text{ out} | J_b(0) | \sigma \rangle \cdot \langle \sigma | J_a(0) | 0 \rangle \neq 0$ respectively. The over-all weight function, Φ , depends on u^2 , u_1^2 , x_1^2 , x_2^2 , $u_1(p_j + p_k)$, $u_2(p_j + p_k)$, $u_1(p_a + p_b)$, $u_2(p_a + p_b)$, s_{ab} , s_{jk} , m_i^2 . The study of the analyticity properties of $\text{Abs } T_{jk-}$ is greatly facilitated by considering the pair of frames O_{jk}^i and O_{ab}^i (see Subsection 2.B) for each retarded commutator respectively. As long as we deal with invariant functions and we multiply two four-vectors in the same frame, the above choice is permissible. We shall parametrize $\bar{u}_1$ in O_{jk}^i as $\bar{u}_1 = |\bar{u}_1| (\sin \beta \cos \alpha_1, \sin \beta \sin \alpha_1, \cos \beta)$ and $\bar{u}_2$ in O_{ab}^i as $\bar{u}_2 = |\bar{u}_2| (\sin \beta_2 \cos \alpha_2, \sin \beta_2 \sin \alpha_2, \cos \beta_2)$. Then, we realize that Φ depends on $|\bar{u}_1|$, $|\bar{u}_2|$, u_1^0 , u_2^0 , x_1 , x_2 , β_1 , β_2 , $\cos(\alpha_1 - \alpha_2)$, s_{ab} , s_{jk} and m_i^2 but neither on α_1 , α_2 nor on $\sin(\alpha_1 - \alpha_2)$. Of course, this simplification is due to the fact that the frames O_{jk}^i and O_{ab}^i are related through a Lorentz transformation along the z axis. Thus, upon performing the integral over α_1 by the same formula as Lehmann utilized for the case of the absorptive part of the four-point function ³⁾, and operating, we get

$$\text{Abs } T_{jk-} = \int_{L_1, \min}^{+\infty} dL_1 \int_{L_2, \min}^{+\infty} dL_2 \int_{-1}^{+1} d(\cos \beta_1) d(\cos \beta_2) \int_0^\pi d\alpha \tilde{\Phi}(L_1, L_2, \beta_1, \beta_2, \cos \alpha, s_{ab}, s_{jk}, m_i) \\ \frac{\left[\frac{L_1 - \cos \beta_1 \bar{z}_i}{\sqrt{(L_1 - \cos \beta_1 \bar{z}_i)^2 - \sin^2 \beta_1 \bar{r}_i^2}} + \frac{L_2 - \cos \beta_2 \bar{z}}{\sqrt{(L_2 - \cos \beta_2 \bar{z})^2 - \sin^2 \beta_2 X^2}} \right]}{D(L_1, L_2, \beta_1, \beta_2, \alpha, \bar{z}_i, \bar{z}, \Psi_i)} \cdot \left\{ (L_1 - \cos \beta_1 \bar{z}_i)(L_2 - \cos \beta_2 \bar{z}) + \right. \\ \left. + \sqrt{(L_1 - \cos \beta_1 \bar{z}_i)^2 - \sin^2 \beta_1 \bar{r}_i^2} \sqrt{(L_2 - \cos \beta_2 \bar{z})^2 - \sin^2 \beta_2 X^2} - \cos \alpha \sin \beta_1 \sin \beta_2 X \bar{r}_i \cos \Psi_i \right\}$$

$$D(L_1, L_2, \beta_1, \beta_2, \tau_i, \tau, \Psi_i) = D = \left[(L_1 - \cos \beta_1 \tau_i)(L_2 - \cos \beta_2 \tau) + \right. \\ \left. + \sqrt{(L_1 - \cos \beta_1 \tau_i)^2 - \sin^2 \beta_1 \tau_i^2} \sqrt{(L_2 - \cos \beta_2 \tau)^2 - \sin^2 \beta_2 \tau^2} - \cos \alpha \sin \beta_1 \sin \beta_2 \tau_i \tau \cos \Psi_i \right]^2 - \\ - \sin^2 \alpha \sin^2 \beta_1 \sin^2 \beta_2 \tau_i^2 \tau^2 \sin^2 \Psi_i \quad (4.C.7)$$

$$L_1 = \frac{\bar{u}_1^2 + x_1^2 + \bar{q}_1^2 - (u_1^0 - q_1^0)^2}{2|\bar{u}_1| |\bar{q}_1|}, \quad L_2 = \frac{\bar{u}_2^2 + x_2^2 + \bar{p}_a^2 - (u_2^0 - \frac{m_a^2 - m_b^2}{2\sqrt{s_{ab}}})^2}{2|\bar{u}_2| |\bar{p}_a|}$$

$$x^2 + \tau^2 = 1, \quad \tau_i^2 + \tau_i^2 = 1 \quad (4.C.8)$$

where $\tilde{\varphi}$ is the integral of the former over-all weight function, φ , over $|\bar{u}_1|$, $|\bar{u}_2|$, u_1^0 , u_2^0 , x_1^2 and x_2^2 and $L_{1,\min}$ and $L_{2,\min}$ are the minima of L_1 , L_2 . The only reason for having used the two Lorentz frames O_{jk}^i and O_{ab}^i is that $L_{1,\min}$, $L_{2,\min}$ can be found easily in them. In fact, it is easy to realize that $L_{1,\min}$ is given by Eq. (4.A.11), while

$$L_{2,\min} = \left\{ 1 + \frac{(M_{ab}^2 - m_a^2)(M_{ba}^2 - m_b^2)}{\bar{p}_a^2 [s_{ab} - (M_{ab} - M_{ba})^2]} \right\}^{1/2} \quad (4.C.9)$$

M_{ab} and M_{ba} being the minima of $M_{\alpha,ab}$ and $M_{\alpha,ba}$ as $|\alpha\rangle$ varies.

Since $\tilde{\varphi}$ is independent on z , z_i and $\cos \Psi_i$, we see that $\text{Abs } T_{jk}^-$ has analyticity properties in the regions of the complex z , z_i and $\cos \Psi_i$ planes where neither D nor the square roots in Eqs. (4.C.6) and (4.C.7) vanish when L_1 , L_2 , β_1 , β_2 and α vary over their respective integration ranges. We remark that $\text{Abs } T_{jk}^-$ has singularities whenever xr_i , which appears in the second factor in the numerator of Eq. (4.C.6) and inside D , vanishes. However, this singularity can be separated quite easily. In fact, upon performing some manipulations we arrive at the following decomposition, which makes explicit the singularity

$$\text{Abs } T_{jk}^- = \widetilde{\text{Abs } T_{jk}^-} + x r_i \cdot \widetilde{\widetilde{\text{Abs } T_{jk}^-}} \quad (4.C.10)$$

where each piece $\widetilde{\text{Abs } T_{jk}^-}$ and $\widetilde{\text{Abs } T_{jk}^-}$ has integral representations similar to Eq. (4.C.6), which we shall not write as they are long and not very illuminating. Two interesting properties of $\widetilde{\text{Abs } T_{jk}^-}$ and $\widetilde{\text{Abs } T_{jk}^-}$ are: (i) they are free of the singularity at $x r_i = 0$; (ii) instead of D they contain a new denominator

$$\begin{aligned} \widetilde{D} = & \left\{ \left[(L_1 - \cos \beta_1 z_i)(L_2 - \cos \beta_2 z) + \sqrt{(L_1 - \cos \beta_1 z_i)^2 - \sin^2 \beta_1 r_i^2} \right. \right. \\ & \left. \left. \cdot \sqrt{(L_2 - \cos \beta_2 z)^2 - \sin^2 \beta_2 x^2} \right] + \sin^2 \beta_1 \sin^2 \beta_2 x^2 r_i^2 (\cos^2 \Psi_i - \sin^2 \alpha) \right\} - \\ & - 4 \left[(L_1 - \cos \beta_1 z_i)(L_2 - \cos \beta_2 z) + \sqrt{(L_1 - \cos \beta_1 z_i)^2 - \sin^2 \beta_1 r_i^2} \sqrt{(L_2 - \cos \beta_2 z)^2 - \sin^2 \beta_2 x^2} \right]^2 \\ & \cdot \cos^2 \alpha \sin^2 \beta_1 \sin^2 \beta_2 r_i^2 x^2 \cos^2 \Psi_i \end{aligned} \quad (4.C.11)$$

In what follows, we shall be dealing with either $\text{Abs } T_{jk}^-$ and D or $\widetilde{\text{Abs } T_{jk}^-}$, $\widetilde{\text{Abs } T_{jk}^-}$ and $\widetilde{D}$, as suitable.

