Fundamental Structure of Loop Quantum Gravity

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Abstract

In recent twenty years, loop quantum gravity, a background independent approach to unify general relativity and quantum mechanics, has been widely investigated. The aim of loop quantum gravity is to construct a mathematically rigorous, background independent, nonperturbative quantum theory for Lorentzian gravitational field on four dimensional manifold. In the approach, the principles of quantum mechanics are combined with those of general relativity naturally. Such a combination gives us a picture of, so called, quantum Riemannian geometry, which is discrete at fundamental scale. Imposing the quantum constraints in analogy from the classical ones, the quantum dynamics of gravity is being studied as one of the most important issues in loop quantum gravity. On the other hand, the semi-classical analysis is being carried out to test the classical limit of the quantum theory.

In this review, the fundamental structure of loop quantum gravity is presented pedagogically. Our main aim is to help non-experts to understand the motivations, basic structures, as well as general results. We will focus on the theoretical framework itself, rather than its applications, and do our best to write it in modern and precise language while keeping the presentation accessible for beginners. After reviewing the classical connection dynamical formalism of general relativity, as a foundation, the construction of kinematical Ashtekar-Isham-Lewandowski representation is introduced in the content of quantum kinematics. In the content of quantum dynamics, we mainly introduce the construction of a Hamiltonian constraint operator and the master constraint project. It should be noted that this strategy of quantizing gravity can also be extended to obtain other background independent quantum gauge theories. There is no divergence within this background independent and diffeomorphism invariant quantization programme of matter coupled to gravity.

Keywords: loop quantum gravity, quantum geometry, quantum dynamics, background independence.
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1 Why Quantum Gravity

Nowadays, in traditional view there are four elementary interactions widely understood by community of physicists: strong interaction, weak interaction, electromagnetic interaction and gravitational interaction. The description for the former three kinds of forces is quantized in the well known standard model. The interactions are transmitted by exchanging mediate particles. However, the last kind of interaction, gravitational interaction, is described by Einstein’s theory of general relativity, which is absolutely a classical theory which describes the gravitational field as a smooth metric tensor field on a manifold, i.e., a 4-dimensional spacetime geometry. There is no $\hbar$ and hence no discrete structure of spacetime. Thus there is a fundamental inconsistency in our current description of the whole physical world. Physicists widely accept the assumption that our world is, so called, quantized at fundamental level. So all interactions should be brought into the framework of quantum mechanics fundamentally. As a result, the gravitational field should also have "quantum structure".

Throughout the last century, our understanding of the nature has considerably improved from macroscale to microscale, including the phenomena in molecule scale, atom scale, sub-atom scale, and elementary particle scale. The standard model of particle physics coincides almost with all present experimental tests in laboratory (see e.g. [134]). However, because unimaginably large amount of energy would be needed, no one has understood how the physical process happens near the Planck scales $\ell_p \equiv (G\hbar/c^3)^{1/2} \sim 10^{-33} \text{cm}$ and $t_p \equiv (G\hbar/c^5)^{1/2} \sim 10^{-43} \text{s}$, which are viewed as the most fundamental scales. The Planck scale arises naturally in attempts to formulate a quantum theory of gravity, since $\ell_p$ and $t_p$ are unique combinations of speed of light $c$, Planck constant $\hbar$, and gravitational constant $G$, which have the dimensions of length and time respectively. The dimensional arguments suggest that at Planck scale the smooth structure of spacetime should break down, where the well-known quantum field theory is invalid since it depends on a fixed smooth background spacetime. Hence we believe that physicists should go beyond the successful standard model to explore the new physics near Planck scale, which is, perhaps, a quantum field theory without a background spacetime, and this quantum field theory should include the quantum theory of gravity. Moreover, current theoretical physics is thirsting for a quantum theory of gravity to solve at least the following fundamental difficulties.

- Classical Gravity - Quantum Matter Inconsistency

The most crucial equation to perform the relation between the matter field and gravitational field is the famous Einstein field equation:

$$R_{\alpha\beta}[g] - \frac{1}{2} R[g] g_{\alpha\beta} = \kappa T_{\alpha\beta}[g], \quad (1)$$

where the left hand side of the equation concerns spacetime geometry which has classical smooth structure, while the right hand side concerns also matter field which is fundamentally quantum mechanical in standard
model. In quantum field theory the energy-momentum tensor of matter field should be an operator-valued tensor $\hat{T}_{\alpha\beta}$. One possible way to keep classical geometry consistent with quantum matter is to replace $T_{\alpha\beta}[g]$ by the expectation value $< \hat{T}_{\alpha\beta}[g] >$ with respect to some quantum state of the matter on a fixed spacetime. A primary attempt is to consider the vacuum expectation. However, in the solution of this equation the background $g_{\alpha\beta}$ has to be changed due to the non-vanishing of $< \hat{T}_{\alpha\beta}[g] >$. So one has to feed back the new metric into the definition of the vacuum expectation value etc. The result of the iterations does not converge in general [60]. This inconsistency motivates us to quantize the background geometry to arrive at an operator formula also on the left hand side of Eq. (1).

- **Singularity in General Relativity**

Einstein’s theory of General Relativity is considered as one of the most elegant theories in 20th century. Many experimental tests confirm the theory in classical domain [135]. However, Penrose and Hawking proved that singularities are inevitable in general spacetimes with several tempered conditions on energy and causality by the well-known singularity theorem (for a summary, see [75] [132]). Thus general relativity can not be valid unrestrictedly. One naturally expects that, in extra strong gravitational field domains near the singularities, the gravitational theory would probably be replaced by an unknown quantum theory of gravity.

- **Infinity in Quantum Field Theory**

It is well known that there are infinity problems in quantum field theory in Minkowski spacetime. In curved spacetime, the problem of UV divergence is even more serious because of the interacting fields. Although much progress one the renormalization for interacting fields have been made [76], a fundamentally satisfactory theory is still far from reaching. So it is expected that some quantum gravity theory, playing a fundamental role at Planck scale, would provide a natural cut-off to cure the UV singularity in quantum field theory. The situation of quantum field theory is just like that of the theory of quantum mechanics for particles in electromagnetic field (for example, $H_2$ atom model without Lamb shift and Landau Level) before the establishing of quantum electrodynamics, where the particle mechanics (actress) is quantized but the background electromagnetic field (stage) is classical. