Global Existence of Solutions for the Einstein-Boltzmann system in a Bianchi Type I Space-Time (Detailed paper)

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Abstract

We prove a global in time existence theorem, for the initial value problem for the Einstein-Boltzmann system, with arbitrarily large initial data, in the homogeneous case, in a Bianchi Type I space-time.

1 Introduction

The Einstein equations are the basic equations of the General Relativity theory funded in 1916 by Albert Einstein and according to which: "the geometric structure of the space-time is determined by its material and energetic contents". These equations link, on one hand, the geometry of the space-time, which is determined by a fundamental tool: the metric tensor, standing for the gravitational field, and, on the other hand, all the material and energetic contents, summarized by the stress-energy-matter tensor (we will call it the matter tensor), acting as the sources for the gravitational field. Solving the Einstein equations is determining both the gravitational field and its sources. Naturally, the problem varies with the matter contents, and this is why, in the mathematical study of General Relativity, one of the main problems is to solve the Einstein equations coupled to various field equations, in order to determine both the gravitational field and its sources in different cases. A fundamental problem today is to establish the existence and to give the properties of global solutions to such coupled systems. We are interested in global dynamics of relativistic kinetic matter, and our goal is to prove such results in that domain. In the case of collisionless matter, the governing system is the Einstein-Vlasov system, in the pure gravitational case, or this system coupled to other field equations, if these fields are involved in the sources of the Einstein equations. In the collisionless case, several authors proved global results, see \cite{17, 21} for reviews, \cite{12, 22} and \cite{10} for scalar matter fields, also see \cite{19, 18} for the Einstein-Vlasov system with cosmological constant, which turns out to be a useful tool.
for the proof of the fact that, the expansion of the universe is accelerating, see [10] for more details on this question.

Now, in the case of collisional matter, the Einstein-Vlasov system is replaced by the Einstein-Boltzmann system, which seems to be the best approximation available and that describes the case of instantaneous, binary and elastic collisions. In contrast with the abundance of papers in the collisionless case, the literature seems very poor in the collisional case. If, due to its importance in collisional kinetic theory, several authors proved global results for the single Boltzmann equation, see [8], [5], [7] for the non-relativistic case, and [9] for the full relativistic case, very few authors studied the Einstein-Boltzmann system, see [4] for a local existence theorem. It then seems interesting to us, to extend to the collisional case some global results obtained in the collisionless case. This was certainly the objective of the author in [13], [14], in which he studied the existence of global solutions for the Einstein-Boltzmann system. Unfortunately, several points of the work are far from clear, such as, the use of a formulation which is valid only for the non-relativistic Boltzmann equation, or, concerning the Einstein equations, to abandon the evolution equations, which are really relevant in the spatially homogeneous case he considers, to concentrate on the constraints equations, that reduce, as we will see, to a simple question of choice of the initial data. In [24], the authors prove a recent result on the global existence of solutions for the spatially homogeneous Einstein-Boltzmann system on a Robertson-Walker space-time, in the case of a strictly positive Cosmological constant $\Lambda > 0$.

In this papers, we study the collisional evolution of a kind of uncharged massive particles, under the only influence of their common gravitational field, represented by the metric tensor we denote by $g$, and which is a function of the position of the particles, and whose components $g_{\alpha \beta}$, sometimes called "gravitational potentials", are subject to the Einstein equations. Here, the sources of the Einstein equations are generated by the only matter contents which are the massive particles, through the matter tensor which is a symmetric 2-tensor, we denote $T_{\alpha \beta}$. The particles are statistically described in terms of their distribution function, we denote by $f$, and which is a non-negative real-valued function of both the position and the momentum of particles. The scalar function $f$ which is physically interpreted as the "probability of the presence density" of the massive particles, during their collisional evolution, is subject to the Boltzmann equation, defined by a non-linear operator called the "collision operator". In the binary and elastic scheme due to Lichnerowicz and Chernikov (1940), we adopt, at a given position, only 2 particles collide each other, in an instantaneous shock affecting only the momentum of each particle, only the sum of the 2 momenta being preserved. We then study the coupled Einstein-Boltzmann system in $(g, f)$. The system is coupled in the sense that each unknown appears in the equation determining the other unknown: $f$ which is subject to the Boltzmann equation generates the sources of the Einstein equations through the matter tensor $T_{\alpha \beta}$, whereas the metric tensor $g$ which is subject to the Einstein equations is in both sides of the Boltzmann equation in $f$. If the particles were charged, the above system would be coupled to the Maxwell
equations that would describe the electromagnetic effects; this is our project in a future work.

We now specify the geometric framework, i.e. the kind of space-time we are looking for in the present work. An important part of the General Relativity is Cosmology, which is the study of the universe at large scale; in such a viewpoint, even cluster of galaxies are assimilable to "particles". A. Einstein and W. de Sitter introduced cosmological models in 1917.

Let us point out that the Einstein equations are overdetermined and physically meaning symmetry assumptions reduce the number of unknown. The surface symmetry assumptions which are the spherical, plane and hyperbolic symmetries assumptions constitute the major part of the models studied in the works we quote in the references. Robertson and Walker showed in 1944 that "Exact spherical symmetry about every point would imply that the universe is spatially homogeneous" see [3], p. 135. In the Robertson-Walker models, the spatial geometry has constant curvature which is positive, zero and negative respectively. We look for a spatially homogeneous, locally rotationally symmetric (LRS) Bianchi type I space-time, which is a direct generalization of the Robertson-Walker space-time with zero curvature and which is known, in Cosmology, to be the basic model for the study of the expanding universe. In the model we are looking for, the metric tensor $g$ has only two unknown components $a$ and $b$, and the spatial homogeneity means that $a$ and $b$ depend only on the time $t$, and the distribution function $f$ depends only on the time $t$ and the momentum $p$ of the particles. The study of the Einstein-Boltzmann system then turns out to the determination of the triplet of scalar functions $(a, b, f)$, $a$ and $b$ being strictly positive functions.

We now sketch the strategy we adopt to prove the global in time existence of a solution $(a, b, f)$ of the initial value problem for the coupled Einstein-Boltzmann system, for arbitrarily large initial data $(a_0, b_0, f_0)$ at the initial time $t = 0$. In the spatially homogeneous case we consider, the Einstein equations are a system of three non-linear o.d.e in $a$ and $b$. The Boltzmann equation is a non-linear first order p.d.e for the distribution function $f$.

In the first step, we suppose $a$ and $b$ given with the only assumption to be bounded away from zero; and we give, following Glassey R.T. in [11], the correct formulation of the relativistic Boltzmann equation in $f$ on a Bianchi type I space-time. We then prove that, on any bounded time interval $I = [t_0, t_0 + T]$, with $t_0 \in \mathbb{R}^+$, $T \in \mathbb{R}^+$, the initial value problem for the Boltzmann equation, with initial data $f_{t_0} \in L^1_{\gamma}(\mathbb{R}^3)$, $f_{t_0} \geq 0$ a.e; has a unique solution $f \in C[I; L^1_{\gamma}(\mathbb{R}^3)]$, $f(t) \geq 0$ a.e, $\forall t \in I$, where $L^1_{\gamma}(\mathbb{R}^3)$ is a weighted subspace of $L^1(\mathbb{R}^3)$, whose weight, is imposed by the manner in which the sources terms $T_{\alpha\beta}$ of the Einstein equations depend on $f$. We follow the method developed in [23] to prove the global existence of the solution $f \in C[0, +\infty; L^1(\mathbb{R}^3)]$ for the initial value problem for the Boltzmann equation, but here the weighted norm of $L^1_{\gamma}(\mathbb{R}^3)$ could allow us to prove the existence theorem only on bounded time intervals $I = [t_0, t_0 + T]$, but this was enough for the coupling with the Einstein equations and to obtain the global existence by another method.

In a second step, we suppose $f$ given in $C[I; L^1_{\gamma}(\mathbb{R}^3)]$ and we consider the Ein-
stein equations in $a$ and $b$, that split into the constraint equations and the evolution equations. The constraint equations contain the momentum constraints which are automatically satisfied in the homogeneous case we consider, and the Hamiltonian constraint which reduces to a question of choice for the initial data. The main problem is then to solve the evolution equations which are a system of two non-linear o.d.e in $a$ and $b$. We set, following Rendall, A.D., and Uggla, C. in [15]

$$H = -\frac{\text{tr} k}{3}; \quad z = \frac{1}{a^{-2} + 2b^{-2} + 1}; \quad s = \frac{b^2}{b^2 + 2a^2}; \quad \Sigma_+ = -\frac{3}{\text{tr} k} \frac{b}{b} - 1.$$  

in which $k$ is the second fundamental form on the space-like 3-manifold that foliate the space-time in the 3+1 formulation of the Einstein equations. We then prove that the Einstein evolution system in $a$ and $b$ is equivalent to a first order system in $(H, z, s, \Sigma_+)$, for which standard theorems apply. From there, we deduce the existence of the solutions $a, b$ of the initial value problem for the Einstein evolution equations, on any interval $I = [0, T], T > 0$, which are increasing function of $t$, and hence bounded from below, satisfying consequently the assumptions of the first step.

In a third step, we use the change of variables indicated above together with the characteristic method for the first order p.d.e that reduces the Boltzmann equation to a non linear o.d.e in $f$ with values in the Banach space $L^1_2(R^3)$, to prove the local existence of solutions for the coupled Einstein-Boltzmann system by applying standard results.

In a fourth step, we use the results of the two first steps to construct an appropriate functional framework, allowing us to prove, using the fixed point theorem that, the local solution of the coupled system, whose existence is established in the third step, is in fact, a global solution.

Our method preserves the physical nature of the problem that imposes to the distribution function $f$ to be a non-negative function, and this is why we had to consider spaces of functions $f$ which are positive everywhere. Nowhere we had to require smallness assumptions on the initial data, which can, consequently, be taken arbitrarily large.

An aspect of the results of the present paper is an extension of the results of [24], to the case of the zero cosmological constant $\Lambda = 0$. Notice that the present paper follows papers [23] and [24] and the proofs for the Boltzmann equation are similar, but profound differences arise with [24] on the Einstein equations.

This paper is organized as follows:
In section 2, we introduce the Einstein equations, the Boltzmann equation and the coupled Einstein-Boltzmann system, on a Bianchi type I space-time.
In section 3, we study the Boltzmann equation in $f$.
In section 4, we study the Einstein equations in $a$ and $b$.
In section 5, we prove the existence of local solutions to the coupled Einstein-Boltzmann system.
In section 6, we prove the existence of global solutions, to the coupled Einstein-Boltzmann system.
2 The Boltzmann equation, the Einstein equations and the Einstein-Boltzmann system on a LRS Bianchi type I space-time.

2.1 Notations and function spaces

A greek index varies from 0 to 3 and a Latin index from 1 to 3, unless otherwise specified. We adopt the Einstein summation convention $\sum_{\alpha \in \sigma} a_{\alpha} b_{\alpha}$. We consider a locally rotationally symmetric (LRS) Bianchi type I space-time denoted $(\mathbb{R}^4, g)$ where, for $x = (x^\alpha) = (x^0, x^i) \in \mathbb{R}^4$, $x^0 = t$ denotes the time and $\bar{x} = (x^i)$ the space; $g$ stands for the metric tensor with signature $(-, +, +, +)$ that can be written:

$$g = -dt^2 + a^2(t)dx^1 + b^2(t)[dx^2 + dx^3]$$  \hspace{1cm} (2.1)

in which $a$ and $b$ are strictly positive functions of $t$. We consider the collisional evolution of a kind of uncharged massive relativistic particles in the time-oriented space-time $(\mathbb{R}^4, g)$. The particles are statistically described by their distribution function $f$, which is a non-negative real-valued function of both the position $(x^\alpha)$ and the 4-momentum $p = (p^\alpha)$ of the particles, and that coordinatize the tangent bundle $T(\mathbb{R}^4)$ i.e:

$$f : T(\mathbb{R}^4) \simeq \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}_+, \hspace{1cm} (x^\alpha, p^\alpha) \mapsto f(x^\alpha, p^\alpha) \in \mathbb{R}_+$$

For $\bar{p} = (p^i), \bar{q} = (q^i) \in \mathbb{R}^3$, we set, as usual $|\bar{p}| = \sqrt{\sum_i (p^i)^2}$, and we define the scalar product:

$$\langle \bar{p} , \bar{q} \rangle = a^2p^1q^1 + b^2[p^2q^2 + p^3q^3]$$  \hspace{1cm} (2.2)

We suppose the rest mass $m > 0$ of the particles normalized to unity, i.e $m = 1$. The relativistic particles are then required to move on the future sheet of the mass-shell, whose equation is $g(p, p) = -1$; from which we deduce, using expression (2.1) of $g$:

$$p^0 = \sqrt{1 + a^2(p^1)^2 + b^2[(p^2)^2 + (p^3)^2]}$$  \hspace{1cm} (2.3)

where the choice of $p^0 > 0$ symbolizes the fact that, in a natural manner, particles eject towards the future. (2.3) shows that, in fact, $f$ is defined on the 7-dimensional subbundle of $T(\mathbb{R}^4)$ coordinatized by $(x^\alpha), (p^i)$. In the spatially homogeneous case we consider, $f$ depends only on $t$ and $\bar{p} = (p^i)$. The framework we will refer to is the subspace of $L^1(\mathbb{R}^3)$, denote $L^1_2(\mathbb{R}^3)$ and defined by :

$$L^1_2(\mathbb{R}^3) = \{ f \in L^1(\mathbb{R}^3), \| f \| := \int_{\mathbb{R}^3} \sqrt{1 + |\bar{p}|^2}|f(\bar{p})|d\bar{p} < +\infty \}$$  \hspace{1cm} (2.4)

$\| . \|$ is a norm on $L^1_2(\mathbb{R}^3)$ and $(L^1_2(\mathbb{R}^3), \| . \|)$ is a Banach space. We set for $r$ an arbitrary strictly positive real number:

$$X_r = \{ f \in L^1_2(\mathbb{R}^3), \hspace{0.5cm} f \geq 0 \hspace{0.5cm} a.e, \hspace{0.5cm} \| f \| \leq r \}$$  \hspace{1cm} (2.5)
One verifies easily that, endowed with the metric induced by the norm \( \| \cdot \| \), \( X_r \) is a complete and connected metric subspace of \( (L^1_2(\mathbb{R}^3), \| \cdot \|) \). Let \( I \) be a real interval; we set:

\[ C[I; L^1_2(\mathbb{R}^3)] = \{ f : I \to L^1_2(\mathbb{R}^3), f \text{ continuous and bounded } \} \]

endowed with the norm:

\[ \|f\| = \sup_{t \in I} \| f(t) \| \quad (2.6) \]

\( C[I; L^1_2(\mathbb{R}^3)] \) is a Banach space. \( X_r \) being defined by \( \| \cdot \| \); we set:

\[ C[I; X_r] = \{ f \in C[I; L^1_2(\mathbb{R}^3)], f(t) \in X_r, \forall t \in I \} \quad (2.7) \]

Endowed with the metric induced by the norm \( \| \cdot \| \) defined by \( \| \cdot \| \), \( C[I; X_r] \) is a complete metric subspace of \( (C[I; L^1_2(\mathbb{R}^3)], \| \cdot \|) \).

### 2.2 The Boltzmann equation

In its general form, the Boltzmann equation in \( f \), on the curved space-time \( (\mathbb{R}^4, g) \) can be written:

\[ p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^\alpha_{\mu \nu} p^\mu p^\nu \frac{\partial f}{\partial p^\alpha} = Q(f, f) \quad (2.8) \]

Where \( (\Gamma^\alpha_{\lambda \mu}) \) are the Christoffel symbols of \( g \), and \( Q \) is a non-linear integral operator called the "Collision Operator". We specify this operator in details in next section. Now, since \( f \) depends only on \( t \) and \( (p^i) \), \( p^0 \), \( (p^i) \) writes:

\[ p^0 \frac{\partial f}{\partial t} - \Gamma^i_{\mu \nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = Q(f, f) \quad (2.9) \]

We now express the Christoffel symbols in the case of the metric \( g \) defined by \( \| \cdot \| \). Their general expression is:

\[ \Gamma^\lambda_{\alpha \beta} = \frac{1}{2} g^{\lambda \nu} \left[ \partial_\alpha g_{\nu \beta} + \partial_\beta g_{\nu \alpha} - \partial_\nu g_{\alpha \beta} \right] \quad (2.10) \]

in which \( (g^{\lambda \nu}) \) denotes the inverse matrix of \( (g_{\lambda \mu}) \); We have \( \Gamma^\lambda_{\alpha \beta} = \Gamma^\lambda_{\beta \alpha} \). The definition \( \| \cdot \| \) of \( g \) gives at once:

\[
\begin{align*}
\{ & \quad g^{00} = g_{00} = -1; \quad g_{11} = a^2; \quad g_{22} = g_{33} = b^2; \quad g^{11} = a^{-2}; \quad g^{22} = g^{33} = b^{-2} \\
& \quad g_{0i} = g^{0i} = 0; \quad g_{ij} = g^{ij} = 0 \quad \text{for} \quad i \neq j
\end{align*}
\]

A direct computation, using \( \| \cdot \| \) and \( \| \cdot \| \), gives, with \( \dot{a} = \frac{da}{dt} \):

\[
\begin{align*}
\{ & \quad \Gamma^{0}_{01} = \dot{a} a; \quad \Gamma^{0}_{22} = \Gamma^{0}_{33} = \dot{b} b; \quad \Gamma^{1}_{10} = \frac{\dot{a}}{a}; \quad \Gamma^{2}_{20} = \Gamma^{3}_{30} = \frac{\dot{b}}{b}; \quad \Gamma^{0}_{00} = 0; \\
& \quad \Gamma^{0}_{\alpha \beta} = 0 \quad \text{for} \quad \alpha \neq \beta; \quad \Gamma^{i}_{ij} = 0
\end{align*}
\]

\( \| \cdot \| \)
The Boltzmann equation (2.10) then writes, using (2.12):

$$\frac{\partial f}{\partial t} - 2\frac{\dot{a}}{a} p_1 \frac{\partial f}{\partial p_1} - 2\frac{\dot{b}}{b} p_2 \frac{\partial f}{\partial p_2} - 2\frac{\dot{b}}{b} p_3 \frac{\partial f}{\partial p_3} = \frac{1}{p^0} Q(f, f) \quad (2.13)$$

in which $p^0$ is given by (2.3). (2.13) is a non-linear p.d.e in $f$ we study in next section.

