Deformed Coherent and Squeezed States of Multiparticle Processes

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ABSTRACT

Deformed squeezed states are introduced and it is shown that the boundary vectors in the matrix-product states approach to multiparticle diffusion processes are deformed coherent or squeezed states of a deformed harmonic-oscillator algebra. A coherent states solution to the n -species boundary problem is proposed and studied.

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1. Introduction

Coherent states have a wide range of applications to various problems in many different areas of physics. Introduced by Schrödinger [1] in the early days of quantum mechanics, the harmonic-oscillator coherent states were developed for the first time by Glauber for quantized electromagnetic radiation [2]. They were generated by the action of the displacement operator on the ground state, or equivalently defined as eigenstates of the annihilation operator, and turned out to be orbits of the Weyl–Heisenberg group. This important property led to group-theoretical generalizations by Perelomov [3] and Gilmore [4] for arbitrary Lie group and to the formulation of coherent states as orbits of the group with respect to a stationary subgroup.

With the invention of the quantum groups and the hopes that rich non-commutative structures will amount to new results in field theory and statistical physics, generalized coherent states [5, 6] for the deformed Heisenberg algebra and for compact quantum groups [6] were introduced and studied.

Coherent states exhibit two basic characteristics, namely continuity and resolution of unity, which are the minimum requirements for a set of vectors to be referred to as generalized coherent states. According to Klauder [7], a coherent state $|l\rangle$ where the (complex) label l is an element of an appropriate label space $\mathcal{L}$, endowed with the notion of topology, is a vector of a Hilbert space $\mathcal{H}$ such that (i) the vector $|l\rangle$ is strongly continuous in the label l , (ii) there exists a positive measure δl on $\mathcal{L}$ so that the unit operator on $\mathcal{H}$ admits a resolution of unity $I = \int |l\rangle\langle l| \delta l$. Consequently any quantum state $|\psi\rangle$ can be represented by its projections onto the different coherent states $\psi(l) = \langle l|\psi\rangle$, and similarly any operator A can be represented by its coherent-states matrix elements $\frac{\langle l|A|l'\rangle}{\langle l|l'\rangle}$.

By origin the coherent states are quantum states, but at the same time they are parametrized by points in the phase space of a classical system. This makes them very suitable for the study of systems where one encounters a relationship between classical and quantum descriptions. From this point of view, interacting many-particle systems with stochastic dynamics provide an appropriate playground to enhance the utility of generalized coherent states.

A stochastic process is described in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1, 2, \dots, n-1$ at a site $i = 1, 2, \dots, L$ of a linear chain. A configuration on the lattice at a time t is determined by the set of occupation numbers $s_1, s_2, \dots, s_L$ and a transition to another configuration s' during an infinitesimal time step dt is given by the probability $\Gamma(s, s')dt$. The time evolution of the stochastic system is governed by the master equation

$$\frac{dP(s, t)}{dt} = \sum_{s'} \Gamma(s, s') P(s', t) \quad (1)$$

for the probability $P(s, t)$ of finding the configuration s at a time t . With the restriction of dynamics that changes of configuration can only occur at two adjacent sites, the rates for such changes depend only on these sites. The two-site rates $\Gamma \equiv \Gamma_{jl}^{ik}$, $i, j, k, l = 0, 1, 2, \dots, n-1$ are assumed to be independent from the position in the bulk. At the boundaries, i.e. sites 1 and L , additional processes can take place with single-site rates L_k^i and R_k^i , $i, k = 0, 1, \dots, n-1$. For processes where each lattice site can be occupied by a finite number of different-type particles,

the master equation can be mapped to a Schrödinger equation in imaginary time of an n -state quantum spin- S Hamiltonian ($n = 2S + 1$ distinct states) with nearest-neighbour interaction in the bulk and single-site boundary terms

$$\frac{dP(t)}{dt} = -HP(t), \quad H = \sum_j H_{j,j+1} + H^{(L)} + H^{(R)}. \quad (2)$$

The probability distribution thus becomes a state vector in the configuration space of the quantum spin chain and the ground state of the Hamiltonian, in general non-Hermitian, corresponds to the stationary probability distribution of the stochastic dynamics. The mapping provides a connection with integrable quantum spin chains and allows for exact results of the stochastic dynamics with the formalism of quantum mechanics.

A different description, which is also based on the relationship of a Markov process probability distribution with the quantum Hamiltonian picture, is the matrix-product states approach to stochastic dynamics [8, 9]. The idea is that the stationary probability distribution, i.e. the ground state of a quantum Hamiltonian with nearest-neighbour interaction in the bulk and single-site boundary terms can be expressed as a product of (or a trace over) matrices that form a representation of a quadratic algebra

$$\Gamma_{jl}^{ik} D_i D_k = x_l D_j - x_j D_l, \quad i, j, k, l = 0, 1, \dots, n-1, \quad (3)$$

determined by the dynamics of the process. For diffusion processes that will be considered in this paper, $\Gamma_{ki}^{ik} = g_{ik}$ and the n -species diffusion quadratic algebra has the form

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k, \quad (4)$$

where g_{ik} and g_{ki} are positive (or zero) probability rates, x_i are c -numbers and $i, k = 0, 1, \dots, n-1$. (No summation over repeated indices in eq. (4).) The algebra has a Fock representation in an auxiliary Hilbert space where the n generators D act as operators. For systems with periodic boundary conditions, the probability distribution is given by the expression

$$P(s_1, \dots, s_L) = \text{Tr}(D_{s_1} D_{s_2} \dots D_{s_L}). \quad (5)$$

When boundary processes are considered the probability distribution is given by a matrix element in the auxiliary vector space

$$P(s_1, \dots, s_L) = \langle w | D_{s_1} D_{s_2} \dots D_{s_L} | v \rangle \quad (6)$$

with respect to the vectors $|v\rangle$ and $\langle w|$, determined by the boundary conditions

$$\langle w | (L_i^k D_k + x_i) = 0, \quad (R_i^k D_k - x_i) | v \rangle = 0, \quad (7)$$

where the x -numbers sum up to zero, because of the form of the boundary rate matrices

$$L_i^i = - \sum_{j=0}^{n-1} L_j^i, \quad R_i^i = - \sum_{j=0}^{n-1} R_j^i, \quad \sum_{i=0}^{n-1} x_i = 0. \quad (8)$$

These relations simply mean that one associates with an occupation number s_i at position i a matrix $D_{s_i} = D_k$ ($i = 1, 2, \dots, L$; $k = 0, 1, \dots, n - 1$) if a site i is occupied by a k -type particle. The number of all possible configurations of an n -species stochastic system on a chain of L sites is n^L and this is the dimension in the configuration space of the stationary probability distribution as a state vector; each component of this vector is a trace or an expectation value in the auxiliary space given by (5) or (6). The quadratic algebra reduces the number of independent components to only monomials symmetrized upon using the relations (4).