Since $L_{1,\min} > 1$, $L_{2,\min} > 1$, it is not difficult to see from Eq. (4.C.7) that $D > 0$ (and hence $\widetilde{D} > 0$) whenever all z , z_i and $\cos \Psi_i$ vary inside their respective physical regions ($-1 \leq z, z_i$, $\cos \Psi_i \leq +1$). This shows clearly that $\widetilde{\text{Abs } T_{jk}^-}$ and $\widetilde{\text{Abs } T_{jk}^-}$ are well defined on the physical region, which is thus contained inside a larger analyticity domain in the corresponding three complex variables. Clearly, the same happens to $\text{Abs } T_{jk}^-$, if care is taken of the singularity at $x r_i = 0$.

Some general properties of the analyticity domain are the following.

- a) As a necessary condition both z_i and z should be such that none of the square roots in Eq. (4.C.7) vanish, which implies that they should lie respectively inside the "minimal" ellipses

$$\frac{(\text{Re } z_i)^2}{L_{1,\min}^2} + \frac{(\text{Im } z_i)^2}{L_{1,\min}^2 - 1} = 1, \quad \frac{(\text{Re } z)^2}{L_{2,\min}^2} + \frac{(\text{Im } z)^2}{L_{2,\min}^2 - 1} = 1 \quad (4.C.12)$$

- b) When both z_i and z lie on their "minimal" ellipses, there is no domain of analyticity in $\cos \Psi_i$ at all (as, for instance, D becomes proportional to $\cos \Psi_i \pm \cos \alpha$). Even more, if only one of them lies on its minimal ellipse, there are no analyticity properties in either the other one or $\cos \Psi_i$, since the first factor in the numerator of Eq. (4.C.6) becomes infinite.

c) Let z_i and z be real and such that $|z_i| < 1$, $|z| < 1$. Then, the equation $D = 0$ implies readily

$$\frac{(\operatorname{Re} \cos \Psi_i)^2}{g^2} + \frac{(\operatorname{Im} \cos \Psi_i)^2}{g^2 - 1} = 1 \quad (4.C.13)$$

$$g = \frac{L_1 - \cos \beta_1 z_i}{\sin \beta_1 \sqrt{1 - z_i^2}} \cdot \frac{L_2 - \cos \beta_2 z}{\sin \beta_2 \sqrt{1 - z^2}} + \sqrt{\left(\frac{L_1 - \cos \beta_1 z_i}{\sin \beta_1 \sqrt{1 - z_i^2}}\right)^2 - 1} \cdot \sqrt{\left(\frac{L_2 - \cos \beta_2 z}{\sin \beta_2 \sqrt{1 - z^2}}\right)^2 - 1} \quad (4.C.14)$$

Upon finding out the minimum of g , we conclude that $\operatorname{Abs} T_{jk}^-$ is analytic in $\cos \Psi_i$ inside an ellipse like Eq. (4.C.13) corresponding to

$$g = \sqrt{\frac{L_{1,\min}^2 - z_i^2}{1 - z_i^2}} \sqrt{\frac{L_{2,\min}^2 - z^2}{1 - z^2}} + \sqrt{\frac{L_{1,\min}^2 - 1}{1 - z_i^2}} \sqrt{\frac{L_{2,\min}^2 - 1}{1 - z^2}} \quad (4.C.15)$$

When both z and z_i are real but $1 < |z_i| < L_{1,\min}$, $1 < |z| < L_{2,\min}$, a similar argument shows that $\operatorname{Abs} T_{jk}^-$ is analytic in $\cos \Psi_i$ inside an analogous ellipse for

$$g = \sqrt{\frac{L_{1,\min}^2 - z_i^2}{z_i^2 - 1}} \sqrt{\frac{L_{2,\min}^2 - z^2}{z^2 - 1}} + \sqrt{\frac{L_{1,\min}^2 - 1}{z_i^2 - 1}} \sqrt{\frac{L_{2,\min}^2 - 1}{z^2 - 1}} \quad (4.C.16)$$

When z and z_i are still real but $1 < |z_i| < L_{1,\min}$, $|z| < 1$, it can be shown through analogous considerations that $\operatorname{Abs} T_{jk}^-$ is analytic in $\cos \Psi_i$ inside the ellipse

$$\frac{(\operatorname{Re} \cos \Psi_i)^2}{f^2 + 1} + \frac{(\operatorname{Im} \cos \Psi_i)^2}{f^2} = 1 \quad (4.C.17)$$

$$f = \sqrt{\frac{L_{1,\min}^2 - z_i^2}{z_i^2 - 1}} \sqrt{\frac{L_{2,\min}^2 - z^2}{1 - z^2}} + \sqrt{\frac{L_{1,\min}^2 - 1}{z_i^2 - 1}} \sqrt{\frac{L_{2,\min}^2 - 1}{1 - z^2}} \quad (4.C.18)$$

and similarly for z, z_i real, $|z_i| < 1, 1 < |z| < L_{2,\min}$. Notice that when both z_i and z reach $L_{1,\min}$ and $L_{2,\min}$ (that is, their "minimal" ellipses), the ellipse of analyticity in $\cos \Psi_i$ shrinks and collapses completely.

- d) Using both D and $\tilde{D}$, it is not difficult to see that for physical $\cos \Psi_i$ both $\widetilde{\text{Abs } T_{jk}^-}$ and $\widetilde{\text{Abs } T_{jk}^-}$ are analytic functions of both z_i and z , whenever the latter are contained inside their respective "minimal" ellipses.

Summarizing, we have achieved a decomposition of the absorptive part $\text{Abs } T_{jk}^-$ into two parts, $\widetilde{\text{Abs } T_{jk}^-}$ and $\widetilde{\text{Abs } T_{jk}^-}$, each of which is an analytic function of the three variables $\cos \Psi_i, z$ and z_i in a domain which contains the physical region inside of it, for fixed s_{ab} and s_{jk} ($(m_j + m_k)^2 \leq s_{jk} < m_{jk,\text{in}}^2$).

Consequently, a partial wave expansion of both $\widetilde{\text{Abs } T_{jk}^-}$ and $\widetilde{\text{Abs } T_{jk}^-}$ into those three variables converges not only in the physical region but also inside the analyticity domain.

By using the corresponding kinematical equations summarized in Section 2, the above analyticity properties imply also analyticity properties for $\text{Abs } T_{jk}^-$ in the sets (t_{ai}, t_{bj}, s_{ik}) and (t_{ai}, t_{bj}, ω) for fixed s_{ab} and s_{jk} . They also imply analyticity properties in transverse-momenta-like variables like $\vec{r}_j^2, \vec{r}_k^2, \vec{r}_j \vec{r}_k$ for fixed s_{ab} and $p_{i,L}^2 + r_i^2$.