### 2.3 The Einstein Equations

The Einstein equations in $g = (g_{\alpha\beta})$ can be written:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi G T_{\alpha\beta} \quad (2.14)$$

in which:
- $R_{\alpha\beta}$ is the Ricci tensor, contracted of the curvature tensor of $g$ we specify below;
- $R = g^{\alpha\beta} R_{\alpha\beta} = R^\alpha_\alpha$ is the scalar curvature;
- $T_{\alpha\beta}$ is the matter tensor that represents the matter contents, namely the massive particles in our case. $T_{\alpha\beta}$ is generated by the distribution function $f$ of the particles by:

$$T_{\alpha\beta}(t) = \int_{\mathbb{R}^3} \frac{p^\alpha p^\beta f(t, \bar{p}) |g|^{\frac{1}{2}}}{p^0} d\bar{p} \quad (2.15)$$

where $|g|$ is the determinant of $g$: (2.11) gives $|g|^{\frac{1}{2}} = ab^2$. Recall that $f$ is a function of $t$ and $\bar{p} = (p^\mu)$; hence $T_{\alpha\beta}$ is a function of $t$.

$G$ is the universal gravitational constant. We take $G = 1$.

The contraction of the Bianchi identities gives the identities $\nabla_\alpha S^{\alpha\beta} = 0$, where $S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$ is the Einstein tensor. The Einstein equations (2.14) then imply that the tensor $T^{\alpha\beta}$ must satisfy the four relations $\nabla_\alpha T^{\alpha\beta} = 0$ called the conservation laws. But it is proved in [2] that these laws are satisfied for all solution $f$ of the Boltzmann equation. We have:

$$\begin{cases}
R_{\alpha\beta} = R^\lambda_{\alpha, \lambda\beta} \\
where \\
R^\lambda_{\mu, \alpha\beta} = \partial_\alpha \Gamma^\lambda_{\mu\beta} - \partial_\beta \Gamma^\lambda_{\mu\alpha} + \Gamma^\lambda_{\nu\alpha} \Gamma^\nu_{\mu\beta} - \Gamma^\lambda_{\nu\beta} \Gamma^\nu_{\mu\alpha}
\end{cases}$$

(2.16)

in which $\Gamma^\lambda_{\mu\beta}$ is given by (2.10) with the only non zero components given by (2.12). (2.12) and (2.16) show that the Einstein equations (2.14) are a system of second order non-linear o.d.e in $a$ and $b$. In order to have things fresh in mind at the adequate moment, we leave the expression of (2.14) in term of $a$ and $b$ to section 4, which is devoted to the study of the Einstein equations.

### 2.4 The coupled Einstein-Boltzmann system

(2.14)-(2.13) is the Einstein-Boltzmann system we study. The system is coupled in the sense that, $f$ which is subject to the Boltzmann equation (2.13) generates
through (2.15), the sources terms $T_{\alpha\beta}$ of the Einstein equations; it appears clearly that the coefficients of the Boltzmann equation (2.13) in $f$ depend on $a$ and $b$ which are subject to the Einstein equations (2.14), and we will see, in next section, that $a$ and $b$ also appear in the collision operator $Q$, which is the r.h.s of the Boltzmann equation (2.13).

3 Existence Theorem for the Boltzmann Equation

In this section, we suppose that the components $a$ and $b$ of the metric tensor $g$ are given and fixed, and we prove an existence theorem for the initial values problem for the Boltzmann equation (2.13), on every bounded interval $I = [t_0, t_0 + T]$ where $t_0 \in \mathbb{R}_+, T \in \mathbb{R}_+^*$. We begin by specifying the collision operator $Q$ in (2.13).

3.1 The Collision Operator

In the instantaneous, binary and elastic scheme, due to Lichnerowicz and Chernikov we consider, at a given position $(t, \bar{x})$, only 2 particles collide instantaneously, without destroying each other, the collision affecting only the momentum of each particle that changes after the shock, only the sum of the 2 momenta being preserved, following the scheme:

\[
\begin{array}{c}
p \quad \downarrow \quad p' \\
(t, \bar{x}) \\
q \quad \uparrow \\
\end{array}
\]

\[p + q = p' + q',\]

The collision operator $Q$ is defined, using functions $f, g$ on $\mathbb{R}^3$, $p, q$ standing for the momenta before the collision and $p', q'$ the momenta after the collision by:

\[Q(f, g) = Q^+(f, g) - Q^-(f, g)\] (3.1)

where:

\[Q^+(f, g)(\bar{p}) = \int_{\mathbb{R}^3} \frac{ab^2d\bar{q}}{q^0} \int_{S^2} f(\bar{p'}) g(\bar{q'}) A(a, b, \bar{p}, \bar{q}, \bar{p}', \bar{q}') d\omega \] (3.2)

\[Q^-(f, g)(\bar{p}) = \int_{\mathbb{R}^3} \frac{ab^2d\bar{q}}{q^0} \int_{S^2} f(\bar{p}) g(\bar{q}) A(a, b, \bar{p}, \bar{q}, \bar{p}', \bar{q}') d\omega \] (3.3)

whose elements we now introduce step by step, specifying properties and hypotheses we adopt:
1) $S^2$ is the unit sphere of $\mathbb{R}^3$, whose volume element is denoted $dw$.

2) $A$ is a non-negative real-valued regular function of all its arguments, called the collision kernel or the cross-section of the collisions, on which we require the following boundedness, symmetry and Lipschitz continuity assumptions:

$$0 \leq A(a, b, \bar{p}, \bar{q}, \bar{p}', \bar{q}') \leq C_0 \tag{3.4}$$

$$A(a, b, \bar{p}, \bar{q}, \bar{p}', \bar{q'}) = A(a, b, \bar{p}', \bar{q}', \bar{p}, \bar{q}) \tag{3.5}$$

$$A(a, b, \bar{p}, \bar{q}, \bar{p}', \bar{q'}) = A(a, b, \bar{q}, \bar{p}', \bar{p}', \bar{q}') \tag{3.6}$$

$$|A(a_1, b, \bar{p}, \bar{q}, \bar{p}', \bar{q}') - A(a_2, b, \bar{p}, \bar{q}, \bar{p}', \bar{q}')| \leq k_0 |a_1 - a_2| \tag{3.7}$$

$$|A(a, b_1, \bar{p}, \bar{q}, \bar{p}', \bar{q}') - A(a, b_2, \bar{p}, \bar{q}, \bar{p}', \bar{q}')| \leq k_0 |b_1 - b_2| \tag{3.8}$$

where $C_0$ and $k_0$ are strictly positive constants.

3) The conservation law $p + q = p' + q'$ splits into:

$$p^0 + q^0 = p'^0 + q'^0 \tag{3.9}$$

and shows, using (2.3), the conservation of the quantity:

$$e = \sqrt{1 + a^2(p^1)^2 + b^2[(p^2)^2 + (p^3)^2]} + \sqrt{1 + a^2(q^1)^2 + b^2[(q^2)^2 + (q^3)^2]} \tag{3.11}$$

called the elementary energy of the unit rest mass particles; we can interpret (3.10) by setting, following Glassey, R. T., in [11]:

$$\begin{cases} 
\bar{p}' = \bar{p} + R(\bar{p}, \bar{q}, \omega)\omega \\
\bar{q}' = \bar{q} - R(\bar{p}, \bar{q}, \omega)\omega; \quad \omega \in S^2 
\end{cases} \tag{3.12}$$

in which $R(\bar{p}, \bar{q}, \omega)$ is a real-valued function. We prove, by a straightforward computation (using (2.3) to express $p'^0$, $q'^0$ in terms of $\bar{p}'$, $\bar{q}'$, and next (3.12) to express $p'$, $p'$ in terms of $\bar{p}$, $\bar{q}$, $\omega, R$, that equation (3.9) leads to a quadratic equation in $R$, that solves to give the only non trivial solution:

$$R(\bar{p}, \bar{q}, \omega) = \frac{2 p^0 q^0 e < \omega, (\hat{q} - \hat{p}) >}{e^2 - |< \omega, (\bar{p} + \bar{q})|^2} \tag{3.13}$$

in which $\hat{p} = \frac{\bar{p}}{p^0}$, $e$ is given by (3.11), and $<,>$ is the scalar product defined by (2.2). Another direct computation gives, using the properties of the
We solve the initial value problem on $I$.

**Remark 3.1**

1) Formulae (3.13) and (3.14) are generalizations to the case to simplify notations, we will explicit the dependence on $t$ to practice, we will consider functions of $\bar{f}$. Formulae (3.12) shows, using once more (2.3) and the implicit functions theorem, that the change of variables (3.12) is invertible and also allows to compute $\bar{p}, \bar{q}$, and (3.3) completely express in terms of $\bar{p}, \bar{q}$. Finally, formulae (2.3) and (3.12) show that the functions to integrate in (3.2) leave functions $A, \bar{p}$ in terms of $\bar{p}', \bar{q}'$.

Finally, formulae (3.13) and (3.14) show that the functions to integrate in (3.2) and (3.3) completely express in terms of $\bar{p}, \bar{q}, \omega$, and the integrations with respect to $\bar{q}$ and $\omega$ leave functions $Q^+(f,g), Q^-(f,g)$ in terms of the single variable $\bar{p}$. In practice, we will consider functions $f : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$, that induce, for $t$ fixed in $\mathbb{R}$, functions $f(t)$ on $\mathbb{R}^3$, defined by $f(t)(\bar{p}) = f(t, \bar{p})$. Even in such cases, in order to simplify notations, we will explicit the dependence on $t$ only if necessary.

**Remark 3.1**

1) Formulae (3.13) and (3.14) are generalizations to the case of the LRS Bianchi type I space-time, of analogous formulae established by Glussey, R.T, in [11], in the case of the Minkowski space-time, which corresponds to the case $a(t) = b(t) = 1$ in the metric defined by (2.1), in which case the scalar product $<,>$ defined by (2.2) reduces to the usual scalar product in $\mathbb{R}^3$.

2) In (3.13), (3.14), $\omega_q = |g|^\frac{k}{2} \sum_{i} \frac{d\bar{p}_i}{d\bar{q}_i} = ab^2 \sum_{i} \frac{d\bar{p}_i}{d\bar{q}_i}$ is a Leray form, which is in fact, the canonical volume element in the momenta space.

3) $A = ke^{-a^2 - b^2 - |\bar{p}|^2 - |\bar{q}|^2}, k > 0$, is a simple example of collision kernel satisfying assumptions (3.4), (3.5), (3.6), (3.7) and (3.8).

### 3.2 Resolution of the Boltzmann equation

We consider the Boltzmann equation (2.13) on $I = [t_0, t_0 + T], t_0 \in \mathbb{R}^+, T > 0$.

The functions $a$ and $b$ are supposed to be defined on $I$. Equation (2.13) is a first order p.d.e in $f$ and its resolution is equivalent to the resolution of the associated characteristic system, which can be written, taking $t$ as parameter:

$$
\frac{dp^1}{dt} = -2\frac{\dot{a}}{a}p^1; \quad \frac{dp^2}{dt} = -2\frac{\dot{b}}{b}p^2; \quad \frac{dp^3}{dt} = -2\frac{\dot{b}}{b}p^3; \quad \frac{df}{dt} = \frac{1}{p^0}Q(f, f) \quad (3.15)
$$

We solve the initial value problem on $I = [t_0, t_0 + T]$, with initial data:

$$
p^i(t_0) = y^i; i = 1, 2, 3; \quad f(t_0) = f_{t_0} \quad (3.16)
$$

The equations in $\bar{p} = (p^i)$ solve directly to give, setting $y = (y^i) \in \mathbb{R}^3$;

$$
\bar{p}(t_0 + t, y) = D(t)y, \quad \text{with} \quad D(t) = Diag \left( \frac{a^2(t_0)}{a^2(t_0 + t)}, \frac{b^2(t_0)}{b^2(t_0 + t)}, \frac{b^2(t_0)}{b^2(t_0 + t)} \right), \quad t \in [0, T] \quad (3.17)
$$
which shows that, the initial value problem for the Boltzmann equation (2.13) is finally equivalent to the following integral equation in which, for simplicity, we conserve the notation $\bar{p}$, which stands this time for any independent variable in $\mathbb{R}^3$:

$$f(t_0 + t, \bar{p}) = f_{t_0}(\bar{p}) + \int_{t_0}^{t_0 + t} \frac{1}{\bar{p}^0} Q(f, f)(s, \bar{p}) ds \quad t \in [0, T]$$  (3.18)

We prove:

**Theorem 3.1** Let $a$ and $b$ be strictly positive continuous functions of $t$, such that $a(t) \geq \frac{3}{2}$, $b(t) \geq \frac{3}{2}$, whenever $a$ and $b$ are defined. Let $f_{t_0} \in L^1_2(\mathbb{R}^3)$, $f_{t_0} \geq 0$, a.e, and $r \in \mathbb{R}^*_+$ such that $r > \|f_{t_0}\|$. Then, the initial value problem for the Boltzmann equation on $[t_0, t_0 + T]$, with initial data $f_{t_0}$, has a unique solution $f \in C[t_0, t_0 + T; X_r]$. Moreover, $f$ satisfies the estimation:

$$\sup_{t \in [t_0, t_0 + T]} \|f(t)\| \leq \|f_{t_0}\|$$  (3.19)

**Proof:** Theorem 3.1 is a direct consequence of the following result:

**Proposition 3.1** Assume hypotheses of theorem 3.1 on: $a$, $b$, $f_{t_0}$ and $r$.

1) There exists an integer $n_0(r) \geq 1$ such that, for every integer $n \geq n_0(r)$ and for every $v \in X_r$, the equation:

$$\sqrt{n} u - \frac{1}{\bar{p}^0} Q(u, u) = v$$  (3.20)

has a unique solution $u_n \in X_r$.

2) Let $n \in \mathbb{N}$, $n \geq n_0(r)$

i) For every $u \in X_r$, define $R(n, Q)u$ to be the unique element of $X_r$ such that:

$$\sqrt{n} R(n, Q)u - \frac{1}{\bar{p}^0} Q[R(n, Q)u, R(n, Q)u] = u$$  (3.21)

ii) Define operator $Q_n$ on $X_r$ by:

$$Q_n(u, u) = n \sqrt{n} R(n, Q)u - nu$$  (3.22)

Then

a) The integral equation

$$f(t_0 + t, \bar{p}) = f_{t_0}(\bar{p}) + \int_{t_0}^{t_0 + t} Q_n(f, f)(s, \bar{p}) ds \quad t \in [0, T]$$  (3.23)

has a unique solution $f_n \in C[t_0, t_0 + T; X_r]$. Moreover, $f_n$ satisfies the estimation:

$$\sup_{t \in [t_0, t_0 + T]} \|f_n(t)\| \leq \|f_{t_0}\|$$  (3.24)
We deduce from (3.9), using $a > d$.

We make the change of variable (3.12) and we deduce from (3.14) that gives $C > 0$ in which the above inequality yields:

Lemma 3.1

We use:

Expression (3.2) of $Q$.

As we pointed it out, the above result is similar to those of [23] and [24]. The time, we establish useful formulae for next steps of the present paper. Only on points where differences arise with the present case, and, at the same time, we establish useful formulae for next steps of the present paper.

A) Proof of point 1) of prop 3.1

We use:

$$ L(t) = C_{a b}(t) $$

in which $C > 0$ is an absolute constant.

Proof of lemma 3.1

We deduce from [24], using $a > 1$, $b > 1$ and expression [28] of $p^0$:

$$ \sqrt{1 + |\bar{p}|^2} \leq \sqrt{1 + \frac{1}{a^2} + \frac{1}{b^2} p^0} \leq 2(p^0 + q^0) = 2(p^0 + q^0) $$

Expression [34] of $Q^+(f, g)$ then gives, since by [34], $|A| \leq C_0$:

$$ \left\| \frac{1}{p^0} Q^+(f, g) \right\| = \int_{R^3} \left\| \frac{1}{p^0} \sqrt{1 + |\bar{p}|^2} Q^+(f, g) \right\| d\bar{p} $$

$$ \leq 2a^2 p^0(t) C_0 \int_{R^3} \int_{R^3} \int_{S^2} \frac{p^0 + q^0}{p^0 q^0} |f(\vec{p})| |g(\vec{q})| d\vec{p} d\vec{q} \omega $$

We make the change of variable (3.12) and we deduce from [34] that gives $d\vec{p} d\vec{q} = \frac{p^0 q^0}{p^0 q^0} d\vec{p} d\vec{q}'$, and since by [28] we have $\frac{p^0 + q^0}{p^0 q^0} = \frac{1}{p^0} + \frac{1}{q^0} \leq 2$, that the above inequality yields:

$$ \left\| \frac{1}{p^0} Q^+(f, g) \right\| \leq 8\pi a^2(t) C_0 \int_{R^3} \int_{R^3} |f(\vec{p}')| |g(\vec{q}')| d\vec{p}' d\vec{q}' \leq C a^2 \left\| f \right\| \left\| g \right\| $$
with \( C = 8\pi C_0 \). The estimation of \( \left\| \frac{1}{\sqrt{p}} Q^-(f, g) \right\| \) follows the same way without change of variables and \( \text{3.25} \) follows. The inequalities \( \text{3.26} \) are direct consequences of \( \text{3.25} \) and the bilinearity of \( Q^+ \) and \( Q^- \), which allows us to write, \( P \) standing for \( \frac{1}{\sqrt{p}} Q^+ \) or \( \frac{1}{\sqrt{p}} Q^- \):

\[
P(f, f) - P(g, g) = P(f, f - g) + P(f - g, g).
\]

Finally, \( \text{3.27} \) is a consequence of \( \text{3.26} \) and \( Q = Q^+ - Q^- \). This completes the proof of the lemma 3.1.

Now, since the function \( t \to ab^2(t) \) is positive and continuous on the line segment \([t_0, t_0 + T]\), there exists an absolute constant \( C(t_0, T) > 0 \) such that \( C(t) \leq C(t_0, T), \forall t \in [t_0, t_0 + T] \), where \( C(t) \) is defined by \( \text{3.28} \). We then obtain, from lemma 3.1, the same inequalities with an absolute constant, than the inequalities in proposition 3.1 in [23]. The proof of the point 1) of proposition 3.1 above is then exactly the same as the proof of proposition 3.2 in [23].

