It is pointed out that inhomogeneous condensates or spinodal instabilities suppress the propagation of elementary excitations due to the absorptive zero mode dynamics. This mechanism is shown to be present in the scalar $\phi^4$ model and in Quantum Gravity. It is conjectured that the plane waves states of color charges have vanishing scattering amplitude owing to the color condensate in the vacuum.

1. Introduction

The mechanism of quark confinement continues to be one of the stumbling block in particle physics and one tends to shelve it by the help of more or less ad hoc models motivated by QCD. Instead of a detailed construction of "The Mechanism" which obviously involves some highly technical and non-perturbative issues of QCD we suggest in this talk a rather well known phenomenon, spinodal instability, as a natural ingredient of confinement.

There are actually two mechanisms for quark confinement. One is established by means of studying Wilson-loops in quenched lattice QCD without dynamical quarks and is operating with a linearly rising potential between static fundamental color charges. One may call this hard mechanism because it is based on large energy scales to suppress localized quarks. Another mechanism, proposed by V. Gribov, is conjectured in QCD with dynamical, light quarks and is reminiscent of the supercritical vacuum in QED. The separation of a quark form a hadron triggers an increase in the running coupling constant which in turn generates screening by the anti-quark of a $q - \bar{q}$ vacuum-polarization. We may call this soft mechanism because the screening is achieved by vacuum fluctuations of the energy of a light meson. The soft mechanism is the real one, observed in hadrons and is based on the non-perturbative phenomena in the Dirac-see caused by the increase of the running coupling constant at large distances, the key

phenomenon of the hard mechanism. An inherent difficulty of both mechanisms is the explanation of gluon confinement. We argue below that the colored condensate of the QCD vacuum should drive a spinodal instability which automatically confines color charges, both gluons and quarks.

We shall discuss this mechanism in the framework of two theories. One is the the $\phi^4$ scalar model in the symmetry broken phase. We argue that when a homogeneous external source is coupled to the field $\phi(x)$ and its value is chosen in such a manner that $|\langle \phi(x) \rangle|$ is decreased then the elementary excitations become confined. The second example is Quantum Gravity where confinement of gravitons is expected. QCD is touched upon briefly at the end only.

Figure 1. A qualitative sketch of the pressure $p$ as the function of the density $n$ for the van der Waals equation of state. The density interpolates between the values corresponding to the two phases while the pressure remains unchanged on the horizontal line, the Maxwell-cut.

The basic idea is very simple. Let us start with the spinodal instability shown in Fig. 1 which is usually observed in first order phase transitions. The spinodal unstable region is characterized by the apparent violation of the sum rule which assures that the pressure is a non-decreasing function of the density. The Maxwell-cut restores stability and renders the pressure independent of the density in the mixed phase where domains made up by the two stable phases are formed. The sound velocity,

$$c = \sqrt{\frac{1}{n^2 \partial V/n}}$$

is clearly vanishing in the mixed phase. The vanishing is due to the domain walls which can be displaced without energy absorb the density waves. The density fluctuations are therefore non-propagating and the sound waves become "confined".
2. Scalar model

One can gain more insight into the dynamics of the instabilities of Fig. 1 by considering a scalar model where a homogeneous external source, $j$, coupled linearly to the field $\phi(x)$ is introduced in order to dial a desired value of the condensate. The partition function of the Euclidean model is

$$Z = \int D[\phi] e^{-\int dx \left[ \frac{1}{2} (\partial \phi(x))^2 + U(\phi(x)) \right] + j \int dx \phi(x)}$$

where the potential is an even function, $U(-\phi) = U(\phi)$, and is double-well shaped, as shown in Fig. 2. We shall determine the vacuum which is supposed to be homogeneous for any value of the source $j$ in the tree-level approximation. The vacuum of the theory with $j = 0$ is dominated by a single homogeneous configuration $\langle \phi(x) \rangle = \phi_{\text{vac}} > 0$ satisfying the conditions $U'(\phi_{\text{vac}}) = 0$ and $U''(\phi_{\text{vac}}) > 0$.