The algebra (4) admits an involution through the mapping $D_i \rightarrow D_i^+$, ($D_i \rightarrow -D_i^+$) and $g_{ik}^+ = -g_{ki}$ ($g_{ik}^+ = g_{ki}$) for real parameters $x_i = \bar{x}_i$.

Relations (4) allow an ordering of the elements D_k and, in order to find the stationary probability distribution, one has to compute traces or matrix elements with respect to the vectors $|v\rangle$ and $\langle w|$ of ordered monomials of the form

$$D_{s_1}^{m_1} D_{s_2}^{m_2} \dots D_{s_L}^{m_L} \quad (9)$$

Monomials of given order are the Poincaré–Birkhoff–Witt (PBW) basis for polynomials of fixed degree, as is the probability distribution. The n elements D_k obeying $n(n - 1)/2$ relations (4) generate an associative algebra with a unit e for which the alphabetically ordered monomials (9) form a linear basis, the PBW basis.

In the known example of exactly soluble 2- and 3-species models, through the matrix product ansatz, the solution of the quadratic algebra is provided by a deformed bosonic oscillator algebra, if both g_{ik} and g_{ki} differ from zero, or by infinite-dimensional matrices, if $g_{ik} = 0$. In the general n case, because of the ordering procedure, the solution of the quadratic algebra has to be consistent with the diamond lemma in ring theory, also known as the braid associativity condition in quantum groups. As shown in [10, 11], if all parameters x_i are equal to zero on the RHS of eq. (4), the homogeneous quadratic algebra defines a multiparameter quantized non-commutative space realized equivalently as a q -deformed Heisenberg algebra [12, 13] of n oscillators depending on $n(n - 1)/2 + 1$ parameters (in general on $n(n - 1)/2 + n$ parameters):

$$\begin{aligned} a_i a_i^+ - r_i a_i^+ a_i &= 1 \\ a_i^+ a_j^+ - q_{ji} a_j^+ a_i^+ &= 0 \\ a_i a_j - q_{ji} a_j a_i &= 0 \\ a_i a_j^+ - q_{ji}^{-1} a_j^+ a_i &= 0, \end{aligned} \quad (10)$$

where $i < j$; $i, j = 0, 1, \dots, n - 1$, the deformation parameters r_i, q_{ij} are model-dependent parameters given in terms of the probability rates, and the associative algebra generated by the elements D_i in this case belongs to the UEA of the multiparameter deformed Heisenberg algebra. For a non-homogeneous algebra with x -terms on the RHS of (4), only then is braid associativity satisfied if, out of the coefficients x_i, x_k, x_l corresponding to a triple $D_i D_k D_l$, either one coefficient x is zero or two coefficients x are zero, and the rates are respectively related. The diffusion algebras in this case can be obtained by either a change of basis in the n -dimensional non-commutative space or by a suitable change of basis of the lower-dimensional quantum space realized equivalently as a lower-dimensional deformed Heisenberg algebra. The appearance of

the non-zero linear terms in the RHS of the quantum plane relations leads to a lower-dimensional non-commutative space.

Proposition I

The boundary vectors with respect to which one determines the stationary probability distribution of the n -species diffusion process are generalized, coherent or squeezed states of the deformed Heisenberg algebra underlying the algebraic solution of the corresponding quadratic algebra.

The paper is organized as follows. We first review the basic properties of the deformed oscillator coherent states that are known in the literature. We then define a deformed squeezed state of a pair of deformed oscillators by analogy with the conventional squeezed states. Such a generalization of the undeformed squeezed states is not known. As a physical application we consider the solutions of the two-species asymmetric exclusion process and argue that, depending on the boundary conditions, the boundary vectors are either the deformed boson operator coherent or the suggested deformed squeezed states. We finally propose a coherent state solution of the boundary problem of the n -species asymmetric diffusion process.

2. Coherent states of a q -deformed Heisenberg algebra

The conventional harmonic-oscillator coherent states are defined either (i) directly by the action of the displacement operator $D(z) = \exp(za^+ - \bar{z}a)$ on the vacuum, or (ii) as an eigenstate of the annihilation operator a . It is the displacement operator method that best reveals the group geometric properties of the coherent states; it allowed for generalization to arbitrary Lie groups, but it turned out not to work successfully for quantum groups with conventional complex variables z . The generalization to the deformed boson case went along the annihilation operator method and we review here the main lines of the known results [5].

We consider an associative algebra with generators a , a^+ and $q^{\pm N}$ with defining relations

$$aa^+ - qa^+a = 1, \quad q^Na^+ = qa^+q^N, \quad q^Na = q^{-1}aq^N, \quad (11)$$

where $0 < q < 1$ is a real parameter and

$$a^+a = \frac{1 - q^N}{1 - q} \equiv [N]. \quad (12)$$

A Fock representation is obtained in a Hilbert space spanned by the orthonormal basis $\frac{(a^+)^n}{\sqrt{[n]!}}|0\rangle = |n\rangle$, $n = 0, 1, 2, \dots$ and $\langle n|n'\rangle = \delta_{nn'}$:

$$a|0\rangle = 0, \quad a|n\rangle = [n]^{1/2}|n-1\rangle, \quad a^+|n\rangle = [n+1]^{1/2}|n+1\rangle. \quad (13)$$

The Hilbert space consists of all elements $|f\rangle = \sum_{n=0}^{\infty} f_n|n\rangle$ with complex f_n and finite norm with respect to the scalar product $\langle f|f\rangle = \sum_{n=0}^{\infty} |f_n|^2$. The q -deformed oscillator algebra has a Bargmann–Fock representation on the Hilbert space of entire analytic functions.

