Origin of the Immirzi Parameter

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Abstract

Using quadratic spinor techniques we demonstrate that the Immirzi parameter can be expressed as ratio between scalar and pseudo-scalar contributions in the theory and can be interpreted as a measure of how Einstein gravity differs from a generally constructed covariant theory for gravity. This interpretation is independent of how gravity is quantized. One of the important advantage of deriving the Immirzi parameter using the quadratic spinor techniques is to allow the introduction of renormalization scale associated with the Immirzi parameter through the expectation value of the spinor field upon quantization.
I. INTRODUCTION

One of the most direct ways of approaching the quantization of Einstein’s theory of gravity is to put it into a Hamiltonian form and then try to apply the procedures of canonical quantization. The fact that Einstein’s theory is generally covariant makes the task one of most difficult problems in theoretical physics if not impossible. The complicated non-polynomial structure found for the standard Hamiltonian for general relativity raises another challenge to researchers. However, Ashtekar managed to make progress using new canonical variables to reduce the constraints to polynomial form. The so called Ashtekar connection \[ A = \Gamma + iK \] has a part \( \Gamma \) refers to intrinsic curvature on a spacelike 3-surface \( S \), and another part \( K \) that refers to the extrinsic curvature to the spacetime \( M \). Because of several technical difficulties, in order to make progress, a more general Barbero connection \[ A = \Gamma + \gamma K \] was introduced, where \( \gamma \) is an arbitrary complex number. It is also known as the Immirzi parameter \[ \gamma \] (usually assumed to be a real number for \( SU(2) \) Barbero connection).

The important achievement of quantizing the Ashtekar-Barbero connection variable is the construction of a kinematic Hilbert space using spin networks. With spin networks, the area and volume spectra can be derived. If a spin network intersects a surface \( S \) transversely, then this surface has a definite area in this state, given as a sum over the spins \( j \) of the edges poking through \( S \):

\[
\text{Area}(S) = 8\pi \gamma \sum_j \sqrt{j(j+1)}
\]

in units where the \( \hbar = c = G = 1 \), with a free Immirzi parameter \( \gamma \). Due to the presence of the Immirzi parameter, the famous Bekenstein-Hawking entropy formula, \( S_{BH} = \frac{A}{4} \) could not be uniquely determined. This has been viewed as the main unsatisfactory point of this approach for some years.

Recently Dreyer proposed a way to fix the Immirzi parameter using asymptotic behavior of the quasinormal modes of a Schwarzschild black hole. The result fixed the value \( \gamma = \frac{\ln 3}{2\sqrt{2\pi}} \) with the lowest possible spin \( j_{\text{min}} = 1 \). Domagala, Lewandowski and Meissner fixed an incorrect assumption that only the minimal value of the spin contributes. Their result involves the logarithm of a transcendental number instead of the logarithms of integers; \( \gamma = 0.2375329...(> \frac{\ln 2}{\pi}) \) with \( j_{\text{min}} = \frac{1}{2} \). The calculation works for charged and rotating black holes and black holes coupled to a dilaton field, with the same value of \( \gamma \). There appears to be no clear geometrical reason for a particular choice of the real number value \( \gamma \).
and obscures its physical interpretation.

The appearance of the Immirzi parameter $\gamma$ can be seen in the simplest tetrad-Palatini action of general relativity where one can add an additional term with coupling coefficient $\gamma$. This newly added term does not affect the equations of motion. In the case where torsion free connection that solves the equation of motion is employed to obtain the Einstein-Hilbert gravity action, the additional term in the action becomes identically zero. Arguing with this observation in mind, the effect of $\gamma$ is therefore, not an physical observable in Einstein gravity. Recently Perez and Rovelli [7] had argued that one can observe physical effects of the Immirzi parameter $\gamma$ by coupling Einstein gravity to fermionic degrees of freedom. The presence of matter field induces a torsion term in the connection and the additional term becomes non-vanishing. Freidel, Minic and Takeuchi [8] discussed parity violation and studied the coupling of fermionic degrees of freedom in the presence of torsion from the viewpoint of effective field theory. The importance of these works are to notice that physical effects arise from the Immirzi parameter $\gamma$ is measurable and independent from how gravity is quantized. We believe however, if the Immirzi parameter $\gamma$ is a physical property of the gravity sector, then it should be observable without the introduction of other matter field. In the following, we shall introduce a Quadratic Spinor Representation of General Relativity [9, 10, 11] formalism where the physical meanings and effects of the Immirzi parameter $\gamma$ become transparent in general relativity. In this formalism, the Immirzi parameter becomes a ratio between scalar and pseudo-scalar contributions in the theory and measures how a generally formulated general theory of relativity differs from Einstein gravity. More importantly, one can acquire this ratio a renormalization scale upon quantization.