![Figure 2. The double-well potential of the scalar model.](image)

The gradual decrease of the external source towards more negative values decreases the saddle-point $\phi(x) = \phi_j > 0$ which is defined by the equation $U'(\phi_j) = j$, as shown in Fig. 3. The homogeneous saddle-point vacuum remains stable, $U''(\phi_j) > 0$, as long as

$$\phi_{\text{sp}} \leq \phi_j \leq \phi_{\text{vac}},$$

where the lower bound is defined by the equation $U''(\phi_{\text{sp}}) = 0$. Notice that the vacuum constructed in such a manner is stable against infinitesimal fluctuations but decays into the true vacuum, given by the absolute minimum of $U(\phi) + j\phi$ when fluctuations with sufficiently large amplitude,
Figure 3. The vacuum is dominated by a single homogeneous saddle point in the nucleation phase, given by Eq. (3).

\[ \Delta \phi > \phi_{\text{min}} \] are formed. This regime of the model is called nucleation phase because the large amplitude fluctuations which destabilize the false vacuum are energetically stable above a certain size. All modes are massive in this phase.

Figure 4. The potential energy in the spinodal phase.

As the external source is pushed further towards the negative direction the saddle-point reaches the lower bound in Eq. (3) and we enter into the spinodal instability region, cf. Fig. 4 characterized by instabilities against infinitesimal fluctuations. The saddle-points are inhomogeneous and support domains where \( \phi(x) \) assumes opposite sign. The breakdown of the external, space-time symmetries induces zero modes, the location and
the direction of the domain walls. We shall argue in the next Section that these soft modes make the "sound waves" non-propagating, i.e. remove the plane waves from the asymptotical scattering states of the theory.

3. Instability induced renormalization

The inhomogeneous saddle-points appearing in the spinodal phase require more powerful method than the mean-field approximation. It is a time honored strategy to deal with modes one-by-one in a sequential manner, as in the renormalization group method, instead of facing all of them in the same time. Such a scheme proves to be useful for the saddle-points, too.

Let us consider an infinitesimal decrease of the UV cutoff, \( k \to k - \Delta k \), in momentum space which induces the change \( S_k[\phi] \to S_{k-\Delta k}[\phi] \) in the bare action of the scalar model,

\[
e^{-\frac{\hbar}{\pi}S_{k-\Delta k}[\phi_{IR}]} = \int D[\phi_{UV}] e^{-\frac{\hbar}{\pi}S_k[\phi_{IR}+\phi_{UV}]},
\]

where the split \( \phi(x) = \phi_{IR}(x) + \phi_{UV}(x) \) of the field variable is defined by requiring that the Fourier transform of \( \phi_{UV}(x) \) is non-vanishing within the shell \( k - \Delta k < p < k \) only, as shown in Fig. 5. The tree-level contribution to the evolution equation is

\[
S_{k-\Delta k}[\phi_{IR}] = S_k[\phi_{IR} + \phi_{UV}[\phi_{IR}]].
\]

A non-trivial saddle-point, \( \phi_{UV}[\phi_{IR}] \neq 0 \), induces an evolution in the bare action, \( S_{k-\Delta k}[\phi_{IR}] \neq S_k[\phi_{IR}] \).
Two remarks are in order now about the saddle-points. The first is that the non-vanishing saddle-point is the hallmark of spontaneous symmetry breaking, the appearance of a condensate in the theory. In fact, the inverse propagator, \( \delta^2 S_k[\phi]/\delta \phi \delta \phi \), approaches its renormalized form as \( k \to 0 \). Spontaneous symmetry breaking is driven by the negative values of the renormalized propagator for small momenta computed in the symmetrical vacuum, \( \phi(x) = 0 \). Hence \( \phi(x) = 0 \) ceases to be the absolute minimum of the bare action at sufficiently small values of the cutoff \( k \) in the symmetry broken phase and a non-trivial saddle point is found for Eq. (4). The other remark is about the circumstance that these non-trivial saddle-points appear in a scalar model in any dimension even if the whole partition function supports no solitons or instantons. The saddle-points mentioned again and again in this Section correspond to a constrained functional integral where the blocked variables are held fixed. It is this constrain which induces the block variable-dependent saddle-points and the partition function as a complicated integral does not reveal their existence.