4.D T_{jk}^- HAS ANALYTICITY PROPERTIES IN THREE ANGLES

In fact, coming back to Eq. (4.A.9), replacing the second curly bracket in it by the curly bracket in Eq. (4.A.2) [much in the same way as when going from Eq. (4.A.1) to Eq. (4.A.2)], applying the JLD representation to both retarded commutators as when going from Eq. (4.C.3) to Eq. (4.C.4), we end up with an equation similar to (4.C.4), where now the new weight function depends on $p_j + p_k, p_i$, that is, also on $p_a + p_b$ and $p_j + p_k$. The resulting equation leads again to choose the frames O_{ab}^i and O_{jk}^i and implies also analyticity properties of T_{jk}^- in the same three variables z, z_i and $\cos \Psi_i$. The domains of analyticity of T_{jk}^- are essentially the same as those of $\text{Abs } T_{jk}^-$, with only one minor difference: although $L_{1,\min}$ is here the same as for $\text{Abs } T_{jk}^-$, [being given in Eq. (4.A.11)], here $L_{2,\min}$ should be replaced by the expression of Eq. (4.A.7), which is different from $L_{2,\min}$ as $M_{ab} \neq M_{i,ab}, M_{ba} \neq M_{i,ba}$ in general.

We have presented the analyticity properties of T_{jk}^- into two and three angles separately as: (i) the domains of analyticity and the methods to find them out are different and worth to be treated; (ii) the above explicit discussion on $\text{Abs } T_{jk}^-$ will be needed afterwards in Subsection 5.B. By virtue of the generalization of property d) concerning the analyticity domains in the preceding subsection T_{jk}^- , we see that the partial wave T_{jk}^-, m [which is a function of s_{ab} , s_{jk} and z or t_{ai} , by virtue of Eq. (4.C.12)] is analytic in z inside the corresponding "minimal" ellipse given by Eq. (4.C.12) $\sqrt{L_{2,\min}}$ being replaced by Eq. (4.A.7)], except for a square-root singularity at $x = 0$ induced by the generalization of Eq. (4.C.10) to T_{jk}^- , which can be easily explicitated as in Eq. (4.C.10). On the other hand, the analyticity properties in angles related to $p_i - p_j$ and $p_i - p_k$ should not be expected to be simple which is no surprise as the definition in Eq. (3.B.3) puts the pair j, k on a special footing with respect to particle i . We recall that the only proved analyticity properties of the conventional amplitude refer to two angles defining one initial three-momentum with respect to the final three-momenta in the centre-of-mass ^{1),2)}. Due to the form of $F(s_{jk})$, the F matrix can be a rather singular object [for instance, as $s_{jk} = (m_j + m_k)^2$ by virtue of the effective-range expansion] and, hence, we have stated nothing regarding the convergence of the partial-wave expansions in Eqs. (3.A.10) and (3.B.7). Nevertheless, we expect that the second one has enough convergence properties, on the basis of some final-state interaction models (see the next subsection) and the following fact. If for pure elastic scattering we define a corresponding T_- by means of Eq. (3.B.8), T being replaced by the elastic two particle amplitude T_{el} , we see that T_- becomes precisely $\text{Im } T_{el}$ in this case. Thus, for elastic scattering below the first inelastic threshold, the F matrix transforms $\text{Im } T_{el}$ (which has analyticity properties in one angle inside the large Lehmann ellipse) into T_{el} (which has inside the small Lehmann ellipse) through the analogue of Eq. (3.B.5) (no dispersion relation in the energy being used). Hence, we suspect that a similar phenomenon may happen for five-particle processes concerning T_{jk}^- and T : T_{jk}^- has large analyticity domains in angles, so that even though F_{red} may be rather singular, the series (3.B.7) may have convergence properties. Unfortunately, we shall not prove this here.

4.E FURTHER PROPERTIES AND PHYSICAL INTERPRETATION OF T_{jk}^-

Some qualitative properties of T_{jk}^- and T_{jk}^-, ℓ_m are worth to point out. First, no analyticity properties should be expected for T_{jk}^- in s_{jk} since there is no step function like $\theta(y^0)$ in Eq. (4.1) which could allow one to promote p_k (and, then, s_{jk} , at least in the rest frame of particle j) into the complex plane. Consequently, Eq. (4.1) and hence T_{jk}^- can be expected to have a mathematical meaning only when s_{jk} is real. We shall not bother whether T_{jk}^- can be defined for real s_{jk} outside the interval $(m_j + m_k)^2 \leq s_{jk} < m_{jk,\text{in}}^2$ by means of Eq. (4.1), since T_{jk}^- and T determine each other uniquely through Eqs. (3.B.5), (3.B.8) only when s_{jk} lies in the above (elastic) interval.

One may ask, however, how smooth are T_{jk}^- and T_{jk}^-, ℓ_m as s_{jk} varies in the elastic interval. On the basis of some final state interaction models

[see Cook and Lee ⁶⁾], one may argue that $T_{\ell m}$ [recall Eq. (3.B.7)] should be the product of some Watson factor ¹⁶⁾ $\exp[i\delta_{\ell}(s_{jk})] \sin \delta_{\ell}(s_{jk})$ times a slowly varying function of s_{jk} (which depends on ℓ, m) and, perhaps, a kinematic factor (independent of ℓ, m). We notice that the Watson factor ensures the smallness (enhancement) of $T_{\ell m}$ at those values of ℓ for which $\delta_{\ell}(s_{jk})$ is small (large). Consequently, Eqs. (3.B.7) and (3.B.8) imply that $T_{jk}^-, \ell m$ should be the product of $[\sin \delta_{\ell}(s_{jk})]^2$ times the same slowly varying function of s_{jk} and the same kinematical factor. Hence, for high values of ℓ and small $|\vec{p}_j|$, the effective-range expansion ($|\vec{p}_j|^{2\ell+1} \cotg \delta_{\ell}(s_{jk}) = \frac{1}{a_{\ell}} + \frac{r_{\ell}}{2} |\vec{p}_j|^2 + \dots$) implies that $T_{jk}^-, \ell m$ is more depressed by the final-state interaction of the (j,k) pair than $T_{\ell m}$ is. On the other hand, near a sharply peaked resonance at angular momentum ℓ_{res} , mass $\sqrt{s_{jk, res}}$ and width Γ_{res} , the Breit-Wigner formula

$$\left(\sin \delta_{\ell_{res}}(s_{jk}) \simeq \frac{\frac{1}{2} \Gamma_{res}}{\sqrt{(\sqrt{s_{jk}} - \sqrt{s_{jk, res}})^2 + (\frac{1}{2} \Gamma_{res})^2}} \right)$$

implies that the bump for $|T_{jk}^-, \ell_{res, m}|^2$ is sharper than that for $|T_{\ell_{res, m}}|^2$ (the maximum height being the same for both, of course). Consequently, $T_{jk}^-, \ell m$ can be regarded the inelastic scattering amplitude for the process $a+b \rightarrow i+(j+k)^-, \ell m$, $(j+k)^-, \ell m$ being considered as a composite system of mass $\sqrt{s_{jk}}$ and angular momentum ℓ, m (T_{jk}^- being the coherent sum of all $T_{jk}^-, \ell m$), while $T_{\ell m}$ has less chances to be considered as such. The above interpretation becomes clearer and clearer when $\sqrt{s_{jk}}$ and ℓ coincide with the values for a resonance. We remark that if, following Ishida, we define a new amplitude T_{jk}^+ from Eq. (3.B.1) by replacing $|p_j p_k \text{ out} \rangle$ by the states $|p_j p_k^+ \rangle$ briefly mentioned in Subsection 3.A, it is not difficult to see (by repeating the arguments made above for $T_{jk}^-, \ell m$) that the corresponding partial wave $T_{jk}^+, \ell m$ in O_{jk}^i contains a factor $\sin \delta_{\ell}(s_{jk}) \cos \delta_{\ell}(s_{jk})$. Then, at threshold, $T_{jk}^+, \ell m$ behaves like $T_{\ell m}$, while it vanishes at a resonance. We have not found simple analyticity properties in three angles for T_{jk}^+ . Thus, although $\text{Im } T_{jk}^+ = \text{Abs } T_{jk}^+$, we believe that $T_{jk}^+, \ell m$ has even less chances than $T_{\ell m}$ to be regarded as an inelastic scattering amplitude for a four-particle process with one composite particle, and we remain with $T_{jk}^-, \ell m$ as the best candidate for that. Consequently, the partial-waves corresponding to the states $|p_j p_k^- \rangle$ are also the best candidates for defining true resonance states caused by the interaction (in fact, they vanish identically when there is no interaction). Equation (3.A.7) implies the following relation between those partial waves and the ones corresponding to $|p_j p_k \text{ out} \rangle$:

$$\langle p_j p_k^- | = \sum_{\ell, m} Y_{\ell}^m(\vec{n}) \langle s_{jk}, \ell m^- |$$

$$\langle s_{jk}, \ell m^- | = -i \sin \delta_{\ell}(s_{jk}) \cdot \exp[-i \delta_{\ell}(s_{jk})] \cdot \langle s_{jk}, \ell m \text{ out} | \quad (4.E.1)$$

and the above arguments imply that near a sharp-peaked resonance $\langle s_{jk}, \ell_m^- |$ represents the resonant state, while the corresponding partial wave for $\langle p_j p_k^+ |$ vanishes due to the factor $\cos \frac{\pi}{2} (s_{jk})$. Consequently, in the subspace of the states with angular momentum ℓ, m , the right-hand side of the projection of Eq. (3.A.3) goes to zero near resonance, that is, the resonant state $|s_{jk}, \ell_m^- \rangle$ saturates the completeness relation. The above discussion of the physical interpretation of T_{jk}^- and $\langle p_j p_k^- |$ was not made by either Ishida or Mohan.

Further work on non-relativistic three-body scattering will help to clarify even more the situation, hopefully. In this connection, we remark that if we look at the non-relativistic three-particle scattering problem and regard T and T_{jk}^- as suitable operators acting in the three-particle Hilbert space and interrelated by Eq. (3.B.5), then one realizes that each T_{jk}^- can be independently decomposed into three pieces: $T_{jk}^- = T_{jk}^{(1)} + T_{jk}^{(2)} + T_{jk}^{(3)}$ which satisfy a generalized Faddeev system for each j, k , where the inhomogeneous terms and the kernels can be easily expressed in terms of the F operator and two-particle on- and off-shell T operators. In fact, each $T_{jk}^{(h)}$, $h = 1, 2, 3$, is obtained from a similar piece $T^{(h)}$ in Faddeev's decomposition ($T = T^{(1)} + T^{(2)} + T^{(3)}$), Ref. 17), by means of Eq. (3.B.5).

As pointed out by Professor V. Glaser, the amplitudes T_{jk}^- are nothing but partially absorptive parts of the conventional amplitude in the subenergy s_{jk} .

5. ANALYTICITY PROPERTIES OF T_{jk}^- IN THE ENERGY AND RELATED PROBLEMS

5.A DISPERSION RELATIONS FOR OFF-SHELL PARTICLES

Our starting point is Eq. (3.B.4) [although Eq. (3.B.3) could serve equally well] where now all particle four-momenta are considered in the frame O^{BCP} (see Subsection 2.C), the interesting variables being s_{ik} , s_{jk} , t_{ai} , t_{bj} and ω . We shall fix s_{jk} in $(m_j + m_k)^2 \leq s_{jk} < m_{jk, in}^2$, ω in $0 \leq \omega < 2\pi$ and t_{ai} , t_{bj} at negative values and we shall investigate the analyticity properties of T_{jk}^- in s_{ik} , as the only dependence of Eq. (3.B.4) on it is contained in the exponential. The masses of b, j are fixed necessarily at their physical values and for convenience we do the same with k, i [although it is not strictly necessary as we may use also Eq. (4.1)]. In principle, particle a is off-shell, that is, $p_a^2 \neq m_a^2$. Recalling Eqs. (2.C.1)-(2.C.5) and in order to simplify the exponential, we shall perform the following change of variables

$$x \rightarrow x' = x \mathcal{L}_1(\ell^{jk}) \mathcal{L}_2(\omega) \mathcal{L}_3(q) = \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \quad (5.A.1)$$

where we take x and x' to be column vectors. Then Eq. (3.B.4) becomes

$$T_{jk-} = i \int d^4 x' \exp \left[\frac{i}{2} x' (p_a + p_i)' \right] \theta(x'^0) \langle p_j p_k - 1 [J_i(\frac{x'}{2}), J_a(-\frac{x'}{2})] | p_b \rangle \quad (5.A.2)$$

$$\frac{1}{2} (p_a + p_i)' = L_1(\zeta^{ik}) \begin{pmatrix} \sqrt{\frac{\lambda(t_{ai}, p_a^2, m_i^2)}{-4t_{ai}}} \\ 0 \\ 0 \\ \frac{m_i^2 - p_a^2}{\sqrt{-4t_{ai}}} \end{pmatrix} \quad (5.A.3)$$

We stress that the above change of variables is independent of p_a^2 and m_i^2 and, moreover, it is nothing but a product of successive Lorentz transformations which do not reverse the sign of time. Consequently, the spacelike points $x(x^2 < 0)$ where the commutator in Eq. (3.B.4) vanishes transform into points x' which are also spacelike ($x'^2 < 0$) and hence where a similar property holds for the commutator. Now, noticing that

$$\begin{aligned} x' \frac{1}{2} (p_a + p_i)' &= \frac{\sqrt{-t_{ai}}}{\sqrt{\lambda(t_{ai}, t_{bj}, m_k^2)}} \left\{ x'^0 \left[s_{ik} - t_{bj} - p_a^2 + \frac{(m_i^2 - p_a^2 - t_{bj})(m_k^2 - t_{ai} - t_{bj})}{2t_{ai}} \right] - \right. \\ &\quad \left. - x'_1 \sqrt{\left[s_{ik} - t_{bj} - p_a^2 + \frac{(m_i^2 - p_a^2 - t_{bj})(m_k^2 - t_{ai} - t_{bj})}{2t_{ai}} \right]^2 - \frac{\lambda(t_{ai}, p_a^2, m_i^2) \lambda(t_{ai}, t_{bj}, m_k^2)}{(-2t_{ai})^2}} \right\} \quad (5.A.4) \\ &= x'_3 \cdot \frac{m_i^2 - p_a^2}{\sqrt{-4t_{ai}}} \end{aligned}$$

we realize that if p_a^2 is such that $\lambda(t_{ai}, p_a^2, m_i^2) < 0$, then the curly bracket is always non-negative by virtue of locality and the presence of the step function in Eq. (5.A.2). Moreover, if we take s_{ik} complex in $\text{Im } s_{ik} > 0$, then necessarily $\text{Im} \left[x' \frac{1}{2} (p_a + p_i)' \right] \geq 0$. Consequently, for fixed t_{ai} , t_{bj} , ω and s_{jk} and $\lambda(t_{ai}, p_a^2, m_i^2) < 0$, Eq. (5.A.2) defines an analytic function of s_{ik} in $\text{Im } s_{ik} > 0$ which coincides with the unphysical production amplitude T_{jk-} as $\text{Im } s_{ik} = \epsilon \rightarrow 0^+$. Upon recalling Eq. (2.C.9), we see that when t_{ai} , t_{bj} , ω , s_{jk} and p_a^2 are taken as indicated above, Eq. (5.A.2) also defines an analytic function of s_{ab} in $\text{Im } s_{ab} > 0$. Thus, if we assume that $T_{jk-}(s_{ik}, p_a^2)$ vanishes as $s_{ik} \rightarrow \infty$ in $\text{Im } s_{ik} > 0$ along any real or complex direction, we are able to obtain by standard methods the dispersion relation for it

$$T_{jk-}(s_{ik}, p_a^2) = \frac{1}{\pi} \int_{-\infty}^{+\infty} ds'_{ik} \frac{\text{Im } T_{jk-}(s'_{ik}, p_a^2)}{s'_{ik} - s_{ik}}, \quad \begin{matrix} \lambda(t_{ai}, p_a^2, m_i^2) < 0 \\ \text{Im } s_{ik} > 0 \end{matrix} \quad (5.A.5)$$

and so on when subtractions become necessary. As in the case of the four-point function, the remaining (and difficult) task is to see what happens when p_a^2 increases from the region where $\lambda(t_{ai}, p_a^2, m_i^2) < 0$ up to $p_a^2 = m_a^2$, the physical mass of particle a . Before undertaking that task, some further remarks are worth to be made.