Generalized or q -deformed coherent states are defined as the eigenstates of the deformed annihilation operator a and are labelled by a continuous (in general complex) variable z :

$$a|z\rangle = z|z\rangle, \quad |z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} |n\rangle. \quad (14)$$

These vectors belong to the Hilbert space for $|z|^2 < [\infty] = \frac{1}{1-q}$. The scalar product of two coherent states for different values of the parameter z is non-vanishing

$$\langle z|z'\rangle = \sum_0^{\infty} \frac{(\bar{z}z')^n}{[n]!} = e_q^{\bar{z}z'} \quad (15)$$

and they can be properly normalized with the help of the q -exponent on the RHS of (15):

$$|z\rangle = \exp_q\left(-\frac{|z|^2}{2}\right) \exp_q(za^+)|0\rangle. \quad (16)$$

The q -deformed coherent states reduce to the conventional coherent states of a one-dimensional Heisenberg algebra in the limit $q \rightarrow 1$. These generalized coherent states carry the basic characteristics of the conventional ones, namely continuity and completeness (resolution of unity). We briefly sketch the main properties as they were analysed in relation to the deformed algebra representation on the Hilbert space of entire analytic functions. The representation space is spanned by the orthonormal basis of polynomials

$$u_n = \frac{z^n}{\sqrt{[n]!}}, \quad n = 0, 1, 2, \dots \quad (17)$$

and a scalar product of two elements $g(z)$ and $f(z)$ is given by

$$\langle g|f\rangle = \int \bar{g}(z) f(z) \exp_q(-\bar{z}z) d_q^2 z. \quad (18)$$

The integration in (18) over the complex variable $z = |z| \exp(i\phi)$ is performed as

$$\int d_q^2 z = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{\infty} d_q |z|^2 \quad (19)$$

and the Jackson q -integral of a function $F(x)$ of a real variable x is defined as the inverse of the q -derivative D_q

$$\int_0^{\infty} F(x) d_q x = (1-q)x \sum_{l=0}^{\infty} q^l F(q^l x), \quad D_q f(z) = \frac{f(z) - f(qz)}{z - qz}. \quad (20)$$

The scalar product in (18) allows for a resolution of the unit operator

$$I = \int |z\rangle \langle z| \exp_q(-|z|^2) d_q^2 z. \quad (21)$$

Using the completeness relation one can expand any state $|f\rangle$ in the coherent states:

$$|f\rangle = \int d_q^2 |z\rangle \exp_q(-|z|^2) \langle \bar{z}|f\rangle, \quad (22)$$

where the function $f(z) = \langle \bar{z}|f\rangle$ determines the state completely and is called the symbol of the state. The completeness relation gives rise to a functional representation of operators as well

$$(Af)(z) = \int A(z, z') f(z') \exp_q(\bar{z}z) \exp_q(-\bar{z}'z') d_q^2 z', \quad (23)$$

$A(z, z')$ being the covariant symbol of the operator A

$$A(z, z') = \frac{\langle z|A|z'\rangle}{\langle z|z'\rangle}. \quad (24)$$

The trace of the operator A is given by

$$\text{Tr} A = \int d_q^2 z \exp_q(-|z|^2) \langle z|A|z\rangle. \quad (25)$$

One thus has

$$\begin{aligned} \langle \bar{z}|a^+|f\rangle &= z f(z) \\ \langle \bar{z}|a|f\rangle &= D_q f(z) \\ \langle \bar{z}|N|f\rangle &= z \frac{d}{dz} f(z), \end{aligned} \quad (26)$$

which is the Bargmann–Fock representation of the deformed oscillators and number operator. As shown in [5], in the limit $q \rightarrow 1$, the Bargmann–Segal representation space of the undeformed algebra on a Hilbert space of entire functions is obtained, while in the limit $q \rightarrow 0$, the Hilbert space becomes the Hardy–Lebesgue space of functions on the circle $|z| = 1$.

In view of the eigenvalue properties the coherent-state matrix elements of normally ordered operators $:O(a^+, a):$, i.e. written in such a form that all creation operators are to the left of the annihilation operators, are very simple to evaluate $\langle z'| :O(a^+, a) : |z\rangle = O(\bar{z}', z) \langle z'|z\rangle$, which enhances their utility in a wide variety of physical problems. In particular a polynomial $O(a^+, a)$ in deformed creation and annihilation operators can be brought to the form

$$:O(a^+, a): = \sum_{m,n} O_{m,n}(q) (a^+)^m (a)^n \quad (27)$$

and its matrix element between deformed coherent states is

$$\langle z'| :O(a^+, a) : |z\rangle = \sum_{m,n} O_{mn}(q) \bar{z}'^m z^n \langle z'|z\rangle \quad (28)$$

3. Squeezed states

The conventional squeezed states [4] for the harmonic oscillator operators are obtained directly from a conventional coherent state $|z\rangle$ by applying the squeezed operator $S(\xi)$:

$$S(\xi)D(z)|0\rangle = |z, \xi\rangle, \quad (29)$$

where the unitary operator $S(\xi)$ depends on a (complex) parameter ξ . The squeezed state is an eigenstate of the transformed annihilation operator a

$$S(\xi)aS^\dagger(\xi) = A(a^+, a) \quad (30)$$

so that one has

$$A(a^+, a)|z, \xi\rangle = z|z, \xi\rangle. \quad (31)$$

The unitary transformation leaves the commutator $[a, a^+]$ invariant and can be realized as a linear canonical transformation

$$A(a^+, a) = \mu a + \nu a^+, \quad (32)$$

with $|\mu|^2 - |\nu|^2 = 1$. The unitary operator that leads to such a linear transform has the form

$$S(\xi) = \exp \frac{1}{2}(\xi(a^+)^2 - \bar{\xi}a^2) \quad (33)$$

and with real $\xi = s$ one has $A = a \cosh s + a^+ \sinh s$. The squeezed states are also equivalently defined as coherent states of the group $SU(1, 1)$ [3] by the action of the raising operator $K^+ = \frac{1}{2}(a^+)^2$ on the vacuum:

$$|s\rangle = \exp \frac{1}{2}s(a^+)^2|0\rangle. \quad (34)$$

Attempts at generalizing these definitions to the case of deformed oscillators have not been successful. As for the first definition there were discussions and argumentations that a squeezed operator (as well as a displacement operator $D(z)$) can be consistently defined [14] assuming the variables z, ξ to be non-commuting. On the other hand, the generalization of the second definition to the deformed case gives a state $\exp_q \frac{1}{[2]}\xi(a^+)^2|0\rangle$ that is not normalizable [15]. However, since the action of the (conventional) unitary squeezed operator results in a linear transformation on the oscillators, we are lead by this idea to keep the linear structure of the deformed squeezing operator and assume an analogous definition.