Let us find the tree-level solution of the evolution equation in the local potential approximation where the bare action is assumed to be of the form

\[
S_k[\phi] = \int dx \left[ \frac{1}{2} (\partial \phi(x))^2 + U_k(\phi(x)) \right].
\]

This ansatz allows us to seek the solution only for \( \phi_{\text{IR}}(x) = \Phi \) because

\[
U_{k-\Delta k}(\Phi) = \min_{\phi_{\text{UV}}} \int dx \left[ \frac{1}{2} (\partial \phi_{\text{UV}})^2 + U_k(\Phi + \phi_{\text{UV}}(x)) \right].
\]

As a further simplification, we search for the minimum among the plane wave configurations only,

\[
\phi_{\text{UV}}(x) = \rho_k \cos(k n_k \cdot x + \theta_k).
\]

Note that the unit vector \( n_k \) and the phase \( \theta_k \) are zero modes arising from the breakdown of the rotational and translational symmetry. Eq. (7) now reads as

\[
U_{k-\Delta k}(\Phi) = \min_{\rho_k} \left( \frac{1}{4} k^2 \rho_k^2 + \frac{1}{\pi} \int_0^\pi dy U_k(\Phi + \rho_k \cos y) \right).
\]

In order to obtain the renormalization group flow we impose the initial condition

\[
U_k(\phi) = -\frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4,
\]
Figure 6. Tree-level evolution of the potential $U(\phi)$. The initial condition is set at $k = \Lambda \gg \sqrt{-m^2}$ but the tree-level evolution starts only for $k^2 = -m^2$. The saddle-point is non-vanishing within the area bounded by the outer curve. The naive reasoning would suggest the possibility of separating nucleation and spinodal phases in the unstable regime. This proved to be wrong by the numerical solution, the spinodal region spreads over the whole unstable domain.

with $m^2 < 0$ at $k = \Lambda \gg \sqrt{-m^2}$ for Eq. (9). There is no evolution for $k \approx \Lambda$ because the kinetic energy dominates the inverse propagator

$$G(p) = \frac{1}{p^2 - m^2 + \frac{g}{2} \Phi^2}$$

for $p^2 \gg -m^2$, cf. Fig. 6. The inhomogeneous saddle-point becomes non-trivial and the potential evolves for $k^2 < -m^2$.

The following observations can be made after inspecting the numerical solution of Eq. (9):

1. Eq. (9) goes over a differential equation in the limit $\Delta k \to 0$ and $S_k[\phi]$ is continuous in $k$.
2. The instability is always followed by the appearance of inhomogeneous saddle-points and their zero modes. The zero modes are soft and can be excited by arbitrarily small energies, indicating spinodal phase separation in the whole instability region. The nucleation phase expected naively in between the stable and the spinodal region disappears.
3. The bare action $S_k[\phi]$ is always flat for modes with $p = k$, in the variable $\phi_{\text{UV}}$, due to the cancellation between the kinetic energy and the potential energy which is found to be

$$U_k(\Phi) = -\frac{1}{2} k^2 \Phi^2$$

within the instability region. The resulting flat action, the lack
of restoring force to any equilibrium position, becomes the usual Maxwell-cut as \( k \to 0 \).

(4) The vacuum, \( \phi_{\text{vac}} \), is at the edge of the unstable region.

(5) These tree-level results, Eq. (12) in particularly, hold independently of the choice of the potential at the cut-off.

