The Long-Term Future of Space Travel

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The fact that we apparently live in an accelerating universe places limitations on where humans might visit. If the current energy density of the universe is dominated by a cosmological constant, a rocket could reach a galaxy observed today at a redshift of 1.7 on a one-way journey or merely 0.65 on a round trip. Unfortunately these maximal trips are impractical as they require an infinite proper time to traverse. However, calculating the rocket trajectory in detail shows that a rocketeer could nearly reach such galaxies within a lifetime (a long lifetime admittedly – about 100 years). For less negative values of \( w \) the maximal redshift increases becoming infinite for \( w \geq -1/3 \).

INTRODUCTION

It is happy coincidence that product of the lifespan of a human being and the surface gravity of the Earth \( \tau g/c \) is much larger than unity; this means that at least special relativity presents no impediment to interstellar travel [e.g. 1]. However, the recent discovery that the expansion of the universe is accelerating [2, 3] presents a new set of challenges for long-distance travel. Several authors have pointed out that because the universe is accelerating, we are now seeing as much of the universe as we will ever see [4] and that the long-term future of Galactic astronomy is bleak with only the handful of galaxies bound to the Local Group ultimately being observable [5].

The presence of a horizon that limits what we can observe in the future also limits where we can visit. However, unlike light humans age as they travel and must be accelerated not too roughly to reach relativistic velocities (and it turns out to maintain them as well as we shall see). Several authors have examined the related question of how an accelerating expansion limits the future of computation [5, 6]. However, computers can survive larger accelerations than we can and possibly can operate over a longer proper time.

In first section I shall derive the equations that describe accelerated motion in an expanding universe. The universe is apparently entering an epoch of de Sitter expansion and solutions for this spacetime may be solved analytically. Furthermore, I place interesting analytic limits on the proper time required to travel cosmological distances in more general spacetimes. The second section describes the results of numerical calculations of accelerating trajectories through universes like our own and specifically focusses on paths that will take approximately a human lifetime to traverse.

CALCULATIONS

The metric of a homogeneous, isotropic, flat universe may be given by

\[
ds^2 = g_{\alpha \beta} dx^\alpha dy^\beta = a(\tau)^2 \left[ d\tau^2 - dx^2 - dy^2 - dz^2 \right]
\]  

where \( d\tau = a(\tau)d\tau \) and \( c = 1 \). The symbol \( \tau \) represents the conformal time while \( t \) is the time measured by a co-moving observer. I will generally suppress the functional dependence of \( a(\tau) \), writing \( a \).

A rocket traveller must have a four-velocity \((u^\alpha)\) that satisfies

\[
1 = u^\alpha u^\beta g_{\alpha \beta}.
\]  

Let us assume that the rocket travels in the \( x \)-direction yielding

\[
\frac{d\tau}{ds} = u^0 = \frac{\cosh \chi}{a} \quad \text{and} \quad \frac{dx}{ds} = u^1 = \frac{\sinh \chi}{a}
\]  

that automatically satisfies Eq. 2. The symbol \( \chi \) denotes the rapidity of the rocket that a coincident, co-moving observer would measured.

The acceleration of the rocket measured by the rocketeer is given by

\[
\xi^\alpha = u_\beta^\gamma u^\beta \quad \text{or} \quad u_\beta^\gamma u^\beta = \frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta \gamma} u^\gamma u^\beta
\]  

\[
= \left( \frac{d\chi}{ds} + \frac{da}{d\tau} \frac{1}{a} \right) \begin{bmatrix} u^1 \\ -u^0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u^1 \\ -u^0 \\ 0 \\ 0 \end{bmatrix}
\]  

and

\[
\xi^2 = - \left( \frac{d\chi}{ds} + \frac{da}{d\tau} \frac{1}{a} \right)^2 = - \left( \frac{d\chi}{ds} + H \sinh \chi \right)^2
\]  

where \( H = \frac{da}{d\tau} a^{-2} \) is the Hubble parameter. Let’s fix the amplitude of the proper acceleration \( \xi^\alpha \) to be equal to the gravitational acceleration at the surface of the Earth, \( g \),
and find an equation describing how the rapidity changes with proper time.