Let us see what are the crossing properties of T_{jk-} , which will be also called $T(a+b \rightarrow i+(j+k)_-)$ for a while. For s_{jk} , t_{ai} , t_{bj} , ω , and p_a^2 fixed as indicated above, let us continue $T(a+b \rightarrow i+(j+k)_-)$ from the physical region for $a+b \rightarrow i+(j+k)_-$, that is, from $s_{ik} + i\epsilon$ where $\epsilon > 0$, $s_{ik} > 0$ ($\text{Re} [\frac{1}{2}(p_a + p_i)']^0 > 0$, $\text{Im} [\frac{1}{2}(p_a + p_i)']^0 > 0$) down to the physical region for $\bar{i} + b \rightarrow \bar{a} + (j+k)_-$, that is, to $s'_{ik} + i\epsilon$, where $s'_{ik} < 0$, $\epsilon > 0$, in such a way that the continuation path lies in $\text{Im } s_{ik} > 0$ (see Fig. 8). Then, if, for simplicity, we assume that the field operators for particles a and i are the same as for their antiparticles $\bar{a}$, $\bar{i}$ respectively (for instance, if a and i are π^0 's), using PT invariance, not worrying about the normalization factor and calling $T(\bar{i} + b \rightarrow \bar{a} + (j+k)_-)$ the production amplitude corresponding to $\bar{i} + b \rightarrow \bar{a} + (j+k)_-$, we get easily

$$[T(a+b \rightarrow i+(j+k)_-)]_{\text{continued}}^* = i f^{ai} \int d^4x' \exp[i x' \frac{1}{2}(\hat{p}_a + \hat{p}_i)'] \theta(x'^0).$$

$$\cdot \langle p_j p_k - | [J_a(\frac{x'}{2}), J_i(-\frac{x'}{2})] | p_b \rangle \equiv T(\bar{i} + b \rightarrow \bar{a} + (j+k)_-)$$

$$\frac{1}{2}(\hat{p}_a + \hat{p}_i)' = [-\frac{1}{2}(p_a + p_i)']^* \quad , \quad \text{Re} [\frac{1}{2}(\hat{p}_a + \hat{p}_i)']^0 > 0 \quad , \quad \text{Im} [\frac{1}{2}(\hat{p}_a + \hat{p}_i)']^0 > 0 \quad (5.A.6)$$

which is the desired crossing relation (see Fig. 8), and is also the direct generalization of the crossing relations for four-particle processes.

All developments presented in this section up to here imply the validity of the dispersion relation of Eq. (5.A.5) for the process $\gamma + b \rightarrow i + (j+k)_-$ where b , i , j , k are arbitrary spinless hadrons and γ is an off-shell spinless photon

(coming, for instance, from a scattered electron as shown in Fig. 9), whose mass p_i^2 satisfies $\lambda(t_{ri}, p_i^2, m_i^2) < 0$, no further analytic continuation in p_i^2 being needed in order to get a physical dispersion relation. The generalization of this consequence to spin particles offers no difficulty, in principle. One may ask at this point why have we used the BCP variables ¹⁰⁾ summarized in Subsection 2.C for the study of analyticity properties of T_{jk-} in energies, and not other sets. The reasons are the following: (i) after their introduction ¹⁰⁾, those variables (or slight modifications of them) have been widely used in many studies of production amplitudes; (ii) they have simplified enormously the task of obtaining the unphysical dispersion relation [Eq. (5.A.5)], by virtue of the structure of Eqs. (2.C.1)-(2.C.5), and, moreover, the extension of the arguments leading to Eq. (5.A.5) [but not to the crossing property in Eq. (5.A.6)] to the full production amplitude and to processes with four or more particles in the final state is straightforward; (iii) they are very well suited to study the asymptotic properties of T_{jk-} as we shall see below; (iv) they are appropriate to study dispersion relations when only particle a can be put off-shell, but not particle i [due to the dependence of the weight function in Eq. (4.C.4) on m_i].

Let us assume for simplicity that for fixed p_a^2 , s_{jk} , t_{ai} , t_{bj} and ω and $\lambda(t_{ai}, p_a^2, m_i^2) < 0$, one has

$$\text{Im } T_{jk-}(s'_{ik}, p_a^2) \sim \beta_{\pm} |s'_{ik}|^{\alpha} [\ln |s'_{ik}|]^{\gamma} \quad (5.A.7)$$

$$s'_{ik} \rightarrow \pm \infty$$

$\beta_{\pm}$, α and γ being, in principle, real functions of p_a^2 , s_{jk} , t_{ai} , t_{bj} , ω and $0 > \alpha > -1$. Then, it is not difficult to see that as $s_{ik} \rightarrow \infty$ along any real or complex direction in $\text{Im } s_{ik} > 0$, Ref. 18)

$$T_{jk-}(s_{ik}, p_a^2) \sim -s_{ik}^{\alpha} [\ln |s_{ik}|]^{\gamma} \left\{ \beta_{+} - e^{i\pi\alpha} \beta_{-} \right\} \quad (5.A.8)$$

which is precisely a general Regge pole and cut behaviour if one assumes furthermore that α and γ are independent on t_{bj} , s_{jk} and ω , Ref. 10). Notice that since Eq. (5.A.8) implies

$$\frac{\lim_{s_{ik} \rightarrow +\infty + i\epsilon} |T_{jk-}(s_{ik}, p_a^2)|}{\lim_{s_{ik} \rightarrow -\infty + i\epsilon} |T_{jk-}(s_{ik}, p_a^2)|} = 1 \quad (5.A.9)$$

using the crossing relation of Eq. (5.A.6) and changing notation in the numerator of Eq. (5.A.9), we arrive immediately at a Pomeranchuk-like result when particle a is off-shell

$$\lim_{s_{ab} \rightarrow +\infty + i\epsilon} \frac{|T(a+b \rightarrow i+(j+k)_-, s_{ab}, p_a^2)|}{\lim_{t_{bi} \rightarrow +\infty + i\epsilon} |T(\bar{i}+b \rightarrow \bar{a}+(j+k)_-, t_{bi}, p_a^2)|} = 1 \quad (5.A.10)$$

As indicated above, by using Eq. (3.B.1) instead of Eq. (3.B.4), one can derive a dispersion relation in s_{ab} or s_{ik} for the conventional amplitude, T , for $\lambda(p_a^2, t_{ai}, m_i^2) < 0$ and the same fixed variables. On the other hand, the simple crossing property in Eq. (5.A.6) is not true for T . In fact, by doing similar operations with Eq. (3.B.1) as the ones leading to Eq. (5.A.6), we see that $[T(a+b \rightarrow i+j+k)_{\text{continued}}]^*$ is given by the same integral as in Eq. (5.A.6), but now $|p_j p_k \rangle$ is replaced by $|p_j p_k \text{ in} \rangle$. Since $|p_j p_k \text{ in} \rangle \neq |p_j p_k \text{ out} \rangle$ in general, we conclude that $[T(a+b \rightarrow i+j+k)_{\text{continued}}]^* \neq T(\bar{i}+b \rightarrow \bar{a}+j+k)$. This is another advantage of the amplitudes T_{jk}^- over the conventional one, T . We remark that the above inequality holds for all s_{ab} and, hence, it is in general incompatible with the asymptotic equality ($[T(a+b \rightarrow i+j+k)_{\text{continued}}]^* = T(\bar{i}+b \rightarrow \bar{a}+j+k)$ as $s_{ab} \rightarrow \infty$) investigated in Ref. 19). Notice that: (i) by using $\langle p_j p_k \text{ in} | = \langle p_j p_k \text{ out} | S^+$, $[T_{\text{continued}}]^*$ can be expressed for $(m_j+m_k)^2 \leq s_{jk} \leq m_{jk,\text{in}}^2$ as:

$$[T(a+b \rightarrow i+j+k, \bar{n})_{\text{continued}}]^* = \int d\Omega(\bar{n}') \left\{ \sum_{\ell, m} Y_{\ell}^m(\bar{n}) Y_{\ell}^m(\bar{n}')^* \exp[-2i\delta_{\ell}(s_{jk})] \right\} \cdot T(\bar{i}+b \rightarrow \bar{a}+j+k, \bar{n}') \quad (5.A.11)$$