The unexpected feature is the appearance of zero modes, \( n_k \) and \( \theta_k \) in Eq. (8), within the unstable region in the one-component model with the discrete internal symmetry \( \phi(x) \to -\phi(x) \). These are Goldstone modes arising from the breakdown of translational and rotational symmetry by the inhomogeneous saddle-points. The plane-wave is a poor-man’s approach to the domain wall structure and its phase and direction control the location and the direction of the walls. The integration over the zero modes appearing independently at each value of \( k \) restores the external symmetries and renders the vacuum homogeneous.

It is point 3 which is relevant from the point of view of confinement. What we have obtained here is a layer-by-layer scan of the bare action for this theory. In fact, the dynamics of a mode with wave vector \( p \) is not contained explicitly or appears in a rather approximative manner only in \( S_k[\phi] \) for \( |p| > k \) or \( |p| < k \). This dynamics is best described by the bare action with \( k = |p| \). Let us suppose that the external homogeneous source stabilizes the vacuum at \( |\langle \phi(x) \rangle| < \phi_{\text{vac}} \) and the model is in the spinodal phase. The cancellation between the kinetic and the potential energy should therefor be kept as a feature characterizing all unstable modes and no plane waves exist with small wave numbers. One may say in the more formal language of the reduction formulae that there are no mass-shell singularities in the Green functions for these modes and the scattering amplitude is vanishing, ie. the low energy elementary quanta of the theory are non-propagating, confined.

What happens with the high energy modes which belong to the stable regime of Fig. 6? These modes are supposed to propagate on the background field of the inhomogeneous condensate and the integration over the zero modes appears as a quenched average. One expects that this is a strongly disordered system where the Euclidean analogy of the Anderson localization prevents these states from being extended. In the lack of a detailed study it is sufficient to say that the restored external symmetries impose the cancellation of the scattering amplitude at any energy once it is found vanishing at low energy. In other words, all particle modes are confined as long as there is a finite low energy range with inhomogeneous
4. Savvidy vacuum in Quantum Gravity

There is another model where confinement of the elementary excitations are expected in the semiclassical approximation, namely gravity. Let us consider a point-like body surrounded by its horizon and suppose that the position of the body undergoes small amplitude oscillations. The acceleration sends gravitational waves into the space-time within the horizon. Though the non-propagating features of the solution of the Einstein equations may indicate the motion of the body outside of the horizon, the propagating gravitons are supposed to remain confined by the horizon.

It is not difficult to recognize the trace of spinodal instability in quantum gravity. Let us consider the partition function

\[ Z = \int D[g] e^{-S_E[g]}, \] (13)

for Euclidean signature metric tensor, where the Einstein action is given by

\[ S_E[g] = -\kappa^2 \int dx \sqrt{\mathcal{g}} R. \] (14)

This theory poses serious problems both in the UV and the IR domains.

The difficulty in the UV region is that the theory is non-renormalizable. We shall keep the cutoff \( \Lambda \) at a large but finite value and consider (14) as an effective action valid for distances \( \ell > \ell_{\text{min}} \approx 1/\Lambda \) only. The regularization is carried out by restricting the functional integral for geometries into the vicinity of de Sitter spaces, characterized by their curvature \( R \), and by suppressing modes whose eigenvalue with respect to the operator \(-D^2\) is superior to \( \Lambda^2 \).