In the Friedmann-Robertson-Walker spacetime we get
\[ \frac{d\chi}{ds} = g - H \sinh \chi. \] (8)

The second term is a friction term due to the expansion of the universe. If \( a(\tau) \) is constant (i.e. \( H = 0 \)), we recover Minkowsky spacetime and the usual result that \( \chi \) is the product of the proper time of the rocket and its acceleration. If we equate these two terms we find the maximum value of the rapidity for a fixed proper acceleration
\[ \sinh \chi = \beta \gamma = \frac{g}{H} \approx 1.4 \times 10^{10}. \] (9)

An object that exceeds this rapidity will experience an acceleration that exceeds \( g \), so Eq. (9) is a practical speed limit for humans in an expanding universe.

Let’s specialize to de Sitter space where \( H \) is constant to get
\[ \chi(s) = 2 \tanh^{-1} \left\{ \frac{\tanh \left( \frac{1}{2} (s - C) g' + H \right)}{g} \right\} \] (10)
\[ \tau(\chi) = \frac{\sinh \chi}{g}, \quad x(\chi) = \cosh \chi - 1 \] (11)
\[ t(\chi) = \frac{1}{H} \ln \left( 1 - \frac{H}{g} \sinh \chi \right) \] (12)
\[ l(\chi) = \frac{1}{H} \left[ 2 \frac{g}{g'} \tanh^{-1} \left( \frac{g'}{g} \tanh \frac{\chi}{2} + \frac{H}{g'} \right) - \chi \right] \] (13)
where \( g' = \sqrt{g^2 + H^2} \approx g \), \( C = -2g'^{-1} \tanh^{-1}(H/g') \) and \( l \) is the proper distance \((dl = adx)\) along the path. We have taken \( \chi = \tau = x = t = l = 0 \) at \( s = 0 \). Because \( g \gg H \) we have
\[ \chi(s) = g(s - C) \] (14)
when \( H(g')^{-2} \ll (s - C) \ll g^{-1}\ln(g/H) \). For \( g = 980 \text{ m s}^{-2} \), we have \( s - C \gg 3 \text{ ms} \) and the second term in the numerator of Eq. (10) within the braces dominates. The linear increase of \( \chi \) with proper time stops when the condition given by Eq. (10) is approached; that is, when \( s \approx g^{-1}\ln(g/H) \approx 23 \text{ years} \).

Even if \( H \) varies with time, we can find a solution to Eqs. (8) and (9) for \( g = 0 \): \( a \sinh \chi = a_0 \) where \( a_0 \) is a constant of integration. We see that the value of \( \beta \gamma a \) is constant along a geodesic of a massive particle in the metric given by Eq. (11). We find that a particle travelling along a geodesic with finite rapidity today only has an infinite rapidity as \( a \to 0 \).

Let’s return to the general case with \( g \neq 0 \) and try to obtain a bound on the proper time required to travel between two scale factors. After the initial increase in the rapidity until \( s \approx g^{-1}\ln(g/H) \approx 23 \text{ years} \), the rapidity saturates so we have
\[ \frac{dr}{ds} \approx \frac{dx}{ds} \approx \frac{g}{Ha}. \] (15)

In a general Robertson-Walker spacetime, the Hubble parameter changes with time, but it is reasonable to assume that Eq. (15) still holds as long as \( H \) does not change drastically over the timescale \( 1/H \).

The preceding equation provides a crude estimate of the proper time.

If the Hubble parameter is increasing with time \((w < -1)\) Eq. (16) provides an approximate upper bound on the proper time and conversely it provides an approximate lower bound if \( w > -1 \) and the Hubble parameter decreases with time. The error in this estimate is most dramatic for \( w < -1 \). In this case, the curvature of the universe diverges \((a \to \infty)\) within a finite proper time along geodesics. Because geodesics are paths between events that maximize the proper time, we find in general for \( w < -1 \) the rocketeer reaches \( a \to \infty \) within a finite proper time, in fact within a shorter time than if she stayed at home.

On the other hand if the Hubble parameter is constant or decreasing with time \((w \geq -1)\), the condition \( \sinh \chi = g/H \) is only approached but never reached within a finite proper time, so we have from Eq. (8)
\[ \frac{dr}{ds} < \sqrt{1 + \frac{g^2}{H^2}} \] (17)
\[ \frac{1}{a} \frac{da}{ds} < \sqrt{H^2 + g^2} \] (18)
\[ \frac{1}{a \sqrt{H^2 + g^2}} da < ds. \] (19)

Integrating this for a particular value of \( w = P/\rho \) so that \( H^2 \propto a^{-(3w+3)} \), the equation of state, yields
\[ \Delta s > \left\{ \begin{array}{ll}
\frac{(g')^{-1}\ln\left( \frac{a_f}{a_i} \right)}{a_i} & \text{if } w = -1 \\
\frac{2}{3w+3}g'^{-1}\tanh^{-1}\left( \frac{g'}{a_i} \right) & \text{if } w > -1 .
\end{array} \right. \] (20)

As a reminder \((g')^2 = H_0^2 a^{-(3w+3)} + g^2 \). If we take \( a_f \to \infty \) we find that both expressions diverge, so for \( w > -1 \) the rocketeer requires an infinite proper time to reach \( a = \infty \).

NUMERICAL RESULTS

Before delving into the calculational details for our universe with \( \Omega_1 \approx 0.27, w_1 = 0 \) and \( \Omega_2 \approx 0.73, w_2 \approx -1 \).
and nearly flat spatial hypersurfaces \[ \mathbb{R}^2 \times S^1 \], let’s outline three possible trajectories for the rocket. Fig. 1 depicts these choices for the trips of infinite elapsed proper time if \(-1/3 > w_2 \geq -1\). Because the rocket’s path asymptotically approaches a null geodesic it is straightforward to calculate the maximum comoving distance that our rocket can travel. If it returns or ends at rest with respect to the locally comoving material at a position where it still can communicate with Earth we have \( x_{\text{max}} = 1/(2H) \) (with an error on the order of \( g^{-1} \)). If the rocket continually accelerates it will reach \( x_{\text{max}} = 1/H \) after an infinite proper time has elapsed. If the universe stops accelerating in the future, the rocket can travel arbitrarily far in an arbitrarily long proper time. Conversely, if \( w_2 < -1 \), a comoving observer will find that the universe only will exist for a finite proper time and it is no different for the rocketeer.

Regardless of the details of the cosmology, on the outbound trip the Earth can send messages and in principle energy for a time \( (1 + a_f/a_i)/g \) where \( a_f/a_i \) is the ratio of the initial scale factor at launch to the final scale factor during the deceleration. During the return trip the rocket can receive messages from Earth emitted between a time \( (1 + a_f/a_i)/g \) after launch until the rocket returns. Conversely, all the messages sent from the rocket to Earth during the outbound phase arrive between the launch and a time before \( (a_r/a_f)/g \) of the rocket’s return where \( a_r \) is the scale factor of the universe when the rocket returns. Messages sent during the return trip arrive within a time \( (a_r/a_f)/g \) of the rocket’s return.

In practice because \( g \gg H \) the three possibilities result in a similar comoving distance travelled (at the moment of last contact) – this difference in the comoving distance that the rocketeer travels where it can make a final communication with Earth is only \( (ag)^{-1} \ll H^{-1} \). However, none of these paths are realistic because they take an infinite proper time to traverse. It turns out that paths that travel 90% or even 99% of the maximal comoving distance take about only 100 years to traverse, so paths displayed in Fig. 1 are interesting to study even for realistic journeys.

The maximum comoving distance that one can reach is simply the conformal time remaining until \( a \rightarrow \infty \) \((1.12339)/H_0 \) for \( w = -1, \Omega_{M,0} = 0.27 \) and \( \Omega_{\Lambda,0} = 0.73 \) and half that comoving distance for the round-trip journey. What is more interesting to know is the current age and redshift of objects that we observe today at the particular comoving distances, so one could say “Let’s go to that galaxy” and know whether it is possible.