**B) Proof of the point 2) of prop 3.1**

We use, \( n_0(r) \) being the integer defined in point 1) of proposition 3.1 above:

**Lemma 3.2** We have, for every integer \( n \geq n_0(r) \) and for every \( u \in X_r \)

\[
\| \sqrt{n} R(n, Q)u \| = \| u \| \quad (3.29)
\]

**Proof of the lemma 3.2**

\( \text{3.29} \) is a consequence of:

\[
\int_{\mathbb{R}^3} Q(f, f)(\bar{p})d\bar{p} = 0, \quad \forall f \in L^1_2(\mathbb{R}^3)\quad (3.30)
\]

\( \text{3.30} \) is established exactly as formula (3.30) in the proof of lemma 3.5 in [24], replacing everywhere in that proof, \( a^3 \) by \( ab^2 \), and since we did, in the present paper, the same assumptions and we have the same properties as those used in [24]. Now deduce \( \text{3.31} \) from \( \text{3.30} \) exactly as in the proof of lemma 3.5 in [24], choosing this time \( B = \text{Diag}(a, b, b) \) instead of \( B = \text{Diag}(a, a, a) \) and using \( a \geq \frac{3}{7}, b \geq \frac{3}{7} \), to conclude the proof of lemma 3.2 above.

Now \( \text{3.29} \) is exactly the equality \( \text{3.11} \) in proposition 3.3 in [23]. We then prove exactly as for proposition 3.3 in [23], that all the other relations hold in the present case. The proofs of points 2) a) and 2) b) of the above proposition 3.1 are exactly the same as the proofs of proposition 4.1 and theorem 4.1 in [23], just replacing \([0, +\infty) \) by \([t_0, t_0 + T]\). This completes the proof of proposition 3.1 above, which gives directly theorem 3.1.

### 4 Existence Theorem of the Einstein Equations

#### 4.1 Expression and Reduction of the Einstein Equations

We express the Einstein equations (2.14) in terms of \( a \) and \( b \). We have to compute the Ricci tensor \( R_{\alpha\beta} \) given by (2.16). The expression (2.12) of \( \Gamma_{\alpha\beta} \)
shows, using (2.16), that the only non-zero components of the Ricci tensor are the $R_{00}$ and $R_{22} = R_{33}$. Then, it will be enough to compute $R_{00} = R_{11} = R_{1,1}^\lambda$ and $R_{22} = R_{2,2}^\lambda$. The expression (2.12) of $\Gamma_{\alpha\beta}^\lambda$ and formulae (2.16) give:

$$
R^0_{0,00} = 0; \quad R^1_{0,10} = -\frac{\ddot{a}}{a}; \quad R^2_{0,20} = R^3_{0,30} = -\frac{\ddot{b}}{b}; \\
R^0_{1,01} = a\ddot{a}; \quad R^1_{1,11} = 0; \quad R^2_{1,21} = R^3_{1,31} = a\frac{\dot{b}}{b} \\
R^0_{2,02} = b\ddot{b}; \quad R^1_{2,12} = b\frac{\dot{a}}{a}; \quad R^2_{2,22} = 0; \quad R^3_{2,32} = (\dot{b})^2
$$

We then have:

$$
R_{00} = -\frac{\ddot{a}}{a} - 2\frac{\ddot{b}}{b}; \quad R_{11} = a\ddot{a} + 2a\frac{\dot{a}}{a}; \quad R_{22} = R_{33} = b\ddot{b} + b\frac{\dot{a}}{a} + (\dot{b})^2
$$

We can then compute the scalar curvature $R$ to be:

$$
R = g^{\alpha\beta} R_{\alpha\beta} = 2 \left[ \frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} + 2\frac{\dot{a}\dot{b}}{ab} + \left( \frac{\dot{b}}{b} \right)^2 \right]
$$

The Einstein equations (2.14) in $a$ and $b$ then take the reduced form:

$$
2\frac{\dot{a}\dot{b}}{ab} + \left( \frac{\dot{b}}{b} \right)^2 = 8\pi T_{00} \tag{4.1}
$$

$$
-a^2 \left[ 2\frac{\ddot{b}}{b} + \left( \frac{\dot{b}}{b} \right)^2 \right] = 8\pi T_{11} \tag{4.2}
$$

$$
-b^2 \left[ \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} \right] = 8\pi T_{22} \tag{4.3}
$$

in which $T_{\alpha\beta} = g_{\alpha\lambda} g_{\beta\nu} T^{\lambda\nu}$ is given in terms of $f$ by (2.14).

In this paragraph, we suppose $f$ fixed in $C[I; X_r]$, where $I = [0, T], T > 0$, with $f(0) = f_0 \in L^1_2(\mathbb{R}^3), f_0 \geq 0$ a.e. and $r > \|f_0\|$, $r$ fixed. We study the initial value problem for the non-linear second order system (4.1)-(4.2)-(4.3), in $a$ and $b$, with initial data $a_0$, $b_0$, $\dot{a}_0$, $\dot{b}_0$; i.e

$$
a(0) = a_0; \quad b(0) = b_0; \quad \dot{a}(0) = \dot{a}_0; \quad \dot{b}(0) = \dot{b}_0 \tag{4.4}
$$
4.2 Compatibility

We notice that, if $S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is the Einstein tensor, we have

$$S_{0i} = 0; \quad S_{ij} = 0 \quad \text{for} \quad i \neq j; \quad S_{22} = S_{33}$$

then, the compatibility of the Einstein equations require that:

$$T_{0i} = 0; \quad T_{ij} = 0 \quad \text{for} \quad i \neq j; \quad T_{22} = T_{33} \quad (4.5)$$

But the matter tensor $T_{\alpha\beta} = g_{\alpha\lambda}g_{\beta\mu}T^{\lambda\mu}$ is defined by (2.8) in terms of the distribution function $f$. It then appears that (4.5) are in fact conditions to impose to $f$. We prove:

**Proposition 4.1** Let $G$ be the sub-group of $O(3)$ whose elements are on the form:

$$M_\epsilon(\theta) = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}; \quad \epsilon^2 = 1, \quad \theta \in \mathbb{R}$$

Assume the hypotheses of theorem 3.1 for $t_0 = 0$. Assume in addition that $f_0$ is invariant by $G$, and that the collision kernel $A$ satisfies:

$$A(a, b, \bar{p}, M\bar{q}, M\bar{p}', M\bar{q}') = A(a, b, \bar{p}, \bar{q}, \bar{p}', \bar{q}'), \quad \forall \bar{p}, \bar{q} \in \mathbb{R}^3, \quad \forall M \in G \quad (4.6)$$

then

1) The solution $f$ of the integral equation on $[0, T]$ satisfies:

$$f(t, M\bar{p}) = f(t, \bar{p}) \forall t \in [0, T], \quad \forall \bar{p} \in \mathbb{R}^3, \quad \forall M \in G \quad (4.7)$$

2) The matter tensor $T_{\alpha\beta}$ satisfies the conditions (4.5).

**Proof of Proposition 4.1:** Let $M \in G$

1) One verifies easily, that the scalar product $<,>$ defined by (2.2) and, consequently, $p^0(\bar{p})$ defined by (2.8) are invariant by $G$, which means

$$< M\bar{p}, M\bar{q} > = < \bar{p}, \bar{q} >, \quad p^0(M\bar{p}) = p^0(\bar{p}), \quad \forall M \in G.$$ 

Now (3.18) gives, since $f_0(M\bar{p}) = f_0(\bar{p})$:

$$f(t, M\bar{p}) = f_0(\bar{p}) + \int_0^t \frac{1}{p^0}Q(f, f)(s, M\bar{p})ds, \quad \forall t \in [0, T] \quad (4.8)$$

Next, definition (3.1), (3.2), (3.3) of the collision operator $Q$ gives:

$$Q(f, f)(s, M\bar{p}) = \int_{\mathbb{R}^3} \frac{ab^2}{q^0} d\bar{q} \int_{S^2} [f(s, \bar{p}', f(s, \bar{q}') - f(s, M\bar{p})f(s, \bar{q})]A(a, b, M\bar{p}, \bar{q}, \bar{p}', \bar{q}')d\omega \quad (4.9)$$

Let us set in (4.9) $\bar{q} = M\bar{q}_1; \omega = M\omega_1$. Then, formulae (3.12) give, using expression (3.13) of $R$ and the invariance of the scalar product $<,>$ by the subgroup $G$ of $O(3)$, $\forall M \in G$:

$$\begin{cases} \bar{p}' = M\bar{p} + R(M\bar{p}, \bar{q}, \omega)\omega = M\bar{p} + R(M\bar{p}, M\bar{q}_1, M\omega_1)M\omega_1 = M(\bar{p} + R(\bar{p}, \bar{q}_1, \omega_1)\omega_1) \\
\bar{q}' = \bar{q} - R(M\bar{p}, \bar{q}, \omega)\omega = M\bar{q}_1 - R(M\bar{p}, M\bar{q}_1, M\omega_1)M\omega_1 = M(\bar{q}_1 - R(\bar{p}, \bar{q}_1, \omega_1)\omega_1) \end{cases}$$

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So that $\bar{p}' = M\bar{p}_1'$; $\bar{q}' = M\bar{q}_1'$ where:

$$\bar{p}_1' = \bar{p} + R(\bar{p}, \bar{q}_1, \omega_1)\omega_1; \quad \bar{q}_1' = \bar{q}_1 - R(\bar{p}, \bar{q}_1, \omega_1)\omega_1.$$ 

Then \(4.6\) implies, using assumption \(4.6\) on \(A\), $q^0 = q_1^0$, and the invariance of the volume elements $d\bar{q}$, $d\omega$ by $G$:

$$Q(f, f)(s, M\bar{p}) = Q[f(s) o M, f(s) o M](\bar{p})$$

\(4.8\) then writes:

$$(f(t) o M)(\bar{p}) = f_0(\bar{p}) + \int_0^t \frac{1}{\bar{p}^0}Q[f(s) o M, f(s) o M](\bar{p})ds$$

which shows, by setting $h(s) = f(s) o M$, that $\|h(s)\| = \|f(s)\|$ (by setting $\bar{q} = M\bar{p}$ in the integral defining $\|f(s) o M\|$), and that, $h$ and $f$ are 2 solutions in $C[0, T; X_\nu]$ of the integral equation \(4.13\). The uniqueness theorem 3.1 then implies that $h = f$ and the point 1 of proposition 4.1 is proved.

2) Consider the expression \(2.15\) of $T^{\alpha\beta}$ in which $f$ satisfies \(4.14\) and observe that, if $\bar{p} = M\bar{q}$ where $M \in G$, then $p^0(\bar{p}) = p^0(M\bar{q}) = p^0(\bar{q}) = q^0$

(i) Set in \(2.15\) $\alpha = 0$, $\beta = i$ with $i = 1, 2, 3$; now compute the integral using the change of variables $\bar{p} = M_{-1}(\pi)\bar{q}$; where $M_{\nu}(\theta)$ is defined in prop.4.1; the integral in $\bar{q}$ gives $T^{0i} = -T^{0j}$, $i = 1, 2, 3$; then $T^{0i} = 0$, $i = 1, 2, 3$ and this implies $T_{0i} = 0$, $i = 1, 2, 3$.

(ii) Set in \(2.15\) $\alpha = i$, $\beta = j$, $i \neq j$; now compute the integral using the change of variables $\bar{p} = M_{-1}(\pi)\bar{q}$ if $(i = 1, j = 2)$ or $(i = 1, j = 3)$ and $\bar{p} = M_{1}(\pi)\bar{q}$ if $(i = 2, j = 3)$, to obtain: $T^{ij} = -T^{ij}$ if $i \neq j$; then $T^{12} = T^{13} = T^{23} = 0$ and $T_{12} = T_{13} = T_{23} = 0$.

(iii) Set in \(2.15\) $\alpha = 2$ and compute the integral using the change of variables $\bar{p} = M_{1}(\pi)\bar{q}$. The integral in $\bar{q}$ gives $T^{22} = T^{33}$ and hence $T_{22} = T_{33}$.

This completes the proof of proposition 4.1

In all what follows, we assume that the collision kernel $A$ satisfies assumption \(4.6\), and that $f(t)$ is invariant by $G$. When we will study the coupled Einstein-Boltzmann system, we know by prop.4.1 that it will be enough that $f_0$ be invariant by $G$. Also notice that the kernel $A$ defined in remark 3.1 is an example of kernel satisfying assumption \(4.6\).

### 4.3 The Constraint Equations

The Einstein equations \(2.14\) in which $G = 1$, give, using $S^{\alpha\beta}$:

$$S^0_\alpha - 8\pi T^0_\alpha = 0$$

\(4.10\)

It is proved in \(1\), p.39, that, in the general case, the four quantities:

$H^0_\alpha = S^0_\alpha - 8\pi T^0_\alpha$ do not involve the second derivative with respect to $t$, of the
metric tensor $g_{\alpha\beta}$ and, using the identities $\nabla_\alpha (S^{\alpha\beta} - 8\pi T^{\alpha\beta}) = 0$, that the quantities $H_0^\alpha$ satisfy a linear homogeneous first order differential system. Consequently:

1°) For $t = 0$, the quantities $S^0_\alpha - 8\pi T^0_\alpha$ express uniquely in terms of the initial data $a_0, a_0, \dot{b}_0, b_0$ and $f_0$.

2°) If $(S^0_\alpha - 8\pi T^0_\alpha)(0) = 0$, then $(S^0_\alpha - 8\pi T^0_\alpha)(t) = 0$ in the whole existence domain of the solution $(a, b)$ of the Einstein equations (4.1) - (4.2) - (4.3). In other words, (4.10) is satisfied if the initial data satisfied the constraint

$$S^0_\alpha(0) = 8\pi T^0_\alpha(0)$$

For this reason, the equations (4.10) are called constraint equations. Notice that, since $S^0_i = 0$ and $T^0_i = 0$ by prop.4.1, equations (4.10) for $\alpha = i$ which are called momentum constraints, are automatically satisfied. It then remains equation (4.10) for $\alpha = 0$ which is called the Hamiltonian constraint. Now $S^0_0 - 8\pi T^0_0 = 0$ is equivalent to:

$$S^{00} = 8\pi T^{00}$$

which is exactly (4.11), since $S_{00} = S^{00}$; $T_{00} = T^{00}$. So, (4.11) is the Hamiltonian constraint which is satisfied if it is the case for $t = 0$, i.e., if the initial data $a_0, b_0, a_0, \dot{b}_0, f_0$ satisfy, using (2.15) for $\alpha = \beta = 0$, (4.11), the initial constraint

$$2 \frac{\dot{a}_0 \dot{b}_0}{a_0 b_0} \left( \frac{\dot{b}_0}{b_0} \right)^2 = 8\pi \int_{\mathbb{R}^3} p^0(0)f_0(\bar{p})a_0 b_0^2 d\bar{p}$$

with, using (2.3) and (4.4),

$$p^0(0) = \sqrt{1 + a_0^2 p_1^2 + b_0^2(p)^2 + (\bar{p})^2}.$$ 

It appears that, the choice of the four independent initial data $a_0, b_0, \dot{b}_0, f_0$ uniquely determines $\dot{a}_0$. We will take:

$$a_0 > 0, \quad b_0 > 0, \quad \dot{b}_0 > 0, \quad f_0 \in L^1_2(\mathbb{R}^3), \quad f_0 \geq 0 \text{ a.e}; \quad \dot{a}_0 > 0.$$ (4.12)

where $\dot{a}_0 > 0$ is obtained by taking $b_0$ sufficiently large.

Notice that, in the 3+1 formulation of the Einstein equations, the Hamiltonian constraint (4.11) writes:

$$(trK)^2 - K_{ij}K^{ij} = 16\pi T_{00}$$

where $K = (K_{ij})$ is the second fundamental form induced by the metric $g$ on the hypersurfaces $S_t = \{t\} \times \mathbb{R}^3$. $K_{ij}$ is given in the present case by $K_{ij} = -\frac{1}{2}\partial_i g_{ij}$, which gives, using (2.1)

$$K_{11} = -a\dot{a}; \quad K_{22} = K_{33} = -bb; \quad K_{ij} = 0 \quad if \quad i \neq j$$ (4.13)

and $trK = g^{ij}K_{ij}$ is the trace of $K$; $trK$ which represents the mean curvature of the space-time is then given by:

$$trK = - \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right)$$
4.4 The Evolution Equations

The Einstein evolution equations are the second order non-linear o.d.e (4.2)-(4.3) in $a$ and $b$, we study the initial values problem for this system on $[0, T]$, $T > 0$ with initial data defined by (4.4) and satisfying the initial constraint (4.12). As we explained above, we are funded to consider the Hamiltonian constraint (4.1) or (4.14), as auxiliary equation.

In order to apply standard results, we are going to show that the Einstein evolution equations are equivalent to a non-linear first order system. We consider the system (4.1)-(4.2)-(4.3) and we set in the sources terms, with $T_{\alpha\beta}$ defined by (2.15):

$$\rho = 8\pi T_{00} \tag{4.17}$$

$$P_1 = \frac{8\pi T_{11}}{a^2} \tag{4.18}$$

$$P_2 = \frac{8\pi T_{22}}{b^2} \tag{4.19}$$

$$R = \frac{P_1 + 2P_2}{\rho} \tag{4.20}$$

$$R_+ = \frac{P_2 - P_1}{\rho} \tag{4.21}$$

Next we consider, following Rendall,A.D and Uggla,C. in [15], the change of variables:

$$H = -\frac{trK}{3} \tag{4.22}$$

$$z = \frac{1}{a^{-2} + 2b^{-2} + 1} \tag{4.23}$$

$$s = \frac{b^2}{b^2 + 2a^2} \tag{4.24}$$

$$\Sigma_+ = \frac{b}{Hb} - 1 \tag{4.25}$$

$H$ is called the Hubble variable. Finally we set:

$$\Omega = \frac{\rho}{3H^2} \tag{4.26}$$

$$q = 2\Sigma_+^2 + \frac{\Omega}{2}(1 + R) \tag{4.27}$$

$\Omega$ is called the normalized energy density and $q$, the deceleration parameter.