The IR problems come from the fact the the action (14) is unbounded form below in the conform modes, \( \Omega \), defined as \( \tilde{\mathcal{g}}_{\mu\nu} = g_{\mu\nu} \Omega^2 \) because

\[ S_E[\tilde{g}] = -\kappa^2 \int dx \sqrt{\mathcal{g}} \left( \Omega^2 R + 6\Omega_{\mu\nu} \Omega^{\alpha\beta} \delta^{\mu\nu} \right). \] (15)

Instead of modifying the Wick-rotation, quantization rules, or the functional integral measure, which lead to further problems with general covariance it is more illuminating to recognize the similarity to the instability of the Yang-Mills vacuum. In fact, the interactions among the elementary excitations are attractive in both cases and populate modes macroscopically. These modes are inhomogeneous in order not to break
gauge and external symmetries. The apparent difference, the Yang-Mills action is bounded from below but the gravitational one is not, may not be important because the stabilization of the vacuum can be achieved by quantum fluctuations. An obvious example is the Hydrogen atom where the kinetic energy created by localization balances the attractive, unbounded Coulomb potential.

![Figure 7. The effective potential as the function of the de Sitter curvature in the low and the high cutoff phases.](image)

The one-loop contributions of the stable modes to the effective potential

\[ \gamma_{\text{eff}}(R) = -\ln Z(R) \]

where \( Z(R) \) is the perturbative partition function on the de Sitter space of curvature \( R \) give

\[
\gamma(R) = \begin{cases} 
- \frac{v \kappa^2}{\kappa^2_B} + c_1 \Lambda^4 \ln \frac{c_2 \kappa^2_B}{\Lambda^2} \left( \frac{1}{R^2 R} - \frac{1}{c_3 \Lambda^2} \right) & R \leq c_3 \Lambda^2 \\
- \frac{v \kappa^2}{\kappa^2_B} & R > c_3 \Lambda^2 \end{cases},
\]

with \( c_1 \approx 7.201, c_2 \approx 2.989 \) and \( c_3 \approx 0.665 \), cf. Figs. 7. This result indicates a quantum phase transition at \( \kappa^2_{cr} = \frac{\Lambda^2}{c_2} \) and the curvature

\[
R_{\text{min}} = \begin{cases} 
\frac{2 c_1 \Lambda^4}{v \kappa^2_B} \ln \frac{c_2 \kappa^2_B}{\Lambda^2} & \kappa^2_{cr} \ll \kappa^2_B \\
0 & \kappa^2_B \ll \kappa^2_{cr} \end{cases}
\]

at the minimum of the effective potential. What is left to see whether the conformal modes are stabilized at these vacua. This phase diagram agrees qualitatively with the one found in lattice simulations\(^{12} \) suggesting a radiative corrections induced stabilization of the vacua of Eqs. (17). Notice that whatever stabilization mechanism prevails the vacuum contains inhomogeneous condensate in both phases, made of the inhomogeneous conformal modes, and one expects that the appearing zero modes make the plane waves, gravitons, non-propagating.
5. Conclusions

It has been argued that inhomogeneous condensate, in particular spinodal instability, leads to the confinement of elementary excitations by the absorptive dynamics of the zero modes. Two models have been mentioned, a scalar $\phi^4$ model and Euclidean quantum gravity. The mechanism is established on a qualitative level only, the more satisfactory construction obviously requires a systematic and thorough study of the tree-level contributions to the renormalization group equations.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{The tree-level evolution in the $\phi^4$ model (left) and in QCD (right). The vertical axes show the cutoff, the horizontal ones correspond to the expectation value of the scalar field and some components of the field strength tensor. The inhomogeneous saddle-points appear within the regions bounded by the solid line.}
\end{figure}

Is this mechanism effective in the QCD vacuum? Let us compare the known tree-level renormalization flow of the scalar model with the expected flow in QCD, as shown in Figs. The vacuum of the $\phi^4$ model, shown in the first figure, is at the edge of the spinodal phase. The effects of the eventual zero modes within the unstable region are suppressed and no confinement occurs in the absence of the external source. The color condensate disappears at short distances or in the presence of strong external chromo-magnetic field in QCD. The region with condensate is shown qualitatively in the second figure. What is important is that the true vacuum is within the unstable region. Hence one expects that the zero mode dynamics of the inhomogeneous condensate renders the effective action flat and removes colored plane wave states from the asymptotical sector of QCD.

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