II. QUADRATIC SPINOR REPRESENTATION OF GENERAL RELATIVITY

The canonical formulation of Loop Quantum Gravity can be derived by the Holst action [12],

$$S[\vartheta, \omega] = \alpha \int \ast(\vartheta^a \wedge \vartheta^b) \wedge R_{ab}(\omega) + \beta \int \vartheta^a \wedge \vartheta^b \wedge R_{ab}(\omega),$$

(1)

where the Immirzi parameter is,

$$\gamma = \frac{\alpha}{\beta}.$$  

(2)

In the above, $a, b... = 0, 1, 2, 3$ being the internal indices of the internal orthonormal frame. The field $\vartheta^a$ being the tetrad field; $\omega$ is the $SU(2)$ connection; $R_{ab}$ being the curvature of $\omega$.\]
and \(*R\) being its dual. This is comparable with the Quadratic Spinor Lagrangian \([9, 10]\),

\[
\mathcal{L}_\psi = 2D(\bar{\psi}\vartheta)\gamma_5 D(\vartheta\psi),
\]

(3)

where \(\vartheta = \vartheta^a \gamma_a\) and \(\gamma_a\) being the Dirac gamma matrices. The auxiliary spinor field \(\psi\) in the Quadratic Spinor Lagrangian was first introduced by Witten \([13]\) as a convenient tool used in the proof of positive energy theorem in Einstein gravity. In a more general context, this auxiliary spinor field provides a nice gauge condition to pick up the relevant variables in the theory. The key to these successes is a “spinor-curvature identity”:

\[
2D(\bar{\psi}\vartheta)\gamma_5 D(\vartheta\psi) = \bar{\psi}\psi R_{ab} \wedge * (\vartheta^a \wedge \vartheta^b) + \bar{\psi}\gamma_5 \psi R_{ab} \wedge \vartheta^a \wedge \vartheta^b
\]

\[
+ d[D(\bar{\psi}\vartheta)\gamma_5 \vartheta\psi + \bar{\psi}\gamma_5 D(\vartheta\psi)]
\]

(4)

Note that in the above expression the boundary terms provide an important condition in obtaining a finite action at spatial infinity and consequently a well defined Hamiltonian. Here, the spinor field \(\psi\) plays a key role which allows one to pick up the correct gauges in obtaining a well defined Hamiltonian of the theory. The equation of motion for the connection \(\omega[\vartheta]\) is:

\[
D[\bar{\psi}\psi (\vartheta^a \wedge \vartheta^b) + \bar{\psi}\gamma_5 \psi (\vartheta^a \wedge \vartheta^b)] = 0,
\]

(5)

where \(D\) is the covariant derivative defined by the connection variable \(\omega[\vartheta]\). For \(\bar{\psi}\psi = 1\) and \(\bar{\psi}\gamma_5 \psi = 0\) the torsion free spin connection \(\omega[\vartheta]\) of the tetrad field \(\vartheta\) solves the above field equation. Therefore, if we set

\[
\gamma = \frac{\bar{\psi}\psi}{\bar{\psi}\gamma_5 \psi}
\]

(6)

then the choice of \(\gamma = \infty\) (\(\bar{\psi}\psi = 1\) and \(\bar{\psi}\gamma_5 \psi = 0\)), is the Einstein-Hilbert action in equation(1); and \(\gamma = i\) is the self-dual action in the Ashtekar canonical gravity framework, and \(\gamma = 1\) corresponds to the action for the Hamiltonian considered by Barbero \([3]\). The Immirzi parameter \(\gamma\) in this setting becomes a measure of how Einstein gravity differs from a most generally formulated gravitation theory which satisfies general coordinate covariance. It is also the ratio between scalar and pseudo-scalar contributions in the theory as can be seen from the explicit expression of \(\gamma\). Another important feature revealing in this derivation is the possibility of introducing a renormalization scale \(\mu\) associated with the Immirzi parameter \(\gamma\) upon quantization where expectation of \(<\bar{\psi}\psi>_{\mu}\) and \(<\bar{\psi}\gamma_5 \psi>_{\mu}\) at
some scale $\mu$ should be employed. Thus the “Quadratic Spinor Representation of General Relativity” provides a transparent interpretation of the Immirzi parameter.