We have the following immediate consequences of the above definitions
Lemma 4.1

\[ 0 < z < 1; 0 < s < 1; a^2 = \frac{z}{s(1-z)}; b^2 = \frac{2z}{(1-s)(1-z)}; \Omega \geq 0; \Omega = 1 - \Sigma_+^2 \]

\[ 0 \leq P_1 + 2P_2 \leq \rho; \quad 0 \leq R \leq 1; \quad 0 \leq q \leq 2 \]

\[ p^0(s, z) = \sqrt{1 + \frac{z}{s(1-z)}(p^1)^2 + \frac{2z}{(1-s)(1-z)}[(p^2)^2 + (p^3)^2]} \]

\[ P_1 = \frac{16\pi z^2}{s^2(1-s)(1-z)^z} \int_{\mathbb{R}^3} \frac{(p^1)^2 f}{p^0(s, z)} d\vec{p} \]

\[ P_2 = \frac{32\pi z^2}{s^2(1-s)^2(1-z)^z} \int_{\mathbb{R}^3} \frac{(p^2)^2 f}{p^0(s, z)} d\vec{p} \]

Proof

• i) \[ (4.23)-(4.24) \] imply \( 0 < z < 1, 0 < s < 1 \) and solving this system in \( a^2, b^2 \), yields the solutions given in \( 4.28 \). Next, \( f(t) \geq 0, a.e. \) implies \( \rho = 8\pi T_{00} \geq 0; (4.26) \) then shows that \( \Omega \geq 0. \) For the last result in \( 4.28 \), consider \( 4.13 \) that writes, using \( 4.17 \) and \( 4.26 \):

\[ (trK)^2 - K_{ij}K^{ij} = 6\Omega H^2, \]

in which, by \( 4.15 \):

\[ K_{ij}K^{ij} = \left( \frac{\dot{a}}{a} \right)^2 + 2 \left( \frac{\dot{b}}{b} \right)^2 \]

Now use \( 4.10, 4.22 \) and \( 4.26 \) which give \( \frac{\dot{a}}{a} = -(trK) - 2\frac{\dot{b}}{b}, trK = -3H, \)
and \( \frac{\dot{b}}{b} = (\Sigma_+ + 1)H \) to express \( \frac{\dot{a}}{a}, \frac{\dot{b}}{b} \) in \( 4.31 \) and obtain \( \Omega = 1 - \Sigma_+^2 \).

• ii) To obtain the first result in \( 4.29 \), use \( 4.18, 4.19 \), definition \( 4.13 \) of \( T^\alpha_\beta \), and \( 4.26 \) which gives \( T_{22} = T_{33} \). Next, use \( 4.20 \) to obtain \( 0 \leq R \leq 1 \) and \( 4.27 \) to obtain \( 0 \leq q \leq 2 \) since \( \Omega \geq 0 \Rightarrow \Sigma_+^2 \leq 1 \).

• iii) Use expression \( 4.23 \) of \( p^0 \), expressions of \( a^2, b^2 \) in \( 4.28 \), the expressions of \( P_1, P_2 \) given by \( 4.18, 4.19 \) to obtain (4.30). This completes the proof of Lemma 4.1 \( \blacksquare \)

Now the system \( 4.1-4.3 \) writes, using the notations \( 4.17-4.18-4.19 \):

\[ 2 \frac{\dot{a}b}{ab} + \left( \frac{\dot{b}}{b} \right)^2 = \rho \]

\[ - \left[ 2 \frac{\ddot{b}}{b} + \left( \frac{\dot{b}}{b} \right)^2 \right] = P_1 \]

\[ - \left[ \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}b}{ab} \right] = P_2 \]
It shows useful to explicit the second derivatives. (4.34) gives, using (4.33) to express \( \ddot{b} \):

\[
\frac{\ddot{a}}{a} = -P_2 + \frac{P_1}{2} + \frac{1}{2} \left( \frac{\dot{b}}{\dot{b}} \right)^2 - \frac{\dot{a}\dot{b}}{ab} = -P_2 + \frac{P_1}{2} + \frac{1}{2} \left( \frac{\dot{b}}{\dot{b}} \right)^2 - 2 \frac{\dot{a}\dot{b}}{3ab} - \frac{1}{3} \frac{\dot{a}\dot{b}}{3ab}
\]

(4.35)

and we write \( \ddot{b} \) given by (4.33) on the form:

\[
\frac{\ddot{b}}{b} = -\frac{P_1}{2} \frac{1}{2} + \frac{1}{2} \left( \frac{\dot{b}}{\dot{b}} \right)^2 = -\frac{P_1}{2} \frac{1}{2} + \frac{1}{3} \left( \frac{\dot{b}}{\dot{b}} \right)^2 - \frac{1}{6} \left( \frac{\dot{b}}{\dot{b}} \right)^2
\]

(4.36)

Now, if we use (4.32) to express the last terms in (4.35) and in (4.36), we obtain the Einstein evolution equations in the form:

\[
\frac{\ddot{a}}{a} = \frac{2}{3} \left[ \left( \frac{\dot{b}}{\dot{b}} \right)^2 - \frac{\ddot{a} \dot{a}}{ab} \right] - \frac{\rho}{6} + \frac{1}{2} (P_1 - 2P_2)
\]

(4.37)

\[
\frac{\ddot{b}}{b} = \frac{1}{3} \left[ \frac{\ddot{a} \dot{a}}{ab} - \left( \frac{\dot{b}}{\dot{b}} \right)^2 \right] - \frac{\rho}{6} - \frac{P_1}{2}
\]

(4.38)

We will also use the following relation, deduced from (4.16), (4.22), (4.25):

\[
\dot{a} = (1 - 2\Sigma_+) \frac{\dot{b}}{a}
\]

(4.39)

**Remark 4.1** Assume \( H > 0 \). Then:

If \( \dot{b}(t_0) = 0 \) at a given time \( t = t_0 \), then the problem of the global existence becomes trivial.

The reason is the following: (4.30) shows that \( \dot{b} < 0 \), then \( \ddot{b} \) is a decreasing function. By (4.28), \( (\Omega = 1 - \Sigma_+^2 \geq 0) \Rightarrow (-1 \leq \Sigma_+ \leq 1) \); hence, \( \Sigma_+ + 1 \geq 0 \), and by (4.28), \( \dot{b} \geq 0 \).

But \( \dot{b} \geq 0, \ddot{b} \) decreasing, \( \ddot{b}(t_0) = 0 \) \( \Rightarrow \dot{b}(t) = 0 \) for \( t \geq t_0 \). Hence \( \ddot{b} \) is constant and \( \ddot{b}(t) = \ddot{b}(t_0) \) for \( t \geq t_0 \).

Then, by (4.25), \( \Sigma_+(t) = -1 \) for \( t \geq t_0 \), and by (4.28), \( \Omega(t) = 0 \) for \( t \geq t_0 \); (4.26) then implies \( \rho(t) = 0 \), for \( t \geq t_0 \), and since \( 0 \leq P_1 + 2P_2 \leq \rho \), \( P_1(t) = P_2(t) = 0 \) for \( t \geq t_0 \). The Einstein equations (4.32), (4.33) are trivially satisfied for \( t \geq t_0 \) (since \( \dot{b} = 0 \)), and by (4.34), \( \ddot{a}(t) = 0 \) for \( t \geq t_0 \). Then: \( a(t) = C_1 t + C_2 \); \( C_1, C_2 \) constants.

Notice that \( \rho(t) = 0 \) for \( t \geq t_0 \) \( \Rightarrow \) \( f(t) = 0 \) for \( t \geq t_0 \).

**Conclusion:** For \( t \geq t_0 \), the space-time becomes empty and the problem of global existence becomes trivial.

In what follows, we will show that \( H > 0 \) if \( H(0) > 0 \), and we look for \( b \) such that \( \dot{b} > 0 \); consequently, by (4.26), \( \Sigma_+ + 1 > 0 \).
Proposition 4.2 The Einstein evolution equations (4.36)-(4.37) in a, b, are equivalent to the following non-linear first order system in \((H, s, z, \Sigma_+):\)

\[\begin{align*}
\frac{dH}{dt} &= -(1 + q)H^2 \\
\frac{ds}{dt} &= 6s(1 - s)\Sigma_+H \\
\frac{dz}{dt} &= 2z(1 - z)(1 + \Sigma_+ - 3s\Sigma_+)H \\
\frac{d\Sigma_+}{dt} &= -(2 - q)\Sigma_+H + \Omega R + H
\end{align*}\]

(IV.1)

**Proof:**

1. Suppose we have (4.37)-(4.38).

   - i) (4.22) and (4.16) give:
     \[
     \frac{dH}{dt} = \frac{1}{3} \left( \frac{\ddot{a}}{a} + 2 \frac{\ddot{b}}{b} - \left( \frac{\dot{a}}{a} \right)^2 - 2 \left( \frac{\dot{b}}{b} \right)^2 \right)
     \]
     which gives, using (4.37)-(4.38), (4.31) and (4.14):
     \[
     \frac{dH}{dt} = \frac{1}{3} \left( \rho + \frac{P_1}{2} + \frac{P_2}{2} \right) + \frac{1}{3} \left[ (trK)^2 - 2\rho \right].
     \]
     (4.40) then follows from (4.20), (4.22), (4.26) and (4.27).

   - ii) Definition (4.24) of \(s\) gives 1 \(-\) \(s = \frac{a^2}{a^2 + 2a^2} \) and we can write:
     \[
     \frac{ds}{dt} = 2s(1 - s)\frac{\dot{b}}{b} \left[ 1 - \frac{\dot{a}}{a} \left( \frac{\dot{b}}{b} \right)^{-1} \right]
     \]
     we then deduce from (4.25) and (4.39) that:
     \[
     \frac{ds}{dt} = 2s(1 - s)(\Sigma_+ + 1)H \left( 1 - \frac{(1 - 2\Sigma_+)H}{(\Sigma_+ + 1)H} \right) = 6s(1 - s)\Sigma_+H
     \]

   - iii) Definition (4.23) of \(z\) gives:
     \[
     \frac{dz}{dt} = 2z^2 \left( \frac{\dot{a}}{a} \frac{1}{a^2} + 2 \frac{\dot{b}}{b} \frac{1}{b^2} \right).
     \]
     (4.42) then follows from (4.39), (4.25) and (4.28).

   - iv) We can write, using (4.25) and (4.40):
     \[
     \frac{d\Sigma_+}{dt} = \frac{d}{dt} \left( \frac{b}{H} \right) = \frac{\dot{b}}{b}(1 + q) + \frac{1}{H} \left[ \frac{\dot{b}}{b} - \left( \frac{\dot{b}}{b} \right)^2 \right]
     \]
Now the evolution equation (4.38) gives, using (4.25) and (4.39)

\[
\frac{\ddot{b}}{b} - \left(\frac{\dot{b}}{b}\right)^2 = \frac{\dot{b}}{b} \left(\frac{1}{3} \frac{\dot{a}}{a} - 4 \frac{\dot{b}}{b}\right) - \rho \frac{\dot{b}}{6} - \frac{P_1}{2} = \frac{\dot{b}}{b} (-1 - 2\Sigma_+)H - \rho \frac{\dot{b}}{6} - \frac{P_1}{2}
\]

so that, (a) gives, using once more (4.25):

\[
\frac{d\Sigma_+}{dt} = \frac{\dot{b}}{b} (q - 2\Sigma_+) - \rho \frac{\dot{b}}{6H} - \frac{P_1}{2H} = (\Sigma_+ + 1)(q - 2\Sigma_+)H - \rho \frac{\dot{b}}{6H} - \frac{P_1}{2H}
\]

Now express \( q \) in the third term by (4.27), use (4.26) that gives \( \frac{1}{H} = \frac{\Omega H}{\rho} \), and definitions (4.20) and (4.21) of \( R \) and \( R_+ \) to obtain (4.43).

2. Suppose we have (4.40)-(4.41)-(4.42)-(4.43).

- i) Define the quantities \( a^2, b^2 \) as in (4.28) and the other quantities the same way as we did. Differentiating both sides of the corresponding formula (4.39) yields:

\[
\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \dot{H}(1 - 2\Sigma_+) - 2H\dot{\Sigma}_+
\]

which gives, using (4.40) and (4.41):

\[
\frac{\ddot{a}}{a} = -(1 + q)(1 - 2\Sigma_+)H^2 - 2H^2[(q - 2)\Sigma_+ + \Omega R_+] + \left(\frac{\dot{a}}{a}\right)^2
\]

\[
= -H^2 + 6\Sigma_+H^2 + \left(\frac{\dot{a}}{a}\right)^2 - qH^2 - 2\Omega R_+H^2
\]

Now, use expression (4.27) of \( q \) to obtain

\[
\frac{\ddot{a}}{a} = \left(\frac{H}{2}\right)^2 \left(1 + R\right) + 2\Omega H^2 R_+ \]

(b)

For the second term of (b), use (4.26) which gives \( \Omega H^2 = \frac{\rho}{2} \) and the definitions (4.20) and (4.21) of \( R \) and \( R_+ \) to obtain:

\[
\frac{\Omega H^2}{2} (1 + R) + 2\Omega H^2 R_+ = -\rho + \frac{1}{2}(P_1 - 2P_2)
\]

Next, for the first term of (b), use (4.25) to express \( \Sigma_+ \), (4.16) and (4.22) that give \( H = \frac{1}{3}(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}) \) to obtain:

\[
-H^2 + 6\Sigma_+H^2 - 2\Sigma_+^2H^2 + \left(\frac{\dot{a}}{a}\right)^2 = \frac{2}{3} \left[\left(\frac{\dot{b}}{b}\right)^2 - \frac{\dot{ab}}{ab}\right]
\]

and (4.37) follows.
ii) Differentiating both sides of $b = (\Sigma_+ + 1)H$ yields:

$$\frac{\dot{b}}{b} - \left(\frac{\dot{b}}{b}\right)^2 = \dot{\Sigma}_+ H + (\Sigma_+ + 1)\dot{H}$$

which gives, using (4.40) and (4.43) and simplifying:

$$\frac{\dot{b}}{b} = (-3\Sigma_+ - 1)H^2 - qH^2 + \Omega \Sigma_+ H^2 + \left(\frac{\dot{b}}{b}\right)^2$$

Now this writes using definition (4.27) of $q$:

$$\frac{\dot{b}}{b} = \left[(-2\Sigma_+^2 - 3\Sigma_+ - 1)H^2 + \left(\frac{\dot{b}}{b}\right)^2 - \frac{\Omega H^2}{2}(1 + R) + \Omega H^2 R_+ \right]$$

(c)

For the second term of (c), use (4.26) which gives $\Omega H^2 = \frac{R}{3}$ and definitions (4.20) and (4.21) of $R$ and $R_+$ to obtain:

$$-\frac{\Omega H^2}{2}(1 + R) + \Omega H^2 R_+ = -\frac{\rho}{6} - \frac{P_1}{2}$$

Next, for the first term of (c), use (4.25) to express $\Sigma_+$ and $H = \frac{1}{3} \left(\frac{a}{\rho} + 2\frac{b}{\rho}\right)$ to obtain:

$$(-2\Sigma_+^2 - 3\Sigma_+ - 1)H^2 + \left(\frac{\dot{b}}{b}\right)^2 = \frac{1}{3} \left[\dot{a}b - \left(\frac{\dot{b}}{b}\right)^2\right]$$

and (4.38) follows. This completes the proof of Proposition 4.2.

Now observe, using expressions (4.20), (4.21) and (4.27) of $R$, $R_+$ and $q$, that these quantities express in terms of $\frac{1}{p}$ and we have nothing to bound such quantities. This is why, we give a new formulation of equations (4.40) and (4.43), that raise this problem, as follows:

**Proposition 4.3** Equations (4.40) and (4.43) can write respectively:

$$\frac{dH}{dt} = -\frac{3}{2}(1 + \Sigma_+^2)H^2 - \frac{P_1 + 2P_2}{6} \quad (4.44)$$

$$\frac{d\Sigma_+}{dt} = -\frac{3}{2}(1 - \Sigma_+^2)H\Sigma_+ + \frac{P_1}{6H}(\Sigma_+ - 2) + \frac{P_2}{3H}(\Sigma_+ + 1) \quad (4.45)$$

**Proof:**

i) Write (4.40) using definition (4.27) of $q$ in which $R$ is given by (4.20); next, use (4.28) and (4.26) which give $\Omega = 1 - \Sigma_+^2$ and $\frac{\Omega H^2}{p} = \frac{1}{3}$ to obtain (4.44).
• ii) Write (4.43) using definition (4.27) of \( q \) and apply definition (4.20) and (4.21) of \( R \) and \( R_+ \); next use once more (4.28) and (4.26) which give \( \Omega = 1 - \Sigma_+^2 \) and \( \frac{\partial H}{\partial \tau} = \frac{1}{3\tau} \) to obtain (4.45)

By prop.4.3 system (IV.1) is then equivalent to:

\[
\begin{align*}
\frac{dH}{dt} &= -\frac{1}{3}(1 + \Sigma_+^2)H^2 - \frac{P_1 + 2P_2}{6} \quad (4.44) \\
\frac{ds}{dt} &= 6s(1 - s)\Sigma_+ H \quad (4.41) \\
\frac{dz}{dt} &= -\frac{2}{3}(1 - \Sigma_+^2)H\Sigma_+ + \frac{P_1}{3\tau}(\Sigma_+ - 2) + \frac{P_2}{3\tau}(\Sigma_+ + 1) \quad (4.42) \\
\frac{d\Sigma_+}{dt} &= -\frac{3}{2}(1 - \Sigma_+^2)H \Sigma_+ + \frac{P_1}{3\tau}(\Sigma_+ + 1) + \frac{P_2}{3\tau}(\Sigma_+ - 2). \quad (4.45)
\end{align*}
\]

It shows useful in what follows to introduce the following notation.

**Definition 4.1** Let \( 0 < s < 1 \). Set:

\[
\alpha(s) = \inf(s, 1 - s) \quad (4.46)
\]

For the global existence theorem we will need the following a priori estimations.