Proposition IIa

Let a, a^+ and q^N generate a deformed Heisenberg algebra with the equivalent form of defining relations

$$[a, a^+] = q^N, \quad q^N a = q^{-1} a q^N, \quad q^N a^+ = q a^+ q^N \quad (35)$$

Then there is a two-parameter-dependent linear map to a pair of “quasi”-oscillators with a “quasiparticle” number operator $\mathcal{N}$

$$A = \mu a + \nu a^+, \quad A^+ = \bar{\mu} a^+ + \bar{\nu} a. \quad (36)$$

These operators generate a deformed Heisenberg algebra with relations

$$[A, A^+] = q^N, \quad q^N A = q^{-1} A q^N, \quad q^N A^+ = q A^+ q^N, \quad (37)$$

provided

$$q^N = (|\mu|^2 - |\nu|^2) q^N \quad (38)$$

In the limit $q \rightarrow 1$ the relation between the parameters of the conventional squeezed state is recovered [17]. In the Fock representation space with a vacuum $|0\rangle_s$ one can define a normalizable coherent state $|\zeta\rangle_s$ as the eigenvector of the annihilation operator A :

$$|\zeta\rangle_s = e_q^{-\frac{1}{2}|\zeta|^2} e_q^{\zeta A^+} |0\rangle_s. \quad (39)$$

In order to generate a squeezed state directly one needs of course to explicitly construct an operator $S_q(\mu, \nu)$, the q -deformed analogue of the squeezed operator whose transformation of the oscillators amounts to the linear map in eq. (36), $S_q a S_q^{-1} = A$. This question remains open despite the encouraging fact that the linear transformation has the proper limit $q \rightarrow 1$.

Proposition IIb

A squeezed state of the deformed creation and annihilation operators is a normalized solution of the eigenvalue equation:

$$(\mu a + \nu a^+) |\zeta, \mu, \nu\rangle_s = \zeta |\zeta, \mu, \nu\rangle_s = A |\zeta\rangle_s. \quad (40)$$

This proposition is motivated by the analogy with the non-deformed case and by the fact that such normalized eigenstate vectors of the written above linear combination of q -deformed oscillators appear in the solution of the boundary problem of a many-particle non-equilibrium system.

Even though in the proposed definition (39 - 40) of the squeezed state as an eigenstate $|\zeta\rangle$ of the annihilation operator A , the μ, ν dependence seems to be suppressed, such states exhibit squeezing properties similar to the conventional ones. To show this we consider the Hermitian quadrature operators

$$x = \frac{1}{\sqrt{2}}(a + a^+), \quad p = \frac{1}{i\sqrt{2}}(a - a^+), \quad (41)$$

where the boson operators obey the relations of the form (35) and consequently the operators x and p satisfy the deformed canonical commutation relation

$$[x, p] = i q^N. \quad (42)$$

Writing down the accepted form of a deformed uncertainty relation for their variances in any state

$$(\delta x)^2 (\delta p)^2 \geq \frac{1}{4} |\langle [x, p] \rangle|^2 \quad (43)$$

where $(\delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle$ and similary $(\delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle$, and making use of expressions (41) for x and p in terms of a and a^+ , we find the uncertainties

$$\begin{aligned} (\delta x)^2 &= \langle x^2 \rangle - \langle x \rangle^2 = u_1 + u_2 \\ (\delta p)^2 &= \langle p^2 \rangle - \langle p \rangle^2 = u_1 - u_2, \end{aligned} \quad (44)$$

where

$$\begin{aligned} u_1 &= \frac{1}{2} \langle a^+ a + a a^+ \rangle - \langle a \rangle \langle a^+ \rangle \\ u_2 &= \langle a^2 \rangle - \langle a \rangle^2 + \langle (a^+)^2 \rangle - \langle a^+ \rangle^2. \end{aligned} \quad (45)$$

If one calculates now the mean values in u_1 and u_2 with respect to the deformed coherent states $|z\rangle$ of the oscillators a and a^+ , one finds that $u_2 = 0$, i.e. the deformed uncertainties are equal $\delta x = \delta p$. This equality of the deformed x, p uncertainties has been discussed in previous works and we shall not comment on it here (see [16] for details). It is our aim to calculate the uncertainties with respect to the squeezed states $|\zeta\rangle_s$ and to show that, analogously to the conventional case, they are not equal since $u_2 \neq 0$. For the purpose we first write the inverse of the linear map in (36)

$$\begin{aligned} a &= \bar{\mu}A - \nu A^+ \\ a^+ &= -\bar{\nu}A + \mu A^+. \end{aligned} \quad (46)$$

The next step is to calculate the quantities u_1 and u_2 with respect to the coherent states of the bosonic pair A, A^+ in the Hilbert space of the Bargmann–Fock representation of these operators. Exploring the eigenvalue properties of the normalized coherent eigenstates of A , we simply have

$$\begin{aligned} \langle \zeta | A | \zeta \rangle_s &= \zeta \\ \langle \zeta | A^+ | \zeta \rangle_s &= \bar{\zeta} \\ \langle \zeta | q^{\mathcal{N}} | \zeta \rangle_s &= \langle \zeta | q^{\zeta \frac{d}{d\zeta}} | \zeta \rangle_s = e^{(q-1)|\zeta|^2}. \end{aligned} \quad (47)$$

To find the mean values in $u_{1,2}$ with respect to $|\zeta\rangle_s$, we use the expressions for a, a^+ in terms of A, A^+ according to eqs. (46) and with the help of eqs. (47) we obtain

$$\begin{aligned} 2u_1 &= (\bar{\mu}\mu + \bar{\nu}\nu) \exp_q((q-1)|\zeta|^2) \\ 2u_2 &= (-\bar{\mu}\nu - \bar{\nu}\mu) \exp_q((q-1)|\zeta|^2). \end{aligned} \quad (48)$$

Since $u_2 \neq 0$, this yields a non-equality of the q -deformed uncertainties $\delta x \neq \delta p$. In the limit $q \rightarrow 1$ the corresponding expressions for the x, p uncertainties with respect to the conventional harmonic oscillator squeezed states [17] are recovered. This analogy with the squeezing properties of the quadratures of the boson creation and annihilation operators justifies, in our opinion, the proposed definition of a q -deformed squeezed state in (40).

4. Physical applications

We consider a diffusion process with n species on a chain of L sites with nearest-neighbour interaction with exclusion, i.e. a site can be either empty or occupied by a particle of a given type. In the set of occupation numbers $(s_1, s_2, \dots, s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, $s_i = 1$ if there is a first-type particle at a site $i, \dots, s_i = n-1$ if there is an $(n-1)$ th-type particle at a site i . On successive sites the species i and k exchange places with probability $g_{ik}dt$, where $i, k = 0, 1, 2, \dots, n-1$. With $i < k$, g_{ik} are the probability

rates of hopping to the left, and g_{ki} to the right. The event of exchange happens if out of two adjacent sites one is a vacancy and the other is occupied by a particle, or each of the sites is occupied by a particle of a different type. The n -species symmetric exclusion process is known as the lattice gas model of particle hopping between nearest-neighbour sites with a constant rate $g_{ik} = g_{ki} = g$. The n -species asymmetric exclusion process with hopping in a preferred direction is the diffusion-driven lattice gas of particles moving under the action of an external field. The process is totally asymmetric if all jumps occur in one direction only, and partially asymmetric if there is a different non-zero probability of both left and right hopping. The number of particles n_i of each species in the bulk is conserved and this is the case of periodic boundary conditions. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density. In most studied examples one considers phase transitions inducing boundary processes when a particle of type k is added with a rate L_k^0 and/or removed with a rate L_0^k at the left end of the chain, and it is removed with a rate R_0^k and/or added with a rate R_k^0 at the right end of the chain.