A technical drawback of the above derivation is that $\psi_{\gamma 5} \psi$ is not in general a real function. This can be easily seen from using a particular representation of the Dirac algebra. In order for $\psi_{\gamma 5} \psi$ to be always real to render a corresponding real Ashtekar-Barbero variable, one has to use anti-commuting spinor. In the next section, we shall develop a model which provides a systematic way of obtaining a Quadratic Spinor Representation with an appropriate symmetry for the action where anti-commuting spinor arises naturally.

III. THE ORIGIN OF IMMIRZI PARAMETER

In this section, we derive the Einstein-Hilbert action in a more systematic way with anti-commuting spinors using $Osp(1, 2C)$ algebra, which is the simplest supersymmetric extension of $SL(2, C)$ (or $su(2)$) algebra. The algebra has bosonic generators $J_{00}, J_{01} = J_{10}, J_{11}$ and fermionic generators $Q_0, Q_1$ which satisfies the following algebra:

$$[J_{AB}, J_{CD}] = \epsilon_{C(A} J_{B)D} + \epsilon_{D(A} J_{B)C},$$
$$[J_{AB}, Q_C] = \epsilon_{C(A} Q_{B)},$$
$$\{Q_A, Q_B\} = J_{AB}.$$  \hfill (7)

Their complex conjugates $J_{A'B'}$ and $Q_{A'}$ satisfy the same algebra as above. The $SO(1, 3)$ generators $J_{ab}$ can be constructed by $J_{ab} = \sigma_a^{AA'} \sigma_b^{AA'} J_{AB} + \sigma_a^{AA'} \sigma_b^{A'B'} J_{A'B'}$ where $\sigma_a^{AA'}$ are the Pauli matrices. The $Osp(1, 2C)$ algebra has a nondegenerate Killing form and the Cartan-Killing metric $\eta_{\alpha\beta} = \text{diag}(\eta_{(AB)(MN)}, \eta_{AB})$ is given by

$$\eta_{(AB)(MN)} = \frac{1}{2}(\epsilon_{AM} \epsilon_{BN} + \epsilon_{AN} \epsilon_{BM}),$$
$$\eta_{AB} = -\epsilon_{AB}. \hfill (10)$$

Now, one can follow a well defined procedure to construct a $Osp(1, 2C)$ invariant Lagrangian which is quadratic in the spinor representation. To each generator $T_\alpha = \{J_{AB}, J_{A'B'}, Q_A, Q_{A'}\}$, we associate a 1-form field $A^\alpha = \{\omega^{AB}, \omega^{A'B'}, \varphi^A, \varphi^{A'}\}$, and construct a super Lie algebra valued connection 1-form,

$$A = A^\alpha T_\alpha = \omega^{AB} J_{AB} + \varphi^A Q_A + \text{c.c.}, \hfill (12)$$

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where $\omega^{AB}$ is the SL(2,C) connection 1-form and $\varphi^A$ is an anti-commuting spinor valued 1-form. The curvature is given by $F = dA + \frac{1}{2}[A,A] = dA + \frac{1}{2}A^\alpha \wedge A^\beta \otimes [T_\alpha, T_\beta]$. Given the $Osp(1,2C)$ connection $A$ defined in equation (12), the curvature $(F = F(J)^{AB}J_{AB} + F(Q)^A Q_A + c.c.)$ contains a bosonic part associated with $J_{AB}$ and $J_{A'B'}$,

$$F(J)^{AB} = d\omega^{AB} + \omega^{AC} \wedge \omega^B + \frac{1}{2} \varphi^A \wedge \varphi^B;$$ (13)

and a fermionic part associated with $Q_A$ and $Q_A'$,

$$F(Q)^A = d\varphi^A + \omega^{AB} \wedge \varphi^B.$$ (14)

The action, quadratic in the curvature, using this $Osp(1,2C)$ connection $A$ is

$$S_T[A^\alpha] = \int F^\alpha \wedge F^\beta \eta_{\alpha\beta}$$

$$= \int F(J)^{AB} \wedge F(J)_{AB} + F(Q)^A \wedge F(Q)_A + c.c.,$$ (15)

where $\eta_{pq}$ is the Cartan-Killing metric of the $Osp(1,2C)$ group. However, this action is a total differential and therefore, is a pure topological action without local dynamics. Hence, similar to the work of MacDowell and Mansouri [14], we break the topological field theory of this $Osp(1,2C)$ symmetry into its bosonic sector and fermionic sector. A way to do this is to choose $i_{\alpha\beta} = \text{diag}(i_{(AB)(MN)}, i_{AB})$ such that

$$i_{(AB)(MN)} = 0,$$ (16)

$$i_{AB} = -\epsilon_{AB}.$$ (17)