(ii) a Pomeranchuk-like relation still holds for suitably integrated (off-shell) production differential cross-sections: in fact, by summing over all allowed p_j, p_k and using the completeness of the states $|p_j p_k \text{ out} \rangle$, we get for any $s_{ab}, t_{ai}, (m_j+m_k)^2 \leq s_{jk} \leq m_{jk,\text{in}}^2, \lambda(t_{ai}, p_a^2, m_i^2) < 0$:

$$\sum_{p_j p_k} \delta^{(4)}(p_{\bar{a}} + p_j + p_k - p_i - p_e) |T(a+b \rightarrow i+j+k)_{\text{continued}}|^2 = \sum_{p_j p_k} \delta^{(4)}(p_{\bar{a}} + p_j + p_k - p_i - p_e) \times |T(\bar{i}+b \rightarrow \bar{a}+j+k)|^2$$

and by generalizing Eq. (5.A.9) for T as $s_{ab} \rightarrow \infty$ in $\text{Im } s_{ab} > 0$ at fixed $t_{ai}, t_{bj}, \omega, s_{jk}$ and p_a^2 , we obtain at fixed t_{ai}, s_{jk}, p_a^2 :

$$\lim_{\substack{\sum_{j,k} p_j p_k, s_{ab} \rightarrow \infty \\ \sum_{j,k} p_j p_k, t_{ai} \rightarrow \infty}} \frac{\delta^4(p_i + p_j + p_k - p_a - p_b) |T(a+b \rightarrow i+j+k)|^2}{\delta^4(p_a + p_j + p_k - p_i - p_b) |T(\bar{i}+b \rightarrow \bar{a}+j+k)|^2} = 1 \quad (5.A.12)$$

It is easy to obtain dispersion relations in s_{ab} for the partial waves $T_{jk}^-, \ell m(s_{ab}, s_{jk}, t_{ai})$ when particle a is off-shell. In fact, restricting ourselves for simplicity to $\ell = m = 0$ and using Eqs. (4.E.1), (3.B.4) and (3.B.6) we have in O_{jk}^i

$$T_{jk}^-, \ell m = i \int d^4x \exp\left\{\frac{i}{2}x(p_a + p_i)\right\} \Theta(x^0) \langle s_{jk}, 00 | [J_i(\frac{x}{2}), J_a(-\frac{x}{2})] | p_b \rangle \quad (5.A.13)$$

Upon performing: (i) a rotation in the z - x plane in O_{jk}^i so that $\bar{p}_b$ lies along the z axis (the "composite" particle $|s_{jk}, 00\rangle$ being still at rest); (ii) a boost along the z axis to a new frame, O_1^{BCP} , so that $\bar{p}_b$ and the three-momentum of the composite particle bear respectively the same form as in Eqs. (2.C.1), t_{bj} and m_j^2 being replaced respectively by t_{ai} and s_{jk} , we see that $\frac{1}{2}(p_a + p_i)$ in Eq. (5.A.13) becomes $\frac{1}{2}(p_a + p_i)'$ as given by Eq. (5.A.3), with a new $\cosh 2\mathcal{B}$ which is related to s_{ab} and the other invariants by

$$s_{ab} = \frac{\lambda(t_{ai}, m_b^2, s_{jk}) + \lambda(t_{ai}, m_a^2, m_i^2) - (m_i^2 - m_a^2 + m_b^2 - s_{jk})^2}{-4t_{ai}} + 2 \sqrt{\frac{\lambda(t_{ai}, m_a^2, m_i^2) \lambda(t_{ai}, m_b^2, s_{jk})}{(-4t_{ai})^2}} \cosh 2\mathcal{B} \quad (5.A.14)$$

Then, by repeating the arguments about Eq. (5.A.4) [but using now Eq. (5.A.14)], we conclude that in the new frame O_1^{BCP} $T_{jk}^-, 00$ is analytic in s_{ab} in $\text{Im } s_{ab} > 0$ for fixed $t_{ai} (< 0)$ and s_{jk} when $\lambda(t_{ai}, p_a^2, m_i^2) < 0$ with crossing properties as in Fig. 7. A dispersion relation for $T_{jk}^-, 00$ in s_{ab} similar to Eq. (5.A.5) can be written readily. A representation for $\text{Abs } T_{jk}^-, 00$ can be obtained by partial-wave analysis of Eq. (4.C.3). The generalization of all this to $\ell \neq 0$, $m \neq 0$ is obvious but somewhat troublesome, as one needs to take into account the transformation under rotations and boosts of the "composite" particle states and it will not be given here. Moreover, we shall not pursue all this further, as the troubles to be found with T_{jk}^- in the next subsection will be also met by $T_{jk}^-, \ell m$.

5.B ANALYTIC CONTINUATION IN THE MASS AND RELATED DIFFICULTIES

Next, we shall study what happens to the dispersion relation Eq. (5.A.5) when p_a^2 varies from the region $\lambda(t_{ai}, p_a^2, m_i^2) < 0$ up to $p_a^2 = m_a^2$. In Subsection 4.C we concluded that for fixed s_{ab} , s_{jk} , the analyticity domain of $\text{Abs } T_{jk}^-$ in z , z_i , $\cos \Psi_i$ is a bounded region about the physical region when all particles are on-shell. Now, we shall take t_{ai} , t_{bj} , ω and s_{jk} fixed at physical values $((m_j+m_k)^2 \leq s_{jk} < m_{jk, \text{in}}^2)$, keeping $p_a^2 = m_a^2$ for simplicity and we let s_{ab} to decrease from $s_{ab} = +\infty$ towards the smallest mass of any intermediate state $|a \text{ out} \rangle$ in Eq. (4.C.3) which is the minimum of M_{ab} , M_{ba} , that is, we follow the curve C discussed in Subsection 2.E (see Fig. 7). Let us have in mind $z = -\cos \Theta_{ai}$, which is given by Eq. (2.A.3) for $h = i$, and Eqs. (4.A.11) and (4.C.9), giving the sizes of the analyticity regions. Let us consider first the case where that smallest mass is such that $m_a+m_b \leq \text{Min}(M_{ab}, M_{ba}) \leq m_i+m_j+m_k$. Then, we see that as $s_{ab} \rightarrow (m_i + \sqrt{s_{jk}})^2$ ($> (m_i+m_j+m_k)^2 > (m_a+m_b)^2$), the parameters $L_{1, \text{min}}$ and $L_{2, \text{min}}$ determining the size of the analyticity domains remain finite, while z increases towards ∞ , thus going outside the analyticity domain necessarily.