4.1. The two-species asymmetric simple exclusion process, with only incoming particles at the left boundary and only outgoing particles at the right one

The system is described by the configuration set $s_1, s_2, \dots, s_L$ where $s_i = 0$ if a site is empty and $s_i = 1$ if a site is occupied by a particle. The particles hop with a probability $g_{01}dt$ to the left and with a probability $g_{10}dt$ to the right. Without loss of generality we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The totally asymmetric exclusion process of particles hopping to the right only is obtained for $q = 0$. At the left boundary a particle can be added with a probability αdt and it can be removed at the right boundary with a probability βdt .

The model is exactly solvable through the matrix-product states approach [8, 18]. One considers an associative quadratic algebra with a unit and two generators obeying the relations: Case A - the partially asymmetric simple exclusion process:

$$D_1 D_0 - q D_0 D_1 = D_0 + D_1, \quad (49)$$

Case B - the totally asymmetric simple exclusion process ($q = 0$):

$$D_1 D_0 = D_0 + D_1 \quad (50)$$

with the same boundary conditions defining in both cases the boundary vectors $\langle w|$ and $|v\rangle$:

$$\langle w|D_0 = \langle w|\frac{1}{\alpha} \quad D_1|v\rangle = \frac{1}{\beta}|v\rangle. \quad (51)$$

For a given configuration $(s_1, s_2, \dots, s_L)$ the stationary probability distribution is related to an ordered product $X_1 X_2 \dots X_L$, at a site i $X_i = D_1$ if a site is occupied and $X_i = D_0$ if a site is empty. The probability distribution is given by the expectation value

$$P(s) = \frac{\langle w|X_1 X_2 \dots X_L|v\rangle}{Z_L}, \quad (52)$$

where $Z_L = \langle w|(D_0 + D_1)^L|v\rangle$ is a normalization factor. To simplify the notations one writes $D_0 + D_1 = C$. Within the matrix-product ansatz, one can also evaluate physical quantities such as the current J through a bond between site i and site $i + 1$, the mean density $\langle s_i \rangle$ at a site i , the two-point correlation function $\langle s_i s_j \rangle$:

$$\begin{aligned} J &= \frac{Z_{L-1}}{Z_L} \\ \langle s_i \rangle &= \frac{\langle w|C^{i-1}D_1C^{L-i}|v\rangle}{Z_L} \\ \langle s_i s_j \rangle &= \frac{\langle w|C^{i-1}D_1C^{j-i-1}D_1C^{L-j}|v\rangle}{Z_L} \end{aligned} \quad (53)$$

and higher correlation functions. The algebraic solutions for the partially and for the totally asymmetric cases with the corresponding boundary problems have been found in [8] and [18], respectively. They are of the form of shifted deformed oscillators for a real parameter $0 < q < 1$ and for $q = 0$, respectively.

Case A

$$\begin{aligned} D_0 &= \frac{1}{1-q} + \frac{a^+}{\sqrt{1-q}}, \\ D_1 &= \frac{1}{1-q} + \frac{a}{\sqrt{1-q}}. \end{aligned} \quad (54)$$

To solve the boundary problem we choose the vector $|v\rangle$ to be the (unnormalized!) eigenvector of the annihilation operator a for a real value of the parameter v and the vector $\langle w|$ to be the eigenvector (unnormalized and different from the conjugated one) of the creation operator for the real parameter w :

$$|v\rangle = e_q^{-\frac{1}{2}vw} e_q^{va^+} |0\rangle \quad \langle w| = \langle 0| e_q^{wa} e_q^{-\frac{1}{2}wv} \quad (55)$$

The factor $e_q^{-\frac{1}{2}vw}$ in (55) is due to the condition $\langle w|v\rangle = 1$, which is a convenient choice in physical applications. According to the algebraic solution, these are also eigenvectors of the shifted operators with the corresponding relations of the eigenvalues

$$\begin{aligned} \frac{1}{\alpha} &= \frac{1}{1-q} + \frac{w}{\sqrt{1-q}}, \\ \frac{1}{\beta} &= \frac{1}{1-q} + \frac{v}{\sqrt{1-q}}. \end{aligned} \quad (56)$$

Hence the boundary vectors $|v\rangle$ and $\langle w|$ are a subset of the coherent states of the q -deformed Heisenberg algebra, labelled by the positive real parameters $v(\alpha, q)$ and $w(\beta, q)$ defined in (56). The relation of the boundary vectors to the coherent states simplifies the calculation of the probability distribution. Since, according to the algebraic solution:

$$\begin{aligned} (D_0 + D_1)^L &= \left(\frac{2}{1-q} + \frac{a^+ + a}{\sqrt{1-q}} \right)^L \\ &= \sum_{m=0}^L \frac{L!}{m!(L-m)!} \frac{2^{L-m}}{(1-q)^{L-m} (\sqrt{1-q})^m} (a^+ + a)^m \end{aligned} \quad (57)$$

in order to find the expectation values with respect to the coherent states, one has to normally order the m -th power of the linear combination $a + a^+$, using $aa^+ - qa^+a = 1$. This is achieved with the help of the Stirling numbers

$$(a^+ + a)^m = \sum_{k=0}^{[m/2]} S_m^{(k)} \sum_{l=0}^{m-2k} \frac{[m-2k]!}{[l]![m-2k-l]!} (a^+)^l a^{m-2k-l} \quad (58)$$

where the q -deformed Stirling numbers $S_m^{(k)}$ satisfy the recurrence relation

$$S_{m+1}^{(k)} = [k]S_m^{(k)} + S_m^{(k-1)} \quad (59)$$

with $S_m^{(0)} = \delta_{0m}$, $S_m^{(1)} = S_m^{(m)} = 1$ and $S_m^{(m-1)} = \sum_{i=1}^{m-1} [i]$. For the correlation functions one also needs the expressions

$$\begin{aligned} a^k a^+ &= q^k a^+ a^k + [k] a^{k-1} \\ a(a^+)^k &= q^k (a^+)^k a + [k] (a^+)^{k-1} \end{aligned} \quad (60)$$

Using these relations one can easily find the relevant physical quantities of the system. Thus for the normalization factor Z^L one obtains

$$\langle w | (D_0 + D_1)^L | v \rangle = \sum_{m=0}^L \frac{L!}{m!(L-m)!} \frac{2^{L-m}}{(1-q)^{L-\frac{m}{2}}} \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)} \frac{[m-2k]!}{[l]![m-2k-l]!} w^l v^{m-2k-l} \quad (61)$$

It can be verified, after rescaling the parameters v and w by $\frac{1}{\sqrt{1-q}}$, that this expression coincides with the one evaluated in [18] up to the factor $\langle w | v \rangle$, which is chosen there to be $\langle w | v \rangle \neq 1$.