**Proposition 4.4** Let \( \delta > 0 \) be given and let \( t_0 \in \mathbb{R}_+ \). Then any solution \((H, s, z, \Sigma_+)\) of the system (IV.2) on \([t_0, t_0 + \delta]\), with \( H(t_0) > 0 \), satisfy the following a priori estimations, \( \forall t \in [0, \delta] \):

\[
H(t_0 + t) = \frac{H(t_0)}{1 + H(t_0) \int_{t_0}^{t_0+t}(1 + q)d\tau} \quad (4.47)
\]

\[
\frac{1}{H(t_0 + t)} \leq \frac{1}{H(t_0)} e^{3H(t_0)t} \quad (4.48)
\]

\[
\frac{1}{\alpha(s(t_0 + t))} \leq \frac{1}{\alpha(s(t_0))} e^{6H(t_0)t} \quad (4.49)
\]

\[
\frac{1}{\alpha(z(t_0 + t))} \leq \frac{1}{\alpha(z(t_0))} e^{10H(t_0)t} \quad (4.50)
\]

\(-1 < \Sigma_+(t_0 + t) \leq \frac{1}{2} \quad (4.51)
\]

**Proof:** Let \( t \in [0, \delta] \). By prop.4.3 for \( H \) we consider (4.40) or (4.44), and for \( \Sigma_+ \) we consider (4.43) or (4.45).

1. (4.40) writes: \(-\frac{dH}{dt} = (1 + q)dt\); integrating both sides on \([t_0, t_0 + t]\) yields (4.47). Notice that, using (4.40) or (4.44), \( H \) is a decreasing function.

2. By (4.29) we have \( 0 \leq q \leq 2 \); so, by (4.47), \((H(t_0) > 0) \Rightarrow (H(t_0 + t) > 0)\) and \( 0 < H(t_0 + t) \leq H(t_0) \). Now (4.40) also writes: \(-\frac{dH}{dt} = (1 + q)Hdt\); integrating both sides on \([t_0, t_0 + t]\) and using \( 0 \leq q \leq 2 \), \( H(t_0 + t) \leq H(t_0) \) yields (4.48).
3. (4.41) gives:
\[
\frac{ds}{s} = 6(1-s)\Sigma_+H|dt; \quad \frac{d(1-s)}{1-s} = |6s\Sigma_+H|dt
\]

Now integrate on \([t_0, t_0 + t]\), using \(|\Sigma_+| \leq 1, 0 < s < 1, 0 < H \leq H(t_0)\) to obtain:
\[
\begin{align*}
& e^{-6H(t_0)t} \leq \frac{s(t_0)}{s(t_0) - 1} \leq e^{6H(t_0)t} \\
& e^{-6H(t_0)t} \leq \frac{1-s(t_0)}{1-s(t_0)} \leq e^{6H(t_0)t}
\end{align*}
\]

We then deduce, using definition (4.46) of \(\alpha(s)\) that:
\[
\alpha(s(t_0))e^{-6H(t_0)t} \leq \alpha(s(t_0) + t))
\]
and (4.49) follows.

4. To obtain (4.50), proceed as above, using this time (4.42) which gives:
\[
\frac{dz}{z} = [2(1-z)(1+\Sigma_+-3s\Sigma_+)H]dt; \quad \frac{d(1-z)}{1-z} = |2z(1+\Sigma_+-3s\Sigma_+)H|dt
\]

Notice that the inequalities (4.48), (4.49) and (4.50) don’t involve \(f\), because we used (4.40) in which \(0 \leq q \leq 2\) and (4.41), (4.42), which contain no source terms of the Einstein equations.

5. To prove (4.51), we show that, for every point \(t_1 \in [t_0, t_0 + \delta]\),
\[
(\Sigma_+(t_1) = \frac{1}{2}) \implies (\dot{\Sigma}_+(t_1) \leq 0)
\]

Since (a) will shows that \(\frac{1}{2}\) is an upper bound for all the values \(\Sigma_+(t)\), \(t \in [t_0, t_0 + \delta]\). We use equation (4.45) in \(\Sigma_+.\) Evaluating both sides of (4.45) at \(t = t_1\) gives, using \(\Sigma_+(t_1) = \frac{1}{2}:\)
\[
\dot{\Sigma}_+(t_1) = \frac{1}{H}(-\frac{9}{16}H^2 - \frac{P_1}{4} + \frac{P_2}{2})
\]

Now, the relation \(\Omega = 1 - \Sigma_+^2\) gives \(\Omega(t_1) = 1 - \frac{1}{4} = \frac{3}{4}\). We then deduce from (4.26) that, for \(t = t_1, H^2 = \frac{4\rho}{3}\), and (b) gives:
\[
\dot{\Sigma}_+(t_1) = \frac{1}{H}\left(-\rho - \frac{P_1}{4} + 2P_2\right)
\]
Now, the inequalities \(0 \leq P_1 + 2P_2 \leq \rho\) in (4.29) give : \(2P_2 \leq \rho - P_1\) so that (c) implies:
\[
\dot{\Sigma}_+(t_1) = \frac{1}{H}(-\frac{P_1}{2}) \leq 0
\]
since \(P_1 \geq 0;\) and (a) follows. This completes the proof of proposition 4.4.

\[\blacksquare\]
We deduce:

**Proposition 4.5** Let \( f \in C(0;T;X_r), T > 0 \), be given. Suppose the initial value problem for the system (IV.2), with initial data \((H_0, s_0, z_0, \Sigma_0)\), with \( H_0 > 0 \), \( 0 < s_0 < 1 \), \( 0 < z_0 < 1 \), \(-1 < \Sigma_0 \leq \frac{1}{2}\), at \( t = 0 \), has a solution \((\tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_+)\) on \([0,t_0]\) with \( 0 \leq t_0 < T \). Then, any solution \((H, s, z, \Sigma_+)\) of the initial value problem (IV.2) on \([t_0, t_0 + \delta]\), \( \delta > 0 \), with initial data \((H, s, z, \Sigma_+) (t_0) = (\tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_+) (t_0)\), at \( t = t_0\), satisfy, for \( t \in [0, \delta] \), the inequalities:

\[
\frac{1}{H(t_0 + t)} \leq \gamma_0 e^{3H_0(T+\delta)} \quad (4.52)
\]

\[
\frac{1}{\alpha(s(t_0 + t))} \leq \gamma_0 e^{6H_0(T+\delta)} \quad (4.53)
\]

\[
\frac{1}{\alpha(z(t_0 + t))} \leq \gamma_0 e^{10H_0(T+\delta)} \quad (4.54)
\]

where

\[
\gamma_0 = \left( \frac{1}{H_0} + \frac{1}{s_0} + \frac{1}{z_0} \right) \quad (4.55)
\]

**Proof:** The hypothesis on \( f \) implies that system (IV.2) is defined on \([0,T]\). Apply prop.4.4. Notice that by (4.40), \( \dot{H} \) is a decreasing function; so (4.48), (4.49), (4.50) give in the present case, since \( \dot{H}(t_0) \leq H_0 \):

\[
\begin{cases}
\frac{1}{H(t_0 + t)} \leq \frac{1}{H(t_0)} e^{3H_0 t} \leq \frac{1}{\alpha(s(t_0 + t))} \leq \frac{1}{\alpha(s(t_0))} e^{6H_0 t} \\
\frac{1}{\alpha(z(t_0 + t))} \leq \frac{1}{\alpha(z(t_0))} e^{10H_0 t} 
\end{cases} \quad t \in [0, \delta] \quad (a)
\]

Now apply prop.4.4 to the solution \((\tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_+)\) of (IV.2) on \([0,t_0]\), with \((\tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_+) (0) = (H_0, s_0, z_0, \Sigma_0);\) then (4.48), (4.49), (4.50) in which we take \( t_0 = 0 \) and \( t = t_0 \) give:

\[
\frac{1}{H(t_0)} \leq \frac{1}{H_0} e^{3H_0 t_0} \quad \frac{1}{\alpha(s(t_0))} \leq \frac{1}{\alpha(s(t_0))} e^{6H_0 t_0} \quad \frac{1}{\alpha(z(t_0))} \leq \frac{1}{\alpha(z(t_0))} e^{10H_0 t_0} \quad (b)
\]

(4.52), (4.53), (4.54) then follow from (a), (b), \( t_0 < T, t \leq \delta \) and the definition (4.55) of \( \gamma_0 \). This completes the proof of prop.4.5 \( \blacksquare \)

**Corollary 4.1** Any solutions \( a, b \) of the Einstein equations (4.1)-(4.2)-(4.3) on \([t_0, t_0 + \delta], t_0 \in \mathbb{R}_+, \delta \geq 0\), are increasing functions.

**Proof:** (4.51) shows that \( 1 - 2\Sigma_+ \geq 0 \) on \([t_0, t_0 + \delta]\); then (4.39) implies \( \dot{a} > 0 \), since \( a > 0 \) and \( H > 0 \); and \( a \) is an increasing function. Next, (4.25) gives: \( \frac{\dot{b}}{b} = (\Sigma_+ + 1)H \) and since \( \Sigma_+ + 1 \geq 0, H \geq 0, b > 0 \) this implies \( \dot{b} > 0 \) and \( b \) is an increasing function. \( \blacksquare \)
Remark 4.2 By corollary 4.2, if we take in (4.4) the initial data \( a_0, b_0 \), such that
\[
a_0 \geq \frac{3}{2}; \quad b_0 \geq \frac{3}{2}
\]
then \( a \) and \( b \) satisfy the hypothesis of the existence theorem 3.1 of solutions for the Boltzmann equation.

We now prove the existence theorem for the Einstein equations, which are equivalent, using prop 4.3 to the system (IV.2). Following Prop 4.4 we will take the initial datum \( H_0 \) for \( H \) such that \( H_0 > 0 \), and denote the initial data for the system (IV.2) at \( t = 0 \), \( (H_0, s_0, z_0, \Sigma_{+0}) \), where:
\[
H_0 > 0; \quad 0 < s_0 < 1; \quad 0 < z_0 < 1; \quad -1 < \Sigma_{+0} \leq \frac{1}{2}
\]
We will apply the standard theory on the first order differential system. For this purpose, we will have to study the function defined by the r.h.s of the system (IV.2), i.e, the function \( F \) defined by:
\[
F(t, H, s, z, \Sigma_+) = \begin{cases} 
-\frac{2}{3}(1 + \Sigma_+^2)H^2 - \frac{E_0 + Ep}{6} & \\
\frac{2}{3}(1 - z)(1 + \Sigma_+ - 3s\Sigma_+)H & \\
-\frac{2}{3}(1 - \Sigma_+^2)H\Sigma_+ + \frac{P_0}{\delta^n}(\Sigma_+ - 2) + \frac{P_0}{\delta^n}(\Sigma_+ + 1)
\end{cases}
\]
\[(4.57)\]
(4.40) shows that \( H \) is a decreasing function, so that \( H(t) \leq H_0, t \geq 0 \), whenever \( H \) exists. We are then funded to suppose, using (4.28), (4.51), that \( F \) is defined for:
\[
(H, s, z, \Sigma_+) \in B = [0, H_0] \times [0, 1] \times [0, 1] \times [-1, \frac{1}{2}]
\]
We will have to prove that \( F \) is continuous in \( t \) and locally Lipschitzian in \( X = (H, s, z, \Sigma_+) \), with respect to the norm of \( \mathbb{R}^4 \) we take to be
\[
\|X\| = |H| + |s| + |z| + |\Sigma_+|.
\]
\( F \) depends on \( t \) through \( f \) in \( P_1 \) and \( P_2 \) (see their expressions in (4.30)); but, by hypothesis, \( f \) is continuous in \( t \) and so is \( F \).

A glance to (4.57) shows that the real problem will be to prove that \( P_1 \) and \( P_2 \) defined in (4.30) are locally Lipschitzian.

We then begin by proving:

Lemma 4.2 Suppose \( f \in C \left( \{t_0, t_0 + \delta\}; L^1_2(\mathbb{R}^3) \right), t_0 \geq 0, \delta > 0. \)

Let \( s, s_1, s_2, z, z_1, z_2 \in [0, 1], H_1, H_2 \in [0, H_0], t \in [t_0, t_0 + \delta]. \) then:
\[
|P_i(s_1, z_1, f) - P_i(s_2, z_2, f)| \leq \frac{C\|f(t)\|(|s_1 - s_2| + |z_1 - z_2|)}{\alpha^3(s_1)\alpha^3(s_2)\alpha^3(z_1)\alpha^3(z_2)}; \quad i = 1, 2 \quad (4.59)
\]
\[
|P_i(s, z, f)| \leq \frac{C\|f(t)\|}{\alpha^3(s)\alpha^3(z)}; \quad i = 1, 2 \quad (4.60)
\]
\[
\left| \frac{P_i(s_1, z_1, f)}{H_1} - \frac{P_i(s_2, z_2, f)}{H_2} \right| \leq \frac{C\|f(t)\|(|s_1 - s_2| + |z_1 - z_2| + |H_1 - H_2|)}{H_2(1 + H_1)\alpha^3(s_1)\alpha^3(s_2)\alpha^3(z_1)\alpha^3(z_2)}; \quad i = 1, 2 \quad (4.61)
\]
where \( C > 0 \) is a constant.
Proof: Consider the expressions of \( p^0, P_1, P_2 \) in (4.30). Let us begin by bounding the differences involved in \( P_i(s_1, z_1, f) - P_i(s_2, z_2, f), i = 1, 2 \). We have:

\[
\frac{1}{p^0(s_1, z_1)} - \frac{1}{p^0(s_2, z_2)} = \frac{\left( \frac{\tilde{z}_1}{s_2(1-z_2)} - \frac{\tilde{z}_1}{s_1(1-z_1)} \right) (p^1)^2}{p^0(s_1, z_1)p^0(s_2, z_2) (p^0(s_1, z_1) + p^0(s_2, z_2))} + \frac{\left( \frac{2z_2}{(1-s_2)(1-z_2)} - \frac{2z_1}{(1-s_1)(1-z_1)} \right) (p^2)^2 + (p^3)^2}{p^0(s_1, z_1)p^0(s_2, z_2) (p^0(s_1, z_1) + p^0(s_2, z_2))}
\]

(a)

But \( p^0(s_1, z_1)p^0(s_2, z_2) (p^0(s_1, z_1) + p^0(s_2, z_2)) \geq (p^0(s_1, z_1))^2 p^0(s_2, z_2) \) and (4.30) shows that \((p^0(s_1, z_1))^2 p^0(s_2, z_2)\) is bounded from below by each one of the following four quantities, that will serve to bound \( P_1 \) and \( P_2 \) respectively:

\[
\left\{ \begin{array}{l}
\frac{\tilde{z}_1}{s_1(1-z_1)} - \frac{\tilde{z}_1}{s_2(1-z_2)} \leq \left( \frac{\tilde{z}_1}{s_1(1-z_1)} - \frac{\tilde{z}_1}{s_2(1-z_2)} \right) \left| p^1 \right|^3 \\
\frac{2z_2}{(1-s_2)(1-z_2)} - \frac{2z_1}{(1-s_1)(1-z_1)} \leq \left( \frac{2z_2}{(1-s_2)(1-z_2)} - \frac{2z_1}{(1-s_1)(1-z_1)} \right) \left| p^1 \right| \left( (p^2)^2 + (p^3)^2 \right)
\end{array} \right.
\]

(b)

\[
\left\{ \begin{array}{l}
\frac{\tilde{z}_1}{s_1(1-z_1)} - \frac{\tilde{z}_1}{s_2(1-z_2)} \leq \left( \frac{\tilde{z}_1}{s_1(1-z_1)} - \frac{\tilde{z}_1}{s_2(1-z_2)} \right) \left| p^1 \right|^2 p^2 \\
\frac{2z_2}{(1-s_2)(1-z_2)} - \frac{2z_1}{(1-s_1)(1-z_1)} \leq \left( \frac{2z_2}{(1-s_2)(1-z_2)} - \frac{2z_1}{(1-s_1)(1-z_1)} \right) \left| p^1 \right| \left( (p^2)^2 + (p^3)^2 \right)
\end{array} \right.
\]

(c)

quantities (b) will serve for \( P_1 \) and (c) for \( P_2 \). Next, we have:

\[
\left\{ \begin{array}{l}
\left| \frac{\tilde{z}_1}{s_1(1-z_1)} - \frac{\tilde{z}_1}{s_2(1-z_2)} \right| \leq C(|s_1 - s_2| + |z_1 - z_2|) \\
\left| \frac{2z_2}{(1-s_2)(1-z_2)} - \frac{2z_1}{(1-s_1)(1-z_1)} \right| \leq C(|s_1 - s_2| + |z_1 - z_2|)
\end{array} \right.
\]

(d)

(a) then gives, using (b)-(d) and (c)-(d) respectively:

\[
\left| \frac{1}{p^0(s_1, z_1)} - \frac{1}{p^0(s_2, z_2)} \right| \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{s_1 s_2 (1-s_1)(1-s_2) z_1 z_2} \left| p^1 \right| \\
\left| \frac{1}{p^0(s_1, z_1)} - \frac{1}{p^0(s_2, z_2)} \right| \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{s_1 s_2 (1-s_1)(1-s_2) z_1 z_2} \left| p^2 \right|
\]

(e)

(f)

We now bound the differences in the coefficients of the integrals in \( P_1, P_2 \) which are given by (4.30). Concerning \( P_1 \), we have:

\[
\frac{1}{s_1^2 (1-s_1)} \left( \frac{z_1}{1-z_1} \right)^{\frac{2}{(1-z_2)}} + \frac{1}{s_2^2 (1-s_2)} \left( \frac{z_1}{1-z_1} \right)^{\frac{2}{(1-z_2)}} = \\
\left( \frac{1}{s_1^2 (1-s_1)} - \frac{1}{s_2^2 (1-s_2)} \right) \left( \frac{z_1}{1-z_1} \right)^{\frac{2}{(1-z_2)}} + \frac{1}{s_2^2 (1-s_2)} \left[ \left( \frac{z_1}{1-z_1} \right)^{\frac{2}{(1-z_2)}} - \left( \frac{z_2}{1-z_2} \right)^{\frac{2}{(1-z_2)}} \right]
\]