The new action is

$$S[A^\alpha] = \int F^\alpha \wedge F^\beta i_{\alpha\beta}$$

$$= \int F(Q)^A \wedge F(Q)_A + c.c.$$ (18)

The field equations can be obtained by varying the Lagrangian with respect to the gauge potentials—the $Osp(1,2C)$ connection. With these gauge potentials fixed at the boundary, the field equations are

$$D^2 \varphi^A = R^{AB} \wedge \varphi_B = 0,$$ (19)

$$D(\varphi^A \wedge \varphi^B) = 0.$$ (20)
plus their corresponding complex conjugate equations. We look for classical torsion free Einstein solution $R_{ab}$, by making the ansatz \[15\] such that

$$\varphi^A = \varphi^{AA'} \xi_{A'} \tag{21}$$

where $\varphi^{AA'}$ is the tetrad 1-form field (assuming to be real) and $\xi_{A'}$ is an arbitrary nonzero anti-commuting spinor field. With the ansatz, the second field equation \[20\] gives

$$D(\varphi^{AA'} \wedge \varphi^{B_A} \xi^{B'} \xi_{B'}) = 0, \tag{22}$$

which together with their complicated conjugate part, implies the connection is torsion free. Thus the field equation \[19\] becomes

$$R^{AB} \wedge \varphi^{B_A} \xi_{A'} = 0, \tag{23}$$

together with their complex conjugate equations

$$R^{A'B'} \wedge \varphi^{A'B'} \xi_A = 0; \tag{24}$$

we derive the Einstein equation

$$R^{ab} \wedge \varphi_b = 0, \tag{25}$$

where $R_{ab} = \sigma_a^{AA'} \sigma_{bA'} R_{AB} + \sigma_a^{AA'} \sigma_{bA'} R_{A'B'}$ and $\varphi^a = \sigma_{AA'}^a \varphi^{AA'}$. For a general formulation which satisfies equation \[21\], the action reduces to

$$S = \int (R_{AB} \wedge \varphi^{AA'} \wedge \varphi^{B_A} \xi^{B'} \xi_{B'}) + c.c. + d(...)$$

$$= \int \overline{\psi} \psi R^{ab} \wedge \ast(\varphi_a \wedge \varphi_b) + \overline{\psi} \gamma_5 \psi R^{ab} \wedge (\varphi_a \wedge \varphi_b)$$

$$+ d[D(\overline{\psi} \psi) \gamma_5 \psi \overline{\psi} \gamma_5 \psi], \tag{26}$$

where $\overline{\psi} \psi = \xi^A \xi_A + \xi^{A'} \xi_{A'}$, $\overline{\psi} \gamma_5 \psi = i(-\xi^A \xi_A + \xi^{A'} \xi_{A'})$ and $d(...)$ are the boundary terms which are essential for the action being well-defined at spatial infinity.

The Immirzi parameter is then given by

$$\gamma = \frac{<\overline{\psi} \psi >}{<\overline{\psi} \gamma_5 \psi >}. \tag{27}$$

Since both $<\overline{\psi} \gamma_5 \psi >$ and $<\overline{\psi} \psi >$ are real for anti-commuting spinor, $\gamma$ is always real. Here we have explicitly used the expectation values of the spinor field. It highlights the renormalization scale dependent nature of $\gamma$ upon quantization in this setting. Same as in
the previous session, from the explicit expression for $\gamma$, it is the ratio between scalar and pseudo-scalar contributions in the theory and can be interpreted as a measure of how a generally formulated covariant theory differs from Einstein gravity. In a future publication we shall introduce the dynamics of the spinor field $\psi$ and investigate how $\gamma$ is connected with other properties of general relativity.

IV. CONCLUDING REMARKS

In summary, we have demonstrated how Quadratic Spinor Representation of general theory of relativity with auxiliary spinor field $\psi$ can provide a systematic way to derive the physical properties of the Immirzi parameter $\gamma$. From the explicit expression for $\gamma$, one can see that it is the ratio between scalar and pseudo-scalar contributions in the theory. An important feature of this derivation is the possibility of introducing a renormalization scale associated with $\gamma$ upon quantization.
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[16] The upper-case Latin letters $A, B, ... = 0, 1$ denote two component spinor indices, which are raised and lowered with the constant symplectic spinors $\epsilon_{AB} = -\epsilon_{BA}$ together with its inverse and their conjugates according to the conventions $\epsilon_{01} = \epsilon^{01} = +1$, $\lambda^A := \epsilon^{AB} \lambda_B$, $\mu_B := \mu^A \epsilon_{AB}$. 