Case B

$$\begin{aligned} D_0 &= 1 + a_{q=0}^+, \\ D_1 &= 1 + a_{q=0}. \end{aligned} \quad (62)$$

As the algebra itself, the solution and the boundary vectors are also obtained as the limit $q \rightarrow 0$ of the q -dependent solution and eigenvectors where the representation of the oscillator operators in (62) is found from eqs. (13) with $q = 0$, namely

$$a^+ |n\rangle = |n+1\rangle, \quad a |n\rangle = |n-1\rangle \quad (63)$$

and

$$w = \frac{1-\alpha}{\alpha}, \quad v = \frac{1-\beta}{\beta}. \quad (64)$$

Hence the boundary vectors have the form

$$\langle w | = \langle n | \sum_{n=0}^{\infty} \left(\frac{1-\alpha}{\alpha} \right)^n \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta} \right)^{\frac{1}{2}} \quad |v\rangle = \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left(\frac{1-\beta}{\beta} \right)^n |n\rangle. \quad (65)$$

The infinitesimal matrices D_0 and D_1 in (62) and the boundary vectors (65) coincide with the corresponding ones found in [8]. The physical quantities of the model are readily obtained from the partially asymmetric case in the limit $q \rightarrow 0$. Equation (58) becomes simply

$$(a + a^+)^L |_{q=0} = \sum_{k=0}^{[m/2]} S_m^{(k)} |_{q=0} \sum_{l=0}^{m-2k} (a_{q=0}^+)^l (a_{q=0})^{m-2k-l}, \quad (66)$$

where now $S_{m+1}^{(k)}|_{q=0} = S_m^{(k)}|_{q=0} + S_m^{(k-1)}|_{q=0}$ and $S_m^{(m-1)}|_{q=0} = m - 1$. The expression for Z_L becomes

$$\langle w|(D_0 + D_1)^L|v\rangle = \sum_{m=0}^L \frac{2^{l-m}L!}{m!(L-m)!} \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)}|_{q=0} w^l v^{m-2k-l} \quad (67)$$

Inserting in eq. (67) the expressions for v and w in terms of α and β from (64), it can be verified, after some algebra, that it coincides with the expression for the normalization factor obtained in [8] (as the current and the correlation functions do coincide too). The coherent-state description thus provides a unified solution of the partially and fully asymmetric simple exclusion models.

4.2 The two-species model with incoming and outgoing particles at both boundaries

We consider now the two-species partially asymmetric exclusion process with different dynamics, as above; namely, at the left boundary a particle can be added with probability αdt and removed with probability γdt , and at the right boundary it can be removed with probability βdt and added with probability δdt . The quadratic algebra is the same as eq. (49), and it is solved by the deformed oscillators a, a^+ . The boundary conditions have the form

$$\begin{aligned} (\beta D_1 - \delta D_0)|v\rangle &= |v\rangle \\ \langle w|(\alpha D_0 - \gamma D_1) &= \langle w|, \end{aligned} \quad (68)$$

which read, in terms of the deformed boson operators:

$$\begin{aligned} (\beta a - \delta a^+)|v\rangle &= \sqrt{1-q} \left(1 - \frac{\beta - \delta}{1-q}\right) |v\rangle \\ \langle w|(\alpha a^+ - \gamma a) &= \langle w| \left(1 - \frac{\alpha - \gamma}{1-q}\right) \sqrt{1-q}. \end{aligned} \quad (69)$$

Hence, according to eq. (40), the boundary vectors $|v\rangle$ and $\langle w|$ are squeezed coherent states, eigenstates of an annihilation and a creation operator A, A^+ :

$$\begin{aligned} (\beta a - \delta a^+)|v\rangle &= A|v\rangle = v|v\rangle \\ \langle w|(\alpha a^+ - \gamma a) &= \langle w|A^+ = \langle w|w \end{aligned} \quad (70)$$

corresponding to the eigenvalues

$$\begin{aligned} v(\beta, \delta) &= \sqrt{1-q} \left(1 - \frac{\beta - \delta}{1-q}\right) \\ w(\alpha, \gamma) &= \sqrt{1-q} \left(1 - \frac{\alpha - \gamma}{1-q}\right) \end{aligned} \quad (71)$$

The explicit form of the (unnormalized) vectors in the oscillator Fock space representation is given by $\langle w| = \sum_{n=0}^{\infty} \frac{w^n(\alpha, \gamma)}{\sqrt{[n]!}} |n\rangle$, $|v\rangle = \sum_{n=0}^{\infty} \frac{v^n(\beta, \delta)}{\sqrt{[n]!}} |n\rangle$. Note that unlike eqs. (36) the operators A and A^+ are not each other's Hermitian conjugate. We shall comment on this in the next

section. To find the expectation values of normally ordered monomials in D_0 and D_1 , we make use of the inverse transformation

$$\begin{aligned} a &= \alpha A + \delta A^+ \\ a^+ &= \beta A^+ + \gamma A. \end{aligned} \quad (72)$$

Hence

$$D_0 + D_1 = \frac{2}{1-q} + \frac{\alpha + \gamma}{\sqrt{1-q}} A + \frac{\beta + \delta}{\sqrt{1-q}} A^+ \quad (73)$$

and the normalization factor $\langle w | (D_0 + D_1)^L | v \rangle$ to the probability distribution can be easily calculated in terms of the operators A and A^+ . One has

$$\begin{aligned} (D_0 + D_1)^L &= \left(\frac{2}{1-q} + \frac{\alpha + \gamma}{\sqrt{1-q}} A + \frac{\beta + \delta}{\sqrt{1-q}} A^+ \right)^L \\ &= \sum_{m=0}^L \frac{L!}{m!(L-m)!} \frac{2^{L-m}}{(1-q)^{L-\frac{m}{2}}} ((\alpha + \gamma)A + (\beta + \delta)A^+)^m. \end{aligned} \quad (74)$$