(g)
We can now establish (4.59). We have, using the expression of
and:
\[
\frac{1}{s_1^x(1-s_1)} - \frac{1}{s_2^y(1-s_2)} = \frac{s_2^x(1-s_2) - s_1^y(1-s_1)}{(s_2s_1)^x(1-s_1)(1-s_2)}
\]
\[
\left| \frac{\sqrt{s_2} - \sqrt{s_1}}{(s_2s_1)^x(1-s_1)(1-s_2)} \right| \leq \frac{C|s_1 - s_2|}{s_1^x s_2^y(1-s_1)(1-s_2)}
\]
and:
\[
\left| \left( \frac{z_1}{1-z_1} \right)^{\frac{1}{2}} - \left( \frac{z_2}{1-z_2} \right)^{\frac{1}{2}} \right| \leq \left| \sqrt{\left( \frac{z_1}{1-z_1} \right)^5} - \sqrt{\left( \frac{z_2}{1-z_2} \right)^5} \right| \leq \frac{C|z_1 - z_2|}{z_1^{\frac{1}{2}}(1-z_1)^{\frac{5}{2}}(1-z_2)^{\frac{5}{2}}}
\]
so that (g) gives:
\[
\left| \frac{1}{s_1^x(1-s_1)} \left( \frac{z_1}{1-z_1} \right)^{\frac{1}{2}} - \frac{1}{s_2^y(1-s_2)} \left( \frac{z_2}{1-z_2} \right)^{\frac{1}{2}} \right| \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{s_1^x s_2^y(1-s_1)^{\frac{5}{2}}(1-s_2)^{\frac{5}{2}}(1-z_1)^{\frac{5}{2}}(1-z_2)^{\frac{5}{2}}}
\]
(h)

An analogous calculation gives, for the coefficient of the integral in \( P_2 \)
\[
\left| \frac{1}{s_1^x(1-s_1)^2} \left( \frac{z_1}{1-z_1} \right)^{\frac{1}{2}} - \frac{1}{s_2^y(1-s_2)^2} \left( \frac{z_2}{1-z_2} \right)^{\frac{1}{2}} \right| \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{s_1(1-s_1)^2 s_2^y(1-s_2)^2 z_1^{\frac{1}{2}}(1-z_1)^{\frac{5}{2}}(1-z_2)^{\frac{5}{2}}}
\]
(i)
We can now establish (4.59). We have, using the expression of \( P_1 \) in (4.30)
\[
P_1(s_1, z_1, f) - P_1(s_2, z_2, f) = 16\pi \left[ \frac{1}{s_1^x(1-s_1)} \left( \frac{z_1}{1-z_1} \right)^{\frac{1}{2}} - \frac{1}{s_2^y(1-s_2)^2} \left( \frac{z_2}{1-z_2} \right)^{\frac{1}{2}} \right] \times
\]
\[
\int_{\mathbb{R}^3} f(p_{1})^2 d\vec{p} + \frac{16\pi}{s_1^2(1-s_1)} \left( \frac{z_2}{1-z_2} \right)^{\frac{1}{2}} \int_{\mathbb{R}^3} f(p_{1})^2 \left( \frac{1}{p^0(s_1, z_1)} - \frac{1}{p^0(s_2, z_2)} \right) d\vec{p}
\]

Notice that:
\[
p^0(s_1, z_1) > \left( \frac{z_1}{s_1(1-z_1)} \right)^{\frac{1}{2}} |p^1| \quad (i)
\]

(4.59) for \( P_1 \) then follows from (h), (e), (j), \(|p^1| < \sqrt{1 + |p|^2}\), the definition
(4.46) of \( \alpha(s) \), \( 0 < s < 1 \), and \( 0 < \alpha(s) < 1 \).
Now, concerning $P_2$, we have, using its expression in (4.30):

\[
P_2(s_1, z_1, f) - P_2(s_2, z_2, f) = 32\pi \left[ \frac{1}{s_1^2(1 - s_1)^2} \left( \frac{z_1}{1 - z_1} \right)^\frac{3}{2} - \frac{1}{s_2^2(1 - s_2)^2} \left( \frac{z_2}{1 - z_2} \right)^\frac{3}{2} \right] \times \int_{\mathbb{R}^3} \frac{f(p^2)^2 d\vec{p}}{p^0(s_1, z_1)} + \frac{32\pi}{s_2^\frac{1}{2}(1 - s_2)^2} \left( \frac{z_2}{1 - z_2} \right)^\frac{3}{2} \int_{\mathbb{R}^3} \frac{f(p^2)^2}{p^0(s_1, z_1)} d\vec{p} - \frac{1}{p^0(s_2, z_2)} d\vec{p}
\]

Notice that:

\[
p^0(s_1, z_1) > \left( \frac{2z_1}{(1 - s_1)(1 - z_1)} \right)^\frac{3}{2} |p^2|
\]

(5.9) for $P_2$ then follows from (i), (k), $|p^2| < \sqrt{1 + |p|^2}$, (f) and (4.46).

Next, (4.60) follows from the expressions of $P_1$, $P_2$ in (4.30), using (j) for the integral in $P_1$ and (k) for the integral in $P_2$.

Finally to obtain (4.61), we write, for $i=1, 2$:

\[
\frac{P_i(s_1, z_1, f)}{H_1} - \frac{P_i(s_2, z_2, f)}{H_2} = \left( \frac{1}{H_1} - \frac{1}{H_2} \right) P_i(s_1, z_1, f) + \frac{1}{H_2} \left( P_i(s_1, z_1, f) - P_i(s_2, z_2, f) \right)
\]

and (4.61) is a direct consequence of $\frac{1}{H_1} - \frac{1}{H_2} = \frac{H_2 - H_1}{H_1 H_2}$, (4.59), (4.60) and $0 < \alpha(s) < 1$ for $0 < s < 1$. This completes the proof of lemma 4.2.

Now, concerning the other terms in (4.57), a usual calculation gives:

\[
\begin{aligned}
&|1 + \Sigma_{+2} H_1^2 - (1 + \Sigma_{+2} H_2^2)| \leq C H_0 (H_0 + 1) |H_1 - H_2| + |\Sigma_{+1} - \Sigma_{+2}|), \\
&|s_1(1 - s_1)| \Sigma_{+1} H_1 - s_2(1 - s_2)| \Sigma_{+2} H_2| \leq C |H_1 - H_2| + |s_1 - s_2| + |\Sigma_{+1} - \Sigma_{+2}|, \\
&|z_1(1 - z_1)(1 + \Sigma_{+1} - 3s_1 \Sigma_{+1}) H_1 - z_2(1 - z_2)(1 + \Sigma_{+2} - 3s_2 \Sigma_{+2}) H_2| \leq \\
&C |H_1 - H_2| + |s_1 - s_2| + |z_1 - z_2| + |\Sigma_{+1} - \Sigma_{+2}|, \\
{|(1 - \Sigma_{+1}^2)H_1 | \Sigma_{+1} - (1 - \Sigma_{+2}^2) H_2 | \Sigma_{+2} | \leq C (1 + H_0) |H_1 - H_2| + |\Sigma_{+1} - \Sigma_{+2}|, \\
&\forall s_1, s_2, z_1, z_2 \in [0, 1] [H_1, H_2 \in [0, H_0] [\Sigma_{+1}, \Sigma_{+2} \in [-1, 1/2]].
\end{aligned}
\]

(4.62)

We then deduce from (4.59), (4.61) and (4.62), using the expression (4.57) of $F$ the inequality:

\[
\| F(t, H_1, s_1, z_1, \Sigma_{+1}) - F(t, H_2, s_2, z_2, \Sigma_{+2}) \| \leq M (|H_1 - H_2| + |s_1 - s_2| + |z_1 - z_2| + |\Sigma_{+1} - \Sigma_{+2}|)
\]

(4.63)

where:

\[
M = M(H_1, H_2, s_1, s_2, z_1, z_2) = C \left[ 1 + \left( 1 + \frac{1}{H_2(1 - H_1)} \right)^\frac{3}{2} \| f(t) \| \right]
\]

(4.64)

We can now state:

**Proposition 4.6** Let $f \in C([0, T]; X_r)$, $T > 0$, be given. Then, the initial value problem for the system (IV.2), with initial data $(H_0, s_0, z_0, \Sigma_{+0})$ at $t = 0$ satisfying (4.56) has a unique solution $(H, s, z, \Sigma_{+})$ on $[0, T]$. 

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Proof: Take, following the definition (4.58) of the domain of $F$:

$$(H^0, s^0, z^0) \in[0, H_0] \times ]0, 1[ \times ]0, 1[ \quad (4.65)$$

and:

$$H_i \in \left[\frac{H^0}{2}, H_0\right]; \ s_i \in \left[\frac{s^0}{2}, \frac{s^0+1}{2}\right]; \ z_i \in \left[\frac{z^0}{2}, \frac{z^0+1}{2}\right]; \ i = 1, 2 \quad (4.66)$$

Then:

$$H_i > \frac{H^0}{2}; \ s_i < \frac{s^0+1}{2}; \ z_i < \frac{z^0+1}{2}; \ i = 1, 2$$

and:

$$\frac{1}{H_i} < \frac{2}{H^0}; \ \frac{1}{\alpha(s_i)} < \frac{1}{\alpha(s^0)}; \ \frac{1}{\alpha(z_i)} < \frac{1}{\alpha(z^0)}; \ i = 1, 2 \quad (4.67)$$

Consequently, the number $M$ defined by (4.64) is bounded on the neighborhood $[\frac{H^0}{2}, H_0] \times [\frac{s^0}{2}, \frac{s^0+1}{2}] \times [\frac{z^0}{2}, \frac{z^0+1}{2}]$ of $(H^0, s^0, z^0)$ in $\mathbb{R}^3$, by a number $M^0$, depending only on $H^0, s^0, z^0, r$, since $\|f(t)\| < r, \forall t \in [0, T]$; the inequality (4.63) then shows that $F$ is locally Lipschitzian with respect to the norm of $\mathbb{R}^4$. By the standard existence theorem for the first order differential system, the initial values problem for system (IV.2) has a unique local solution (standard existence theorem for the first order differential system, the solution $(\tilde{H}, \tilde{s}, \tilde{z}, \Sigma_\pm)$ of (4.60) implies that the functions $P_1, P_2$ are uniformly bounded; so $F$ defined by (4.67) is uniformly bounded and, by the standard theory on the first order differential system, the solution $(H, s, z, \Sigma_\pm)$ is global on $[0, T]$. This ends the proof of prop.4.6.\[\square\]

It shows useful to deduce the following result:

**Proposition 4.7** Let $f \in C([0, T]; X_r)$, $T > 0$, be given. Let $(\tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_\pm)$ be the solution of the initial value problem for the system (IV.2) with the initial data $(H_0, s_0, z_0, \Sigma_{+0})$ at $t = 0$, satisfying (4.56). Let $t_0 \in [0, T]$. Then the initial value problem for the system (IV.2), with initial data $(\tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_\pm)(t_0)$ at $t = t_0$, has a unique solution $(H, s, z, \Sigma_\pm)$ on $[t_0, t_0 + \delta]$ where $\delta > 0$ is independent of $t_0$.

**Proof:** The proof of the existence of a local solution $(H, s, z, \Sigma_\pm)$ on an interval $[t_0, t_0 + \delta]$, for the initial value problem for the system (IV.2) with initial data $(\tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_\pm)(t_0)$ is analogous to that of prop.4.6. Suppose we look for $\delta$ such that $0 < \delta < 1$; then applying the inequalities of prop.4.6 leads to an inequality (4.63) with a constant $M$ independent of the initial data at $t = t_0$, and of $t_0$; from there, the existence of such a number $\delta > 0$.\[\square\]

We end this section by the following result:

**Theorem 4.1** Let $f \in C([0, T]; X_r)$, $T > 0$, be given. Then the initial value problem for the Einstein equations (4.1)-(4.2)-(4.3) with initial data $(\alpha_0, b_0, a_0, b_0, f_0)$ at $t = 0$, satisfying (4.13), and the initial constraint (4.12), has a unique solution $a, b$ on $[0, T]$. 

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Proof: Apply proposition 4.6, choosing by virtue of the change of variables (4.22), (4.23), (4.24), (4.25) and using (4.16), the initial data $H_0, s_0, z_0, \Sigma_0$, as follows:

$$
\begin{align*}
H_0 &= \frac{1}{3} \left( \frac{6}{a_0} + \frac{2 b_0}{b_0} \right); \quad s_0 = \frac{b_0^2}{a_0^2 + 2 b_0^2} \\
z_0 &= \frac{1}{a_0^2 + 2 b_0^2 + 1}; \quad \Sigma_0 = \frac{1}{H_0 b_0} - 1 
\end{align*}
$$

Notice that $\frac{a_0}{b_0} > 0 \Rightarrow \frac{b_0}{a_0} < \frac{1}{2} H_0$; then $-1 < \Sigma_0 = \frac{b_0}{H_0 b_0} - 1 < \frac{1}{2}$; so that $(H_0, s_0, z_0, \Sigma_0)$ defined by (4.68) satisfy the assumption (4.56). Proposition 4.6 then proves the existence of the unique solution $(H, s, z, \Sigma)$ on $[0, T]$, of the initial value problem for the system (IV.2) with the above initial data $(H_0, s_0, z_0, \Sigma_0)$ at $t = 0$. The equivalence of the Einstein evolution equations (4.37)-(4.38) with the system (IV.2), then shows that, $(a, b)$ defined in (4.28) in lemma 4.4 is the unique solution on $[0, T]$ of the initial value problem for the Einstein evolution equations. Now, since the initial constraint (4.12) is satisfied, this implies that the Hamiltonian constraint (4.1) is satisfied on $[0, T]$. This ends the proof of theorem 4.1. \[\square\]

5 Local Existence theorem for the Coupled Einstein-Boltzmann System

As we proved in paragraph 3 and paragraph 4, the Einstein-Boltzmann System in $(a, b, f)$ is equivalent to the following first order differential system in $(f, H, s, z, \Sigma_0)$ given by (3.15) and (IV.2):

$$
\begin{align*}
\frac{df}{dt} &= \frac{1}{p^2} Q(f, f) \\
\frac{dH}{dt} &= -\frac{2}{3} (1 + \Sigma_+^2) H^2 - \frac{P_1 + 2 P_2}{6} \\
\frac{dP_1}{dt} &= 6s(1 - s) \Sigma_+ H \\
\frac{dP_2}{dt} &= 2z(1 - z)(1 + \Sigma_+ - 3s\Sigma_+)H \\
\frac{d\Sigma}{dt} &= -\frac{2}{3} (1 - \Sigma_+^2) H \Sigma_+ + \frac{P_1}{3H} (\Sigma_+ - 2) + \frac{P_2}{3H} (\Sigma_+ + 1).
\end{align*}
$$

in which $Q$ is the collision operator defined by (3.1)-(3.2)-(3.3) and $p^0, P_1, P_2$ are defined in terms of $s, z, f, a, b, f$ by (4.30). To solve system (V), we apply the standard theory of the first order differential system for functions on $\mathbb{R}$, with values in the Banach space $E = L^1(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, whose norm will be taken to be: $||(f, H, s, z, \Sigma)||_E = ||f|| + ||H|| + ||s|| + ||z|| + ||\Sigma||$. Since $f$ is now also an unknown, the problem will be to prove that the function $F$, defined by the r.h.s of (V) i.e:

$$
F(f, H, s, z, \Sigma) = \begin{cases} \\
\frac{1}{p^2} Q(f, f) - \frac{2}{3} (1 + \Sigma_+^2) H^2 - \frac{P_1 + 2 P_2}{6} \\
6s(1 - s) \Sigma_+ H \\
2z(1 - z)(1 + \Sigma_+ - 3s\Sigma_+)H \\
-\frac{2}{3} (1 - \Sigma_+^2) H \Sigma_+ + \frac{P_1}{3H} (\Sigma_+ - 2) + \frac{P_2}{3H} (\Sigma_+ + 1)
\end{cases}
$$
which does not depend explicitly on \( t \), is locally Lipschitzian in \((f, H, s, z, \Sigma_+))\), with respect to the above norm of \( E \). Following (4.58), \( \tilde{F} \) will be defined on: \( L^1_\Sigma([0, H_0]|0, H_0] \times [0, 1] \times [0, 1] \times [1, 1) \). A glance to (5.6) shows that, taking into account the study we did in paragraph 4 for the system (IV.2), the new problem to face here will be the study of all the differences in \( f \), and the differences of \( \frac{Q(f, f)}{p^0} \) in \( s \) and \( z \), which turns out to be the only real problem we meet now, since expressions \( P_1, P_2 \) in (4.30) show that these functions are linear in \( f \). We then begin by proving:

**Lemma 5.1** Let \( f_1, f_2 \in L^1_\Sigma(\mathbb{R}^3) \), \( s_1, s_2, z_1, z_2 \in [0, 1] \) then:

\[
\left\| \left( \frac{1}{p^0(s_1, z_1)} - \frac{1}{p^0(s_2, z_2)} \right) Q(f_1, f_1)(s_2, z_2) \right\| \leq \frac{C\|f_1\|^2(|s_1 - s_2| + |z_1 - z_2|)}{\alpha^4(s_1)\alpha^2(s_2)\alpha^2(z_1)\alpha^2(z_2)} \tag{5.7}
\]

\[
\left\| \frac{1}{p^0(s_2, z_2)} (Q(f_1, f_1)(s_2, z_2) - Q(f_2, f_2)(s_2, z_2)) \right\| \leq \frac{C\|f_1\| + \|f_2\|\|f_1 - f_2\|}{\alpha^2(s_2)\alpha^2(z_2)} \tag{5.8}
\]

where \( C > 0 \) is a constant.

**Proof:** For (5.7), we follow the proof of (4.59) in lemma 4.2, using this time in that proof \((p^0(s_1, z_1))^2 > \frac{2\alpha}{s_1\alpha + z_1} |p^1|^2 \) and \((p^0(s_1, z_1))^2 > \frac{2\alpha}{(1 - s_1)(1 - z_1)} |p^3|^2 \) and, the relations (a) and (d) to obtain:

\[
\left| \frac{1}{p^0(s_1, z_1)} - \frac{1}{p^0(s_2, z_2)} \right| \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{p^0(s_2, z_2)z_2(1 - z_1)(1 - z_2)} \left( \frac{1}{s_1s_2} + \frac{1}{(1 - s_1)(1 - s_2)} \right) \tag{a}
\]

Now, we use (3.27) with \( f = f_1, g = 0, (3.28) \) and (4.28) which gives the expression of \( a, b \) in terms of \( s \) and \( z \) to obtain:

\[
\left\| \frac{Q(f_1, f_1)(s_2, z_2)}{p^0(s_2, z_2)} \right\| \leq \frac{C\|f_1\|^2}{s_2\alpha^2(1 - s_2)(1 - z_2)\alpha^2} \tag{b}
\]

(5.7) then follows from (a) and (b) and definition (4.46) of \( \alpha(s) \). Now (5.8) is given directly by (3.27) with \( f = f_1, g = f_2 \), the expression of \( ab^2 \) in terms of \( s, z \), and once more (4.46). This completes the proof of lemma 5.1.