On the other hand, if $m_a+m_b > m_i+m_j+m_k > \text{Min}(M_{ab}, M_{ba})$ or $m_a+m_b > \text{Min}(M_{ba}, M_{ab}) > m_i+m_j+m_k$ then, as s_{ab} tends to $(m_a+m_b)^2$, and becomes a bit smaller than that, the "minimal" ellipse for z considered in Eq. (4.C.12) also collapses completely, so that the necessary condition a) is never fulfilled and, consequently, the analyticity domain of $\text{Abs } T_{jk}^-$ disappears completely. So, we see that while for sufficiently large values of s_{ab} $\text{Abs } T_{jk}^-$ is nicely well defined, diseases appear inevitably as s_{ab} approaches the thresholds and $\text{Abs } T_{jk}^-$ ceases to be mathematically well-defined by Eq. (4.C.6) when all particles are on-shell. Obviously, the same situation occurs when particle a becomes off-shell, that is, p_a^2 decreases from $p_a^2 = m_a^2$ towards $\lambda(t_{ai}, p_a^2, m_i^2) < 0$. Notice that the only dependence of $\text{Abs } T_{jk}^-$ in p_a^2 is contained in the denominator and in the "size" $L_{2, \text{min}}$ given in Eq. (4.A.11) [when m_a^2 is replaced by p_a^2]. The latter is well defined and larger than unity as p_a^2 varies from $\lambda(t_{ai}, p_a^2, m_i^2) < 0$ up to $p_a^2 = m_a^2$ for fixed s_{ab} , t_{ai} , as in the case of the four-point function. It is not difficult, in principle, to obtain the minimum value of s_{ab} , s_{ab}^{min} , such that for any $s_{ab} > s_{ab}^{\text{min}}$ $\text{Im } T_{jk}^- = \text{Abs } T_{jk}^-$ is well defined mathematically [that is, neither D nor $\tilde{D}$ in Eqs. (4.C.7), (4.C.11) vanish] when p_a^2 varies in the above range, for fixed physical values of t_{ai} , t_{bj} , s_{jk} and ω . One only needs replacing $\cos \Psi_i$, z_i , z in terms of s_{ab} , s_{jk} , t_{ai} , t_{bj} , ω (as indicated at the end of Subsection 2.C) into the expressions for D or $\tilde{D}$ [Eqs. (4.C.7) and (4.C.11)] and, for fixed real and physical values of t_{ai} , t_{bj} , s_{jk} and ω , to seek s_{ab}^{min} so that D or $\tilde{D}$ just vanish as p_a^2 varies as indicated, when L_1 , L_2 , β_1 , β_2 and ω vary in their respective integration ranges. Notice that one may restrict oneself to real values of all variables. Having indicated how the calculation would be done, we shall not go through it, as it is very complicated algebraically. Notice that $s_{ab}^{\text{min}} > \text{Min}(M_{ab}, M_{ba})$.

The same situation occurs whenever s_{ab} increases from $s_{ab} = -\infty$ along the curve C considered in Subsection 2.C. Then, $\text{Im } T_{jk}^- = \text{Abs } T_{jk}^-$ is given by the expression in Eq. (4.C.1) which no longer vanishes while the one given in Eq. (4.C.2) do vanish. The new expression for $\text{Abs } T_{jk}^-$, which obviously refer to the channel $\bar{l}+b \rightarrow \bar{a}+2+3$, has similar analyticity properties in three angular variables, and one meets a similar phenomenon: it is nicely well defined mathematically as p_a^2 varies from $\lambda(t_{ai}, p_{ai}^2, m_i^2) < 0$ up to $p_a^2 = m_a^2$ in $-\infty < s_{ab} < s_{ab}^{\text{min, crossed}}$ (where the minimum $s_{ab}^{\text{min, crossed}}$ would be calculated similarly to s_{ab}^{min}), but as we approach the corresponding thresholds, it ceases to be well defined mathematically. Of course, all what we have said applies as well if we use s_{ik} instead of s_{ab} . Summarizing: the analytic continuation in p_a^2 of $\text{Im } T_{jk}^-(s_{ab}^i, p_a^2)$ in the off-shell dispersion relation [Eq. (5.A.5)] by means of the representation studied in Subsection 4.C has meaning only for $-\infty < s_{ab} < s_{ab}^{\text{min, crossed}}$ and $s_{ab}^{\text{min}} < s_{ab} < +\infty$, but not for $s_{ab}^{\text{min, crossed}} < s_{ab} < s_{ab}^{\text{min}}$. Consequently, we cannot state that Eq. (5.A.5) is valid for $p_a^2 = m_a^2$. Moreover, it seems that complex cuts are present in the s_{ab} complex plane for $p_a^2 = m_a^2$, as we indicate below.

Let us see what happens in $s_{ab}^{\text{min, crossed}} < s_{ab} < s_{ab}^{\text{min}}$ and the relation of this problem to some findings in pure S matrix theory. For simplicity, let us restrict ourselves to values of the fixed variables so that the following situation is met: $(m_a + m_b)^2 \leq \text{Min}(M_{ba}, M_{ab}) \leq s_{ab}^{\text{min}} \leq [\text{squared invariant mass of any } |\alpha \text{ out} \rangle \text{ with three or more particles in Eq. (4.C.3)}]$ and a corresponding series of inequalities for $s_{ab}^{\text{min, crossed}}$, Eq. (4.C.1) and channel $\bar{l}+b \rightarrow \bar{a}+j+k$.

Then, by operating formally, it is not difficult to obtain from Eq. (4.B.1), and using some results of Subsections 3.A and 3.B, the following extended unitarity equation, for $s_{ab}^{\text{min, crossed}} \leq s_{ab} \leq s_{ab}^{\text{min}}$ and $(m_j + m_k)^2 \leq s_{jk} < m_{jk, \text{in}}^2$

$$\text{Abs } T_{jk}^- = \frac{1}{4(2\pi)^4} \cdot \frac{|\vec{p}_j| p_j^0 p_k^0}{\sqrt{s_{jk}}} \cdot \frac{|\vec{p}_a| p_a^0 p_b^0}{\sqrt{s_{ab}}} \cdot \int d\Omega(\vec{n}_\beta, \vec{q}_{jk}^i) d\Omega(\vec{n}_\alpha, \vec{q}_{ab}^i) T_{\alpha, jk \rightarrow \beta}(\vec{n}_\beta, \vec{n}_\alpha)$$

$$T_{ab \rightarrow 123}(\vec{n}_\alpha, \vec{n}_\beta)^* T_{\alpha, ab \rightarrow \beta}(\vec{n}_\alpha, \vec{n}_\beta) \quad - [\text{crossed term, coming from (4.C.1)}] \quad (5.B.1)$$

where j, k, β refer to the frame O_{jk}^i and a, b, α to O_{ab}^i . The subscripts α, β indicate merely the connection with Eq. (4.C.3), no summation over them being implied.

Notice that in some cases one or both terms vanish while in others both could contribute. Furthermore, we are assuming that, if necessary for the above equation to hold, all particles $a, b, 1, 2, 3$ are off-shell [actually, Eq. (4.C.3), for instance, allows one to put particles a, b, j, k off-shell only, but, upon applying the reduction technique to $\langle \beta | J_i(0) | \alpha \rangle$, one may also explicit the dependence on p_i in terms of retarded commutators and, hence, one may put particle i off mass shell, at least formally]. Notice that by using Eq. (3.B.5), one may replace the standard amplitude T by the new amplitudes T_{jk} into the above equation, thus getting integral equations for the latter. We stress that Eq. (5.B.1) has only formal sense, in principle. If: (i) one inserts suitable partial wave expansions for the elastic four-particle amplitudes and expressions of the type $1/(t_{ai}-m^2)$ for T into Eq. (5.B.1); (ii) one does the appropriate integration over angles in Eq. (5.B.1); (iii) one makes continuations in external masses so that all external particles be on-shell at the end for fixed s_{jk} , then one realizes how Eq. (5.B.1) forces one not to restrict oneself to $s_{ab}^{\text{min, crossed}} < s_{ab} < s_{ab}^{\text{min}}$ but, moreover, requires to be continued for complex s_{ab} up to some end point in the s_{ab} plane, whose position depends on the fixed invariants. We shall not insist on this, as similar considerations have been done for the discontinuity of the standard amplitude in the case of triangle graphs in pure S matrix theory by Cook and Lee and Hwa⁶⁾, for instance. This is, in fact, the same mechanism which produces complex circular cuts to the partial waves in πN elastic scattering when the Mandelstam representation is assumed for the invariant amplitudes²⁰⁾.