The difference with the previously evaluated normalization factor for the boundary processes with only incoming particles at the left end and only outgoing particles at the right end is that now one has to normally order boson operators whose deformed commutator is not normalized to unity. This results in a slightly modified formula

$$((\alpha + \gamma)A + (\beta + \delta)A^+)^m = \sum_{k=0}^{[m/2]} S_k^m (\alpha + \gamma)^k (\beta + \delta)^k \sum_{l=0}^{m-2k} \frac{[m-2k]!}{[l]![m-2k-l]!} ((\beta + \delta)A^+)^l ((\alpha + \gamma)A)^{m-2k-l} \quad (75)$$

One explores next the eigenvalue properties of the operators A, A^+ with respect to the vectors $|v\rangle$ and $\langle w|$ and finds the normalization factor $\langle w | (D_0 + D_1)^L | v \rangle = Z_L$ to the probability distribution:

$$Z_L = \sum_{m=0}^L \binom{L}{m} \frac{2^{L-m}}{(1-q)^{L-\frac{m}{2}}} \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)} (\alpha + \gamma)^k (\beta + \delta)^k \binom{m-2k}{l}_q ((\beta + \delta)w)^l ((\alpha + \gamma)v)^{m-2k-l} \quad (76)$$

Using this prescription of normal ordering, one can easily calculate any other quantity of interest, such as density profile, current, correlation functions.

5. Coherent state solution of the boundary problem for the n -species process

The algebra for the n -species open asymmetric exclusion process of a diffusion system coupled at both boundaries to external reservoirs of particles of fixed density has the form

$$\begin{aligned} D_{n-1}D_0 - qD_0D_{n-1} &= \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0 \\ D_0D_k - q_kD_kD_0 &= -\frac{x_0}{g_k}D_k \\ D_kD_{n-1} - q_kD_{n-1}D_k &= \frac{x_{n-1}}{g_k} \\ D_kD_l - q_{kl}^{-1}D_lD_k &= 0, \end{aligned} \quad (77)$$

where $k, l = 1, 2, \dots, n-2$, $x_0 + x_{n-1} = 0$ and

$$q = \frac{g_{0,n-1}}{g_{n-1,0}}, \quad q_{kl} = \frac{g_{kl}}{g_{lk}}, \quad q_k = \frac{g_{k0}}{g_{0k}} = \frac{g_{n-1,k}}{g_{k,n-1}}. \quad (78)$$

The equalities in the last formula, together with the relations

$$g_k = g_{0k} = g_{k,n-1}, \quad g_{0k} - g_{k0} = g_{k,n-1} - g_{n-1,k} = g_{0,n-1} - g_{n-1,0}, \quad (79)$$

yield a mapping to the commutation relations of a q -deformed Heisenberg algebra (see eqs. (10)) of $n-1$ oscillators $a_k, a_k^+, k = 0, 1, 2, \dots, n-2$. A solution is obtained by a shift of the oscillators a_0, a_0^+

$$\begin{aligned} D_0 &= \frac{x_0}{g_{n-1,0}} \left(\frac{1}{1-q} + \frac{a_0^+}{\sqrt{1-q}} \right) \\ D_{n-1} &= \frac{-x_{n-1}}{g_{n-1,0}} \left(\frac{1}{1-q} + \frac{a}{\sqrt{1-q}} \right) \end{aligned} \quad (80)$$

and by the identification of the rest of the generators $D_k, k = 1, 2, \dots, n-2$ with the remaining $n-2$ creation operators a_k^+

$$D_k = a_k^+, \quad k \neq 0. \quad (81)$$

For the phase transition inducing boundary processes, when a particle of type k is added with a rate L_k^0 and removed with a rate L_0^k at the left end of the chain and when it is removed with a rate R_0^k and added with a rate R_k^0 at the right end of the chain, the boundary vectors are defined by the systems of equations:

$$\begin{aligned} \langle w | ((-L_1^0 - L_2^0 - \dots - L_{n-1}^0)D_0 + L_0^1 D_1 + L_0^2 D_2 + \dots + L_0^{n-1} D_{n-1} + x_0) &= 0 \\ \langle w | (L_1^0 D_0 - L_0^1 D_1) &= 0 \\ \langle w | (L_2^0 D_0 - L_0^2 D_2) &= 0 \\ &\vdots \\ \langle w | (L_{n-2}^0 D_0 - L_0^{n-2} D_{n-2}) &= 0 \\ \langle w | (L_{n-1}^0 D_0 - L_0^{n-1} D_{n-1} + x_{n-1}) &= 0 \end{aligned} \quad (82)$$

and

$$\begin{aligned} ((-R_1^0 - R_2^0 - \dots - R_{n-1}^0)D_0 + R_0^1 D_1 + R_0^2 D_2 + \dots + R_0^{n-1} D_{n-1} - x_0) |v\rangle &= 0 \\ (R_1^0 D_0 - R_0^1 D_1) |v\rangle &= 0 \\ (R_2^0 D_0 - R_0^2 D_2) |v\rangle &= 0 \\ &\vdots \\ (R_{n-2}^0 D_0 - R_0^{n-2} D_{n-2}) |v\rangle &= 0 \\ (R_{n-1}^0 D_0 - R_0^{n-1} D_{n-1} - x_{n-1}) |v\rangle &= 0 \end{aligned} \quad (83)$$

The two systems are similar and can be solved by the same procedure. From the second to the last but one equation in (82) and (83), one has

$$\langle w | L_0^k D_k = \langle w | L_k^0 D_0 \quad (84)$$

$$R_0^k D_k |v\rangle = R_k^0 D_0 |v\rangle \quad (85)$$

for $k = 1, 2, \dots, n-2$. Hence one inserts eqs. (84) in the first equation of the system (82) and eqs. (85) in the first equation of the system (83) to obtain in both cases an equation that coincides with the last equation of the correspondings systems. Thus the system for the left and right boundary vectors are reduced to the pair of equations

$$\begin{aligned} \langle w | (L_{n-1}^0 D_0 - L_0^{n-1} D_{n-1}) &= \langle w | \\ (R_0^{n-1} D_{n-1} - R_{n-1}^0 D_0) | v \rangle &= | v \rangle. \end{aligned} \quad (86)$$

Making use of the explicit solution for D_{n-1} and D_0 as shifted deformed oscillators (with $x_0 = -x_1 = 1$), we rewrite eqs. (86) as

$$\begin{aligned} (R_0^{n-1} a_0 - R_{n-1}^0 a_0^+) | v \rangle &= \sqrt{1-q} \left(g_{n-1,0} - \frac{R_0^{n-1} - R_{n-1}^0}{1-q} \right) | v \rangle \\ \langle w | (L_{n-1}^0 a_0^+ - L_0^{n-1} a_0) &= \langle w | \left(g_{n-1,0} - \frac{L_{n-1}^0 - L_0^{n-1}}{1-q} \right) \sqrt{1-q}. \end{aligned} \quad (87)$$