We will also need the following result which is not obvious:

**Lemma 5.2** Let \( f \in L^1_\Sigma(\mathbb{R}^3) \), \( s_1, s_2, z_1, z_2 \in [0, 1] \) then:

\[
\left\| \frac{1}{p^0(s_1, z_1)} (Q(f, f)(s_1, z_1) - Q(f, f)(s_2, z_2)) \right\| \leq \frac{C\|f\|^2(|s_1 - s_2| + |z_1 - z_2|)}{\alpha^3(s_1)\alpha^3(s_2)\alpha^5(z_1)\alpha^3(z_2)} \tag{5.9}
\]

where \( C > 0 \) is a constant.

**Proof:** Here, we split \( Q \) into \( Q^+, Q^- \) using the basic definition (3.1) of \( Q \), i.e \( Q = Q^+ - Q^- \) where \( Q^+, Q^- \) are defined by (3.2) and (3.3) in which \( ab^2 \) is
expressed in terms of $s$ and $z$. It is here that we need the Lipschitz continuity assumptions (3.7), (3.8) on the collision kernel $A$. We write:

$$Q(f, f)(s_1, z_1) - Q(f, f)(s_2, z_2) = [Q^+ (f, f)(s_1, z_1) - Q^+ (f, f)(s_2, z_2)] + [Q^-(f, f)(s_2, z_2) - Q^-(f, f)(s_1, z_1)] \quad (a)$$

We can write, using the expression (3.2) of $Q^+$:

$$Q^+(f, f)(s_1, z_1) - Q^+(f, f)(s_2, z_2) = \int_{\mathbb{R}^3 \times S^2} \Omega(s_1, s_2, z_1, z_2, \tilde{p}, \tilde{q}, \tilde{p}', \tilde{q}') f(\tilde{p}) f(\tilde{q}) dq d\omega \quad (b)$$

where $\Omega$ is a function of the indicated arguments, we write on the form:

$$\Omega = \left( \frac{ab^2}{q'^0} \right) (s_1, z_1) [A(s_1, z_1) - A(s_2, z_2)] + \left( \frac{ab^2}{q'^0} \right) (s_2, z_2) A(s_2, z_2) \quad (c)$$

in which $A(s_i, z_i)$ stands in fact for $A(a(s_i, z_i), b(s_i, z_i), \tilde{p}, \tilde{q}, \tilde{p}', \tilde{q}') \ i = 1, 2$; we now bound the first term in (c). The assumptions (3.7), (3.8) on the collision kernel $A$ give, adding and subtracting $A(a(s_2, z_2), b(s_1, z_1))$:

$$|A(s_1, z_1) - A(s_2, z_2)| \leq k_0 (|a(s_1, z_1) - a(s_2, z_2)| + |b(s_1, z_1) - b(s_2, z_2)|) \quad (d)$$

Now (4.28) gives, by usual calculation:

$$\begin{align*}
|a(s_1, z_1) - a(s_2, z_2)| &= \sqrt{\frac{2}{s_1(s_1-z_1)}} - \sqrt{\frac{2}{s_2(s_2-z_2)}} \\
|b(s_1, z_1) - b(s_2, z_2)| &= \sqrt{\frac{2}{(1-s_1)(1-z_1)}} - \sqrt{\frac{2}{(1-s_2)(1-z_2)}}
\end{align*} \quad (e)$$

We then deduce from (d), (e) and (4.28) which gives:

$$\left( \frac{ab^2}{q'^0} \right) (s_1, z_1) [A(s_1, z_1) - A(s_2, z_2)] \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{\alpha^3(s_1) \alpha^2(s_2) \alpha^3(z_1) \alpha(z_2) q'^0(s_1, z_1)} \quad (g)$$

that:

$$\left( \frac{ab^2}{q'^0} \right) (s_1, z_1) [A(s_1, z_1) - A(s_2, z_2)] \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{\alpha^3(s_1) \alpha^2(s_2) \alpha^3(z_1) \alpha(z_2) q'^0(s_1, z_1)} \quad (g)$$

Now, we bound the second term in (c); we write:

$$\begin{align*}
\left( \frac{ab^2}{q'^0} \right) (s_1, z_1) \left( \frac{ab^2}{q'^0} \right) (s_2, z_2) &= \left( \frac{ab^2}{q'^0} \right) (s_1, z_1) q'^0(s_2, z_2) - \left( \frac{ab^2}{q'^0} \right) (s_2, z_2) q'^0(s_1, z_1) \frac{q'^0(s_1, z_1) q'^0(s_2, z_2)}{q'^0(s_1, z_1) q'^0(s_2, z_2)} \\
&= \frac{1}{q'^0(s_1, z_1)} \left[ \left( \frac{ab^2}{q'^0} \right) (s_1, z_1) - \frac{q'^0(s_1, z_1)}{q'^0(s_2, z_2)} \left( \frac{ab^2}{q'^0} \right) (s_2, z_2) \right] \\
&= \frac{1}{q'^0(s_1, z_1)} \left[ \left( \frac{ab^2}{q'^0} \right) (s_1, z_1) - \left( \frac{ab^2}{q'^0} \right) (s_2, z_2) + \left( 1 - \frac{q'^0(s_1, z_1)}{q'^0(s_2, z_2)} \right) \left( \frac{ab^2}{q'^0} \right) (s_2, z_2) \right]
\end{align*}$$

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Now, concerning the first term in (h), we have, using (f) and by usual calculation:

\[ (ab^2)(s_1, z_1) - (ab^2)(s_2, z_2) = \frac{1}{q^0(s_1, z_1)} \times \]

\[ \left[ (ab^2)(s_1, z_1) - (ab^2)(s_2, z_2) + \left( \frac{q^0(s_2, z_2) - q^0(s_1, z_1)}{q^0(s_2, z_2)} \right) (ab^2)(s_2, z_2) \right] \]

Now, by (4.30):

\[ \frac{q^0(s_2, z_2) - q^0(s_1, z_1)}{q^0(s_2, z_2)} = \left( \frac{\frac{2z_2}{s_2(1-z_2)} - \frac{2z_1}{s_1(1-z_1)}}{q^0(s_2, z_2)} \right) (q^1)^2 \]

\[ + \left( \frac{2z_2}{(1-s_2)(1-z_2)} - \frac{2z_1}{(1-s_1)(1-z_1)} \right) \left[ (q^2)^2 + (q^3)^2 \right] \]

and we proceed as in the proof of (4.59), using this time:

\[ q^0(s_2, z_2) \geq \frac{2s_2}{s_2(1-z_2)} (q^1)^2; \quad \left[ q^0(s_2, z_2) \right]^2 \geq \frac{2s_2}{(1-s_2)(1-z_2)} (q^2)^2 + (q^3)^2 \]

to obtain, using expression (f) of \( ab^2 \):

\[ (ab^2)(s_2, z_2) \left| \frac{q^0(s_2, z_2) - q^0(s_1, z_1)}{q^0(s_2, z_2)} \right| \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{\alpha^2(s_1)\alpha^3(s_2)\alpha^2(z_1)\alpha^3(z_2)} \] (i)

Now, concerning the first term in (h), we have, using (f) and by usual calculation:

\[ |(ab^2)(s_1, z_1) - (ab^2)(s_2, z_2)| = 2 \left| \frac{1}{s_1^{\frac{1}{2}} \left( \frac{1-z_1}{1-z_2} \right)^{\frac{1}{2}}} - \frac{1}{s_2^{\frac{1}{2}} \left( \frac{1-z_1}{1-z_2} \right)^{\frac{1}{2}}} \right| \]

\[ \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{\alpha(s_1)\alpha(s_2)\alpha^3(z_1)\alpha^3(z_2)} \] (j)

We then have for the second term in (c), using (h), (i), (j) and \( 0 \leq A \leq C_0 \):

\[ A(s_2, z_2) \left| \frac{ab^2}{q^0} (s_1, z_1) - \frac{ab^2}{q^0} (s_2, z_2) \right| \leq \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{q^0(s_1, z_1)\alpha^2(s_1)\alpha^3(s_2)\alpha^3(z_1)\alpha^3(z_2)} \] (k)

We then deduce from (h), using (c), (g) and (k) that:

\[ \left\| \frac{1}{p^0(s_1, z_1)} (Q^+(f, f)(s_1, z_1) - Q^+(f, f)(s_2, z_2)) \right\| \leq \]

\[ \frac{C(|s_1 - s_2| + |z_1 - z_2|)}{\alpha^3(s_1)\alpha^3(s_2)\alpha^3(z_1)\alpha^3(z_2)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \frac{\sqrt{1 + |\vec{p}|^2} |f(\vec{p})| |f(\vec{q})|}{p^0 q^0(s_1, z_1)} d\vec{p} d\vec{q} d\omega \] (l)
We compute this integral the same way as in the proof of (3.25) in lemma 3.1 using the change of variables \((\tilde{p}, \tilde{q}) \mapsto (p', q')\) defined by (3.12) to obtain:

\[
\left\| \frac{1}{p^0(s_1, z_1)} (Q^+(f, f)(s_1, z_1) - Q^+(f, f)(s_2, z_2)) \right\| \leq C P^0 \left( \|f\| \left\| Q^+(f, f)(s_1, z_1) \right\| + \|s_1 - s_2\| + |z_1 - z_2| \right)
\]

We now proceed the same way for the second term in (a), i.e. \((Q^- f, f)(s_2, z_2) - Q^- f, f)(s_1, z_1)\); the only difference is that, in the integrals (b) and (l), \(f(p')f(q')\) is replaced by \(f(\tilde{p})f(\tilde{q})\). So, no need this time to change the variables and a direct calculation using \(p^0 q^0 \geq 1\) leads to the same estimation (m), just substituting \(Q^- f, f\) to \(Q^+ f, f\), and lemma 5.2 follows.

Now we deduce from lemma 5.4 and lemma 5.2 the following:

**Lemma 5.3** Let \(f_1, f_2 \in L^1_2(\mathbb{R}^3), s_1, s_2, z_1, z_2 \in [0, 1]\), then:

\[
\left\| \frac{Q(f_1, f_1)(s_1, z_1)}{p^0(s_1, z_1)} - \frac{Q(f_2, f_2)(s_2, z_2)}{p^0(s_2, z_2)} \right\| \leq C \left( \|f_1\|^2 + \|f_2\|^2 \right) \alpha(s) \alpha^3(s_1) \alpha^3(z_1) \alpha^3(z_2) \left( \|f_1 - f_2\| + \|s_1 - s_2\| + |z_1 - z_2| \right)
\]

where \(C > 0\) is a constant.

**Proof:** Write:

\[
\frac{Q(f_1, f_1)(s_1, z_1)}{p^0(s_1, z_1)} - \frac{Q(f_2, f_2)(s_2, z_2)}{p^0(s_2, z_2)} = \frac{Q(f_1, f_1)(s_1, z_1) - Q(f_1, f_1)(s_2, z_2)}{p^0(s_1, z_1)} + \left( \frac{1}{p^0(s_1, z_1)} - \frac{1}{p^0(s_2, z_2)} \right) Q(f_1, f_1)(s_2, z_2) + \frac{Q(f_1, f_1)(s_2, z_2) - Q(f_2, f_2)(s_2, z_2)}{p^0(s_2, z_2)}
\]

and apply: (5.9) with \(f = f_1\) to the first term, (5.7) to the second term, and (5.8) to the third term and \(0 < \alpha(s) < 1\), to obtain (5.10).

Finally, concerning the differences in \(P_i\) and \(\frac{P_i}{H_i}\), we prove:

**Lemma 5.4** Let \(f_1, f_2 \in L^1_2(\mathbb{R}^3), s_1, s_2, z_1, z_2 \in [0, 1]\); \(H_1, H_2 \in [0, H_0]\). Then:

\[
\left| P_i(s_1, z_1, f_1) - P_i(s_2, z_2, f_2) \right| \leq C \left( 1 + \|f_1\| \right) \left( \|f_1 - f_2\| + \|s_1 - s_2\| + |z_1 - z_2| \right) \alpha^\delta(s_1) \alpha^\delta(s_2) \alpha^\delta(z_1) \alpha^\delta(z_2)
\]

\[
\left| \frac{P_i(s_1, z_1, f_1)}{H_1} - \frac{P_i(s_2, z_2, f_2)}{H_2} \right| \leq C \left( 1 + \|f_1\| \right) \left( \|f_1 - f_2\| + \|s_1 - s_2\| + |z_1 - z_2| + |H_1 - H_2| \right) \frac{1}{H_2(1 + H_1) \alpha^\delta(s_1) \alpha^\delta(s_2) \alpha^\delta(z_1) \alpha^\delta(z_2)}
\]

\(i=1, 2; \) where \(C > 0\) is a constant.

**Proof:** The expressions of \(P_i\), \(i=1, 2\) in (4.30) show that \(P_i\) is linear in \(f\).

So we can write:

\[
P_i(s_1, z_1, f_1) - P_i(s_2, z_2, f_2) = P_i(s_1, z_1, f_1 - f_2) + (P_i(s_1, z_1, f_2) - P_i(s_2, z_2, f_2))
\]

(a)
\[
\frac{P_1(s_1, z_1, f_1)}{H_1} - \frac{P_1(s_2, z_2, f_2)}{H_2} = + \left( \frac{P_1(s_1, z_1, f_1)}{H_1} - \frac{P_1(s_2, z_2, f_1)}{H_2} \right) + \frac{P_1(s_2, z_2, f_1 - f_2)}{H_2} \tag{b}
\]

so, to obtain (5.11), apply (4.60) with \(f(t) = f_1 - f_2\) to the first term of (a), and (4.59) with \(f(t) = f_2\) to the second term of (a). To obtain (5.12), apply (4.61) with \(f(t) = f_1\) to obtain the first term of (b), and (4.60) with \(f(t) = f_1 - f_2\) to the second term of (b) \(\blacksquare\)

We can now state:

**Proposition 5.1** There exists a number \(l > 0\), such that, the initial value problem for the differential system \((V)\), with initial data \((f_0, H_0, s_0, z_0, \Sigma_{+0})\) at \(t = 0\), satisfying (4.56) and \(f_0 \in L^1_2(\mathbb{R}^3)\), has a unique solution \((f, H, s, z, \Sigma, \tau)\) on \([0, l]\).

**Proof:** Choose \(H^0, s^0, z^0\) as in (4.65) and \(f^0 \in L^1_2(\mathbb{R}^3)\).

Take \(H_i, s_i, z_i, i = 1, 2\) satisfying (4.66) and \(f_1, f_2 \in B(f^0, 1) = \{f \in L^1_2(\mathbb{R}^3), \|f - f^0\| \leq 1\}\). Then \(H_i, s_i, z_i, i = 1, 2\) satisfy (4.67) and \(\|f_i\| \leq \|f^0\| + 1, i = 1, 2\). Consequently, the number \(N\) defined by (5.14) is bounded in the neighborhood \(B(f^0, 1) \times H^0, H_0, s^0, z^0\) of \((f^0, H^0, s^0, z^0)\) in \(L^1_2(\mathbb{R}^3) \times \mathbb{R}^2\) by a number \(N^0\), depending only on \(f^0, H^0, s^0, z^0\). The inequality (5.13) then shows that \(F\) is locally Lipschitzian with respect to the norm of \(E = L^1_2(\mathbb{R}^3) \times \mathbb{R}^2\) and proposition [7] follows from the standard existence theorem for the first order differential system, for functions with values in a Banach space. Notice that \(f \in C([0, l]; L^1_2(\mathbb{R}^3)) \) \(\blacksquare\)

We end this paragraph by the following result:

**Theorem 5.1** Let \(f_0 \in L^1_2(\mathbb{R}^3)\); \(a_0, b_0, \dot{a}_0, \dot{b}_0\) satisfying (4.13), the initial constraint (4.12), and \(a_0 \geq \frac{3}{2}, b_0 \geq \frac{3}{2}\), be given. Let \(r > \|f_0\|\).

Then, there exists a number \(l > 0\), such that, the initial value problem for the coupled Einstein-Boltzmann system (2.13)-(4.1)-(4.2)-(4.3) has a unique solution \(f, a, b\) on \([0, l]\). The solution \((f, a, b)\) has the following properties:

\[ a \text{ and } b \text{ are increasing functions} \tag{5.15} \]
\[ f \in C([0, l]; X_r) \tag{5.16} \]
\[ \|f\| \leq \|f_0\| \tag{5.17} \]
Proof: Apply proposition 5.1, choosing \( f_0 \in L^2_2(\mathbb{R}^3) \), \( f_0 \geq 0 \) a.e, and define \( H_0, s_0, z_0, \Sigma_+ \) by (4.68), (4.56) is then satisfied. The existence of a unique solution \((f, a, b)\) of (2.13)-(4.1)-(4.2)-(4.3) on \([0, l]\), \( l > 0 \), is a direct consequence of proposition 5.1, the equivalence of the above system and the system \((V)\), and the hypothesis on the initial data at \( t = 0 \).

Now, concerning the properties of the solution \((f, a, b)\):

- i) (5.15) is given by corollary 5.1 with \( t_0 = 0 \), \( \delta = l \).

- ii) The hypothesis \( a_0 \geq \frac{3}{2}, \ b_0 \geq \frac{3}{2} \) and (5.15) show that \( a(t) \geq \frac{3}{2} \), \( b(t) \geq \frac{3}{2} \), \( \forall t \in [0, l] \); \( a \) and \( b \) then satisfy the hypotheses of the existence theorem 5.1 of the Boltzmann equation in \( f \), in which we take \( t_0 = 0 \) and \( T = l > 0 \). (5.16) is then a consequence of the uniqueness and of \( C([0, l]; \mathcal{X}_r) \subset C([0, l]; L^2_2(\mathbb{R}^3)) \)

- iii) (5.17) is given by (3.19) with \( t_0 = 0 \)

6 The Global Existence Theorem for the Coupled Einstein-Boltzmann System

6.1 The Method

We assume, in all what follows, that the initial data \((f_0, a_0, b_0, \tilde{a}_0, \tilde{b}_0)\) at \( t = 0 \), for the Einstein-Boltzmann system (2.13)-(4.1)-(4.2)-(4.3) satisfy (4.12), (4.13), \( a_0 \geq \frac{3}{2}, \ b_0 \geq \frac{3}{2} \), and that, in the initial data \((f_0, H_0, s_0, z_0, \Sigma_+)\) at \( t = 0 \), for the system \((V)\), paragraph 5, \( H_0, s_0, z_0, \Sigma_+ \) are defined by (4.68), and hence satisfy (4.56). Denote \([0, T[, \ T > 0\), the maximal existence domain of the solution, denoted here \((f, \tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_+)\) and given by proposition 5.1 of the initial value problem for \((V)\), with the above initial data at \( t = 0 \).