At this point, a brief description of Mohan's work⁴⁾ and of the relation between it and the present one are in order: Mohan starts from an amplitude quite similar to our T_{jk} , chooses a set of variables different from ours, establishes some off-shell dispersion relation, finds the absorptive part of his amplitude and does some calculation (with some parts of which we disagree) in order to establish an on-shell dispersion relation. However, he stops before arriving at the analogue of our Eq. (4.C.6) (which is the basis of several analytic continuations studied here) and he does not establish any analyticity property in angles for either his amplitude or its absorptive part. On the other hand, it is not difficult to realize that his choice of variables amounts to fixing s_{ik} , s_{jk} , t_{ai} and t_{bj} , his dispersive variable being s_{ab} . In this case, only a finite interval in s_{ab} can be physical (as indicated in Subsection 2.C) and, consequently, Mohan's dispersion relation in s_{ab} contains only a finite physical interval of integration. Finally, we believe that one finds in his case the same trouble as one faces in ours, namely, when $s_{ab} \rightarrow (m_i + \sqrt{s_{jk}})^2$.

It has been proved for several processes to all orders of perturbation theory that any contribution to the standard amplitude T coming from any Feynman diagram is analytic in a certain domain of real values for the kinematic invariants, bounded by surfaces corresponding to normal thresholds and loop diagrams⁹⁾. Using Section 7 of Wu's paper⁹⁾, for instance, it is not difficult to see that such a domain (really, its projection) is contained inside the tetrahedron of Fig. 4, at least

in the equal-mass case (see Fig. 7). This fact completes the understanding of the different regions in the plot treated in Subsection 2.E and explains why Eq. (5.A.5) breaks down for $p_a^2 = m_a^2$: the line C considered in Subsections 2.C., 2.E and 5.A [along which one integrates in Eq. (5.A.5)] does not intersect Wu's region, when it goes from the physical region for $a+b \rightarrow i+j+k$ to the one for $\bar{i}+\bar{b} \rightarrow \bar{a}+j+k$ (see Fig. 7). However, it is not perhaps unreasonable to conjecture that all complex singularities of T_{jk}^- in the complex s_{ab} or s_{ik} planes for fixed t_{ai} , t_{bj} , s_{jk} , ω and $p_a^2 = m_a^2$ are contained in a finite and bounded region, so that T_{jk}^- is analytic outside it, except for the right- and left-hand cuts (at least, when appropriate stability conditions are met). In that case, Eq. (5.A.5) would be replaced by the following, for $+\infty > s_{ab} > s_{ab}^{\min}$ or $-\infty < s_{ab} < s_{ab}^{\min, \text{crossed}}$

$$\begin{aligned} \text{Re } T_{jk}^-(s_{ab} + i\epsilon, p_a^2 = m_a^2) = & \frac{1}{\pi} P \left[\int_{s_{ab}^{\min}}^{+\infty} + \int_{-\infty}^{s_{ab}^{\min, \text{crossed}}} \right] \frac{ds'_{ab} \text{Im } T_{jk}^-(s'_{ab}, m_a^2)}{s'_{ab} - s_{ab}} + \\ & + \text{Re} \left\{ \frac{1}{\pi i} \int_{\Lambda} \frac{ds'_{ab} T_{jk}^-(s'_{ab}, p_a^2 = m_a^2)}{s'_{ab} - s_{ab}} \right\} \end{aligned} \quad (5.B.2)$$

where the complex contour Λ starts at $s_{ab} = s_{ab}^{\min, \text{crossed}}$ ends at $s_{ab} = s_{ab}^{\min}$ and bounds the finite region in $\text{Im } s_{ab} > 0$ which contains the complex cuts^{*)}. Notice that the crossing behaviour in Eq. (5.A.6), the asymptotic behaviour, etc., would remain valid for $p_a^2 = m_a^2$ and, moreover, the contribution of the complex contour Λ would be small at sufficiently large s_{ab} . The contribution of the complex cuts should be estimated from the analytically continued extended unitarity equation which relates four- and five-particle amplitudes. We remark that this situation would be similar to the one rigorously proved for four-particle processes by Bros, Epstein and Glaser²¹⁾. However, in the five-particle case it is only a conjecture and, hence, it may be untrue.

A similar conjecture can be cheaply made concerning the analyticity properties in the s_{ab} plane of the partial waves T_{jk}^-, ℓ_m when all particles are on-shell, and t_{ai} , s_{jk} are fixed. Related problems in the case of non-relativistic three-particle scattering as well as applications to $\pi N \rightarrow \pi \pi N$ are under study.

*) We have checked this conjecture with some triangle graphs, using Hwa's equations, Ref. 6). It turns out to be fulfilled when stability conditions are met at the vertices.

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FIGURE CAPTIONS

Figure 1 : Particle four-momenta and some kinematic invariants for the process $a+b \rightarrow 1+2+3$ ($p_a + p_b = p_1 + p_2 + p_3$).

Figure 2 : Geometry in the frame O_{ab}^{xy} . The final three-momenta lie in the x-y plane.

Figure 3 : Geometry in the frames O_{ab}^i and O_{jk}^i . In O_{ab}^i , $\bar{p}_a$ is taken in the z-x plane. Since the Lorentz transformation leading from O_{ab}^i to O_{jk}^i is performed along the z axis, in O_{jk}^i $\bar{p}_a$ is also contained in the z-x plane. Notice that the components of $\bar{p}_a / |\bar{p}_a|$ in this figure and in Fig. 2 are different, as O_{ab}^i is rotated with respect to O_{ab}^{xy} .

Figure 4 : Tetrahedral co-ordinates. The co-ordinates of the vertex $v(s_{ab})$ are $s_{ab} = +\sum$, $t_{a1} = t_{a2} = t_{a3} = 0$. Inside the V region (s_{ab}), the ranges of the invariants are $s_{ab} > 0$, $t_{a1} < 0$, $t_{a2} < 0$, $t_{a3} < 0$.

Figure 5 : The V region $V(s_{ab})$ and the "physical" surface $\Gamma(s_{ab})$ corresponding to the vanishing of all transverse momenta.

Figure 6 : The S region $S(t_{a1}, s_{ab})$ and the "physical" surface $\Omega(t_{a1}, s_{ab})$ corresponding to the crossed process $a+b+\bar{1} \rightarrow 2+3$, when all particle three-momenta are parallel or antiparallel.

Figure 7 : The physical regions for $a+b \rightarrow 1+2+3$ and for $a+\bar{2}+\bar{3} \rightarrow \bar{b}+1$ (that is, for $b+\bar{1} \rightarrow \bar{a}+2+3$ also). The curve C misses the tetrahedron (and, hence, also Wu's region; see Subsection 5.B). The points 1 and 2 correspond to $s_{ab}^{(1)}$ and $s_{ab}^{(2)}$ respectively.

Figure 8 : The broken line indicates the continuation path when going from $a+b \rightarrow i+(j+k)_-$ to $\bar{i}+b \rightarrow \bar{a}+(j+k)_-$. To complete the crossing, that is, to go from (1) to (2), the true physical region for $\bar{i}+b \rightarrow \bar{a}+(j+k)_-$, a complex conjugation [the one in Eq. (5.A.6)] is also necessary.

Figure 9 : For this process, the dispersion relation of Eq. (5.A.5) is valid when $\lambda(t_{\mathbf{r}i}, p_{\mathbf{r}}^2, m_i^2) < 0$, $(m_j + m_k)^2 \leq s_{jk} < m_{jk, in}^2$.

$i, j, k = 1, 2, 3$
 $i \neq j \neq k, i \neq k$

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6

FIG. 7

FIG.8

FIG. 9