Fig. 2 shows the current redshift of objects observed today at the maximal comoving distance that can be achieved during one-way and round-trip journeys. Essentially for the favoured value of \( w = -1 \), our rocketeer could visit a galaxy at a redshift of 0.65 if she intended to return to the Milky Way (or report her results) or 1.7 if she didn’t intend to come back. As the value of \( w \) approaches \(-1/3 \), she could travel to further and further distances because the acceleration of the universe is weaker. At \( w = -0.6 \) we have \( z = 3.2 \) (round-trip) or \( z = 83 \) (one-way). An important point to keep in mind is that these “one-way” trips are literally one-way. At the end of the rocket’s journey, the rocket lies outside the asymptotic past-light cone of the Milky Way (to the right of the diagonal line in Fig. 1).

To be more realistic we would like to see how long would a journey last that almost reaches the maximum distance (achieving the maximum distance takes an infinite proper time). Specifically, we use 90% and 99% of the maximal comoving distance that can be traversed on one-way and round-trip journeys. Typically, round-trips take approximately 100 years and one-way trips take a bit more than half as long, but the one-way trips of course reach much higher comoving distances. The reason for this is that the distance travelled by the rocket increases exponentially until the rocket is travelling a cosmological distance each year approximately; therefore, only the portion of the journey with \( s \gtrsim g^{-1} \ln(g/H_0) \approx 23 \) years contributes significantly. The round-trip journeys typically accelerate for only slightly more than 23 years (23.3

FIG. 1: Three possible paths (bold curves) for the rocketeer for deSitter space and the light cones that these curves asymptotically approach. On the lowermost path, the rocket is constantly accelerating and sends a signal back to Earth that arrives on Earth after an infinite time. The middle path shows a rocket that decelerates and stops relative to the comoving material and sends a signal back to Earth and the uppermost path shows a rocketeer that returns to Earth albeit after an infinite proper time.
FIG. 2: The currently observed redshift (upper set of curves) and age of galaxies (lower set of curves) located at the maximal comoving distances that can be achieved on a one-way (outer curves) and round-trip journeys (inner curves).

FIG. 3: The upper panel shows the duration of the journey according to the rocketeer. The upper curves are for the round-trip journey lasting 99% and 90% of the remaining conformal time. The lower curves show the currently observed value of redshift at the far end of the journey. The upper set is for one-way trips, and the lower set is for round trips.

and 23.5 for the case of $w = -1$, whereas the one-way journeys accelerate for a few more years before slowing down, so they travel twice the comoving distance.

The journeys that travel to 99% of the maximum distance only last slightly longer in the frame of the rocket than journeys that reach only 90% of the maximal distance. However, the elapsed time on Earth (or whatever remains of the Earth) can differ by up to an order of magnitude for $w \sim -0.6$ or at least a factor of two for $w > -1$. For $w = -1$ the round-trip lasts about 36 Gyr in the frame of the Milky Way for the shorter journey and 72 Gyr for the longer journey. The vast majority of the elapsed conformal time is during the return trip while the rocket skims along the final past light-cone of the Milky Way. The outward journey lasts only 9 Gyr or 11 Gyr in the proper time measured by a comoving observer.

The place where the rocket comes to rest finally depends quite sensitively on the proper time when the rocket jerks. When the rocket goes from accelerating to decelerating, it is typically travelling with $\beta \gamma \approx 1.4 \times 10^{10}$ (see Eq. 9) or even faster if $H$ decreases with time $w > -1$, so an error in the jerk time of one second would result in the rocket coming to a stop about 140 pc off course. In practice this error would not be terribly important because by the time the rocket returned, the sun would have long since become a white dwarf, and presumably the rocketeer would find little familiar upon her return.

CONCLUSIONS

The theoretical treatment of an accelerating rocket in a cosmological setting is only marginally more difficult than in flat spacetime. In the de Sitter spacetime the rocket’s trajectory may be solved analytically. However, unlike in flat spacetime, there is a limit to the rapidity that a rocket can achieve in an expanding spacetime, so after the rocket travels for a time of approximately $g^{-1} \ln(g/H_0)$ or 23 years, the distance travelled by the rocket no longer increases exponentially with the proper time of the rocketeer; in the case of de Sitter space the distance increases linearly with the proper time in this limit.

A more practical question is how much of the universe that we observe today could we visit with future rocket technology. The paths that the rocket can complete in a finite proper time cannot reach all the way to the future light cone of the Earth, but the rocket can go 99% of the way in a reasonable proper time of about a century; consequently, if future technology can bring us to the stars, travelling to distant galaxies could soon follow.