If \( T = +\infty \), the problem of the global existence is solved. We are going to show that, if we suppose \( T < +\infty \), then the solution \((f, \tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_+)\) can be extended beyond \( T \), which contradicts the maximality of \( T \).

The strategy is the following: Suppose \( 0 < T < +\infty \) and let \( t_0 \in [0, T[ \). We will show that there exists a strictly positive number \( \delta > 0 \), independent of \( t_0 \), such that, the initial value problem for the system \((V)\), on \([t_0, t_0 + \delta]\), with initial data \((\tilde{f}(t_0), \tilde{H}(t_0), \tilde{s}(t_0), \tilde{z}(t_0), \tilde{\Sigma}_+(t_0))\) at \( t = t_0 \), has a unique solution \((f, H, s, z, \Sigma_+)\) on \([t_0, t_0 + \delta]\). Then, by taking \( t_0 \) sufficiently close to \( T \), for example, \( t_0 \) such that, \( 0 < T - t_0 < \frac{\delta}{2} \), hence \( T < t_0 + \frac{\delta}{2} \), we can extend the solution \((\tilde{f}, \tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_+)\) to \([0, t_0 + \frac{\delta}{2}]\), which contains strictly \([0, T]\), and this contradicts the maximality of \( T \). In order to simplify the notations, it will be enough to look for a number \( \delta \) such that \( 0 < \delta < 1 \). As in previous paragraphs, we denote \( r \), a real number such that \( r > \|f_0\| \).
6.2 The functional framework

**Proposition 6.1** Let $t_0 \in [0, T]$ and $0 < \delta < 1$. Then, any solution $(f, H, s, z, \Sigma_+)$ for the initial value problem for the system $(V)$ on $[t_0, t_0 + \delta]$, with the initial data at $t = t_0$:

$$ (f, H, s, z, \Sigma_+(t_0) = (\hat{f}(t_0), \hat{H}(t_0), \hat{s}(t_0), \hat{z}(t_0), \hat{\Sigma}_+(t_0)) $$

(6.1)

satisfy the inequalities:

$$ \frac{1}{H(t_0 + t)} \leq M_0; \quad \frac{1}{\alpha(s(t_0 + t))} \leq M_0; \quad \frac{1}{\alpha(z(t_0 + t))} \leq M_0; \quad t \in [0, \delta] $$

(6.2)

where:

$$ M_0 = M_0(a_0, b_0, \dot{a}_0, \dot{b}_0, T) = \left( \frac{1}{H_0} + \frac{1}{s_0} + \frac{1}{z_0} \right) e^{10H_0(T+1)} $$

(6.3)

in which $H_0$, $s_0$, $z_0$ are defined in terms of $a_0, b_0, \dot{a}_0, \dot{b}_0$ by (4.68).

**Proof:** Apply proposition 6.1 to the subsystem (IV.2) of (V), and consider the solution $\hat{(H, \dot{s}, \hat{z}, \hat{\Sigma}_+)} = (\hat{H}, \dot{s}, \hat{z}, \hat{\Sigma}_+)$ of the initial value problem (IV.2) on $[0, T]$, with initial data $(H_0, s_0, z_0, \Sigma_0)$, defined by (4.68) in terms of $a_0, b_0, \dot{a}_0, \dot{b}_0$; since $t_0 \in [0, T]$ and $(H, s, z, \Sigma_+(t_0) = (\hat{H}, \hat{s}, \hat{z}, \hat{\Sigma}_+(t_0) = (\hat{H}, \hat{s}, \hat{z}, \hat{\Sigma}_+(t_0), \dot{(H, s, z, \Sigma_+)}(t_0)$,

(6.2) and (6.3) are given by (4.52), (4.53), (4.54), (4.55), (4.68), and $0 < \delta < 1$.

In all what follows, $M_0$ is the absolute constant defined by (6.3). We deduce from (4.46), (6.2) and the expressions of $a^2, b^2$ in (4.28) that:

$$ \frac{1}{s} \leq M_0; \quad \frac{1}{z} \leq M_0; \quad a^2 \leq M_0^2; \quad b^2 \leq 2M_0^2 $$

(6.4)

We then deduce from (6.4), the definition (4.23), (4.24) of $z$ and $s$, in terms of $a$ and $b$, using $a^2 \geq a_0^2 \geq \frac{9}{4} > 2$, and the inequality for $H$ in (6.2):

$$ \frac{1}{M_0} \leq z \leq \frac{1}{1 + 2M_0^2}; \quad \frac{1}{M_0} \leq s = \frac{1}{1 + 2a^2b^2}; \quad \frac{1}{M_0} \leq \frac{1}{1 + 2M_0^2}; \quad \frac{1}{M_0} \leq H \leq H_0 $$

(6.5)

On the basis of (6.2), (6.3), (6.5) we now introduce the following functions spaces, for $t_0 \in [0, T]$ and $\delta > 0$:

$$ E_{t_0}^\delta = \left\{ \frac{s \in C(t_0, t_0 + \delta)}{M_0} \leq \frac{1}{1 + 2M_0^2}; \quad \frac{1}{\alpha(s(t_0 + t))} \leq M_0; \quad t \in [0, \delta] \right\} $$

$$ F_{t_0}^\delta = \left\{ \frac{H \in C(t_0, t_0 + \delta)}{M_0} \leq H(t_0 + t) \leq H_0; \quad t \in [0, \delta] \right\} $$

$$ G_{t_0}^\delta = \left\{ \frac{\Sigma_+ \in C(t_0, t_0 + \delta)}{M_0} \leq -1 \leq \Sigma_+(t_0 + t) \leq \frac{1}{2}; \quad t \in [0, \delta] \right\} $$

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One verifies easily that $E_{t_0}^\delta$, $F_{t_0}^\delta$, $G_{t_0}^\delta$ are complete metric subspaces of the Banach space denoted $C([t_0, t_0 + \delta])$, of the continuous (and hence bounded) functions on the line segment $[t_0, t_0 + \delta]$, endowed with the norm:

$$
\|u\|_\infty = \sup_{t \in [t_0, t_0 + \delta]} |u(t)|; \quad u \in C([t_0, t_0 + \delta]) \tag{6.7}
$$

### 6.3 The global existence theorem

**Proposition 6.2** Let $t_0 \in [0, T]$. There exists a strictly positive number $\delta > 0$, depending only on the absolute constants $a_0, b_0, a_0, b_0, T$, and $r$, such that, the initial value problem for the system (V), with the initial data $(f(t_0), \tilde{H}(t_0), \tilde{s}(t_0), \tilde{z}(t_0), \tilde{\Sigma}_+(t_0))$ at $t = t_0$, has a unique solution $(f, H, s, z, \Sigma_+) \in C([t_0, t_0 + \delta]; X_r) \times F_{t_0}^\delta \times E_{t_0}^\delta \times E_{t_0}^\delta \times G_{t_0}^\delta$.

**Proof:** By theorem 5.1 we know that, if we fix $\tilde{s}, \tilde{z} \in E_{t_0}^\delta$, and if we define $\tilde{a} = a(\tilde{s}, \tilde{z}), \tilde{b} = b(\tilde{s}, \tilde{z})$ by (4.28), then equation (5.1) in $f$ has a unique solution $f \in C([t_0, t_0 + \delta]; X_r)$ such that $f(t_0) = \bar{f}(t_0)$, and, by (3.19) and (5.17) with $f = \bar{f}$, that:

$$
\|f(t)\| \leq \|\bar{f}(t_0)\| \leq \|f_0\| < r, \quad t \in [t_0, t_0 + \delta] \tag{6.8}
$$

Next, by proposition 4.7 we know that, there exists a number $\delta > 0$, (we can suppose $0 < \delta < 1$), such that, if $\tilde{f}$ is given in $C([t_0, t_0 + \delta]; X_r)$, then the system (IV.2) has a unique solution $(H, s, z, \Sigma_+) \in [t_0, t_0 + \delta] [H, s, z, \Sigma_+]$ such that $(H, s, z, \Sigma_+) = (\tilde{H}, \tilde{s}, \tilde{z}, \tilde{\Sigma}_+)(t_0)$. Now proposition 6.1 and inequalities (4.51) show that $(H, s, z, \Sigma_+) \in \times F_{t_0}^\delta \times E_{t_0}^\delta \times E_{t_0}^\delta \times G_{t_0}^\delta$. This allows us, setting

$$
X_{t_0}^\delta = C([t_0, t_0 + \delta]; X_r) \times (F_{t_0}^\delta \times E_{t_0}^\delta) \quad \text{and} \quad Y_{t_0}^\delta = C([t_0, t_0 + \delta]; X_r) \times (F_{t_0}^\delta \times E_{t_0}^\delta \times E_{t_0}^\delta \times G_{t_0}^\delta),
$$

to define the application:

$$
g : X_{t_0}^\delta \rightarrow Y_{t_0}^\delta; \quad (f, (H, s, z, \Sigma_+)) \mapsto (f, (H, s, z, \Sigma_+)) \tag{6.9}
$$

We are going to show that, we can find $\delta > 0$, such that $g$ defined by (6.9) induces a contracting map of the complete metric space $X_{t_0}^\delta$ (defined above) into itself that will hence have a unique fixed point $(f, s, z)$. This will allow us to find $H$ and $\Sigma_+$ such that $(f, (H, s, z, \Sigma_+))$ be the unique solution in $Y_{t_0}^\delta$ (defined above) of the initial value problem for the system (V) with the initial data (6.1).

So, if we evaluate the r.h.s of (5.1), for $s = \bar{s}, z = \bar{z} \in E_{t_0}^\delta$, and if we set, in (5.3) and (5.5) $P_i = P_i(s, z, f), i = 1, 2$; where $f \in C([t_0, t_0 + \delta]; X_r)$, then there exists a solution $(f, (H, s, z, \Sigma_+))$ of that system, taking the initial data (6.1) at $t = t_0$, or equivalently, a solution of the following integral system:

$$
\begin{align*}
& \frac{f(t_0 + t)}{\int_{t_0}^{t_0 + t} Q(f, f) Q(f, f)(\tau, \sigma) d\tau} = \frac{1}{H(t_0 + t)} + 6s(1-s)\Sigma_+ H(\tau) d\tau \\
& s(t_0 + t) = \bar{s}(t_0) + \int_{t_0}^{t_0 + t} 2z(1-z)(1 + \Sigma_+ - 3s\Sigma_+) H(\tau) d\tau \\
& z(t_0 + t) = \bar{z}(t_0) + \int_{t_0}^{t_0 + t} 2z(1-z)(1 + \Sigma_+ - 3s\Sigma_+) H(\tau) d\tau \\
& \Sigma_+(t_0 + t) = \bar{\Sigma}_+(t_0) + \int_{t_0}^{t_0 + t} \left[ -\frac{1}{2}(1 - \Sigma_+^2) H \Sigma_+ + \frac{1}{6H}(\Sigma_+ - 2) + \frac{7}{3H}(\Sigma_+ + 1) \right] (\tau) d\tau \\
\end{align*}
$$

(6.10)
To $\bar{f}_i = (\bar{s}_i, \bar{z}_i) \in X_{t_0}^\delta$, $i=1, 2$, corresponds the solution $(f_i, (H_i, s_i, z_i, \Sigma_{+i})) \in Y_{t_0}^\delta$, $i=1, 2$, of the above integral system (VI). Writing the equations for $i=1, i=2$ and subtracting yields, using respectively: (5.10) with $s_i = \bar{s}_i$, $z_i = \bar{z}_i$, $||f_i|| \leq r$; (5.11) and (5.12) with $f_i = \bar{f}_i$, $||f_i|| \leq r$; (6.42), definition (6.6) of $E_{t_0}^\delta$, $F_{t_0}^\delta$, (6.7), $0 \leq t \leq \delta$:

$$
||f_1 - f_2|| \leq \delta M_1 [||f_1 - f_2|| + ||\bar{s}_1 - \bar{s}_2|| + ||\bar{z}_1 - \bar{z}_2||]  
$$

(6.10)

$$
||H_1 - H_2||_{\infty} \leq \delta M_2 [||\bar{f}_1 - \bar{f}_2||]  
$$

$$
+ \delta M_2 [||H_1 - H_2||_{\infty} + ||s_1 - s_2||_{\infty} + ||z_1 - z_2||_{\infty} + ||\Sigma_{+1} - \Sigma_{+2}||_{\infty}]  
$$

(6.11)

$$
||s_1 - s_2||_{\infty} \leq \delta M_3 [||H_1 - H_2||_{\infty} + ||s_1 - s_2||_{\infty} + ||z_1 - z_2||_{\infty} + ||\Sigma_{+1} - \Sigma_{+2}||_{\infty}]  
$$

(6.11)

$$
||z_1 - z_2||_{\infty} \leq \delta M_4 [||H_1 - H_2||_{\infty} + ||s_1 - s_2||_{\infty} + ||z_1 - z_2||_{\infty} + ||\Sigma_{+1} - \Sigma_{+2}||_{\infty}]  
$$

(6.12)

$$
||\Sigma_{+1} - \Sigma_{+2}||_{\infty} \leq \delta M_5 [||\bar{f}_1 - \bar{f}_2||]  
$$

$$
+ \delta M_5 [||H_1 - H_2||_{\infty} + ||s_1 - s_2||_{\infty} + ||z_1 - z_2||_{\infty} + ||\Sigma_{+1} - \Sigma_{+2}||_{\infty}]  
$$

(6.13)

where $M_1, M_2, M_3, M_4, M_5$ are absolute constants, depending only on the absolute constants $a_0, b_0, \bar{a}_0, \bar{b}_0, T$ and $r$.

We keep (6.10) and add (6.11)-(6.12)-(6.13)-(6.14) to obtain:

$$
||H_1 - H_2||_{\infty} + ||s_1 - s_2||_{\infty} + ||z_1 - z_2||_{\infty} + ||\Sigma_{+1} - \Sigma_{+2}||_{\infty}  
$$

\leq 4\delta (M_2 + M_3 + M_4 + M_5) [||\bar{f}_1 - \bar{f}_2||]  

$$
+ 4\delta (M_2 + M_3 + M_4 + M_5) [||H_1 - H_2||_{\infty} + ||s_1 - s_2||_{\infty} + ||z_1 - z_2||_{\infty} + ||\Sigma_{+1} - \Sigma_{+2}||_{\infty}]  
$$

(6.15)

If we choose $\delta$ such that:

$$
0 < \delta < \frac{1}{\text{Inf}(1, \frac{1}{16(M_1 + M_2 + M_3 + M_4 + M_5)})}  
$$

(6.16)

which implies that $\delta M_1 < \frac{1}{4}$ and $4\delta (M_2 + M_3 + M_4 + M_5) < \frac{1}{4}$; then (6.10) and (6.15) give:

$$
\left\{ \begin{array}{l}
||f_1 - f_2|| \leq \frac{1}{4} ||\bar{s}_1 - \bar{s}_2||_{\infty} + ||\bar{z}_1 - \bar{z}_2||_{\infty} \\
||H_1 - H_2||_{\infty} + ||s_1 - s_2||_{\infty} + ||z_1 - z_2||_{\infty} + ||\Sigma_{+1} - \Sigma_{+2}||_{\infty} \leq \frac{1}{4} ||\bar{f}_1 - \bar{f}_2|| 
\end{array} \right.  
$$

and by addition:

$$
||f_1 - f_2|| + ||H_1 - H_2||_{\infty} + ||s_1 - s_2||_{\infty} + ||z_1 - z_2||_{\infty} + ||\Sigma_{+1} - \Sigma_{+2}||_{\infty}  
$$

\leq \frac{1}{3} [||\bar{f}_1 - \bar{f}_2|| + ||\bar{s}_1 - \bar{s}_2||_{\infty} + ||\bar{z}_1 - \bar{z}_2||_{\infty}]  
$$

and by addition:
from which we deduce:

$$\|f_1 - f_2\| + \|s_1 - s_2\|_\infty + \|z_1 - z_2\|_\infty \leq \frac{1}{3} \left[ \|\bar{f}_1 - \bar{f}_2\| + \|\bar{s}_1 - \bar{s}_2\|_\infty + \|\bar{z}_1 - \bar{z}_2\|_\infty \right]$$

(6.17)

(6.17) shows that, the map $\bar{(f, (\bar{s}, \bar{z}))} \mapsto (f, (s, z))$ is a contracting map from the complete metric space $X^b_{\delta_0}$ into itself, for every number $\delta$ satisfying (6.16), which shows that $\delta$ depends only on the absolute constants $M_i$, $i=1, 2, 3, 4, 5$. This map has a unique fixed point $(f, (s, z)) \in X^b_{\delta_0}$. Now, to determine $H$ and $\Sigma_+$, since $s$ is known, (5.3) determines the product in $H\Sigma_+$ in terms of $s$; then, substituting this product in (5.4) gives $H$ in terms of $s$ and $z$; once $H$ is known, (5.3) gives $\Sigma_+$, and we obtain the desired unique solution $(f, (H, s, z, \Sigma_+)) \in Y^b_{\delta_0}$. This ends the proof of proposition 6.2.

We can then state:

**Theorem 6.1** The initial values problem for the spatially homogeneous Einstein-Boltzmann system on a locally rotationally symmetric Bianchi type I space-time, has a global solution $(f, a, b)$ on $[0, +\infty[$, for suitable arbitrarily large initial data at $t = 0$.

**Remark 6.1**

1. Nowhere in the proof we had to restrict the size of the initial data, which can then been taken arbitrarily large.

2. In [13], which studies the case $a = b$, the author didn’t study the Einstein evolution equations, which are, as we saw in paragraph 4 the main problem to solve.

3. The present work extends the global existence result established in [24], in the case $a = b$ and strictly positive cosmological constant $\Lambda > 0$, to the case $\Lambda = 0$.

4. In the future, we plan to prove the geodesic completeness and to relax the hypotheses on the collision kernel.

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**References**