The latter equations, in accordance with eq. (40), determine the boundary vectors as squeezed coherent states of the deformed boson operators a_0, a_0^+ corresponding to the eigenvalues

$$\begin{aligned} v &= \sqrt{1-q} \left(g_{n-1,0} - \frac{R_0^{n-1} - R_{n-1}^0}{1-q} \right) \\ w &= \sqrt{1-q} \left(g_{n-1,0} - \frac{L_{n-1}^0 - L_0^{n-1}}{1-q} \right). \end{aligned} \quad (88)$$

The explicit form of these vectors is readily written, namely $\langle w | = \langle n | \sum_{n=0}^{\infty} \frac{w^n}{\sqrt{[n]!}} e_q^{-\frac{1}{2}vw}$ and $| v \rangle = e_q^{-\frac{1}{2}vw} \sum_{n=0}^{\infty} \frac{v^n}{\sqrt{[n]!}} | n \rangle$. We therefore conclude:

The left and right boundary vectors are squeezed coherent states of the shifted deformed annihilation and creation operators D_{n-1} and D_0 , associated with the non-zero boundary parameters x_{n-1} and x_0 , and with eigenvalues depending on the right and left boundary rates:

$$\begin{aligned} (R_0^{n-1} a_0 - R_{n-1}^0 a_0^+) | v \rangle &= A | v \rangle = v | v \rangle \\ \langle w | (L_{n-1}^0 a_0^+ - L_0^{n-1} a_0) &= \langle w | A^+ = \langle w | w, \end{aligned} \quad (89)$$

where the eigenvalues v and w are given by (88). The operators A and A^+ satisfy the same deformed commutation relations as a and a^+ , as was outlined in section 3, with the only difference that they are not Hermitian-conjugate. However their conjugation property is consistent with the involution of the quadratic algebra (4) which reflects the left-right symmetry of the model. From the inverse linear maps we obtain

$$\begin{aligned} a_0 &= L_{n-1}^0 A + R_{n-1}^0 A^+ \\ a_0^+ &= R_0^{n-1} A^+ + L_0^{n-1} A, \end{aligned} \quad (90)$$

with the help of which the mean values of the generators D_0, D_{n-1} and the rest D_k for $k = 1, 2, \dots, n-2$ are readily found

$$\begin{aligned}\langle w|D_0|v\rangle &= (g_{n-1,0})^{-1} \left(\frac{1}{1-q} + \frac{R_0^{n-1}w + L_0^{n-1}v}{\sqrt{1-q}} \right) \\ \langle w|D_{n-1}|v\rangle &= (g_{n-1,0})^{-1} \left(\frac{1}{1-q} + \frac{R_{n-1}^0w + L_{n-1}^0v}{\sqrt{1-q}} \right) \\ \langle w|D_k|v\rangle &= \frac{L_k^0}{L_0^k} \langle w|D_0|v\rangle = \frac{R_k^0}{R_0^k} \langle w|D_0|v\rangle.\end{aligned}\tag{91}$$

With these expressions at hand, it is easy to calculate the expectation value of any monomial of the form $\langle w|X_1X_2\dots X_L|v\rangle$ (where $X_i = D_j$ for $i = 1, 2, \dots, L$, $j = 0, 1, 2, \dots, n-1$), which enters the probability distribution, the current, the correlation functions. One first makes use of the algebra to bring all generators D_k for $k = 1, 2, \dots, n-2$ to the very right or to the very left, which results in an expression of the expectation value as a power in D_0 and D_{n-1} . Then one writes the arbitrary power of D_0, D_{n-1} as a normally ordered product of A and A^+ to obtain, upon using the eigenvalue properties of the latter, an expression for the relevant physical quantity in terms of the probability-rate-dependent boundary eigenvalues v and w .

To summarize we have shown that in the known examples of the exactly solvable, asymmetric simple exclusion process within the matrix-product states approach, the boundary vectors depending on the boundary processes are either deformed coherent or deformed squeezed states of the deformed oscillator algebra used for the solution. The coherent states provide a unified description of both the partially and the fully asymmetric cases, the solution of the fully asymmetric one being obtained in the limit $q \rightarrow 0$ of the deformation parameter q . Generalizing these results, we have proposed and discussed a coherent-state solution of the boundary problem for the n -species stochastic diffusion process.

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References

- [1] E.Schrödinger, *Naturwissenschaften* **14**(1926) 664.
- [2] R.Glauber, *Phys.Rev.* **130** (1963) 188.
- [3] A.M.Perelomov, Generalized Coherent states and their applications, (Springer, NY, 1986)
- [4] R.Gilmore, *Ann. Phys. (NY)* **74** (1972) 391; W.M.Zhang, D.H.Feng and R.Gilmore, *Rev. Mod. Phys.* **62** (1990) 867.

- [5] M.Aric and D.D.Coon, *J. Math. Phys.* **17** (1976) 524.
- [6] B.Jurco, *Lett. Math. Phys.***21** (1991) 51; B.Jurco and P.Stovicek, *Comm. Math. Phys.* **182** (1996) 221.
- [7] J.Klauder and B.Skagerstam, Coherent states (World Scientific, Singapore, 1985).
- [8] B.Derrida, M.R.Evans, V.Hakim and V.Pasquier, *J. Phys.* **A26** (1991) 1493.
- [9] K.Krebs and S.Sandow, *J. Phys.* **A30** (1997) 3163.
- [10] A.Isaev, P.Pyatov and V.Rittenberg, Preprint cond-matt/0103603, (2001)
- [11] B.Aneva, *J. Phys.***A35** (2002) 859.
- [12] J.Wess and B.Zumino, *Nucl. Phys.(Proc. Suppl.)* **B18** (1991) 302.
- [13] D.Fairlie and C.Zachos, *Phys. Lett.* **B256** (1991) 43.
- [14] R.McDermott and A.Solomon *J. Phys.* **A27** (1994) 2037.
- [15] K.Penson and A.Solomon, *J. Math. Phys.* **40** (1999) 2354.
- [16] J.Katriel and A.Solomon, *J. Phys.* **A24** (1991) 2093.
- [17] H.P.Yuen, *Phys. Rev***A13** (1976) 2226.
- [18] R.Blythe, M.R.Evans, F.Colaioni and F.H.L.Essler, *J. Phys.* **A33** (2000) 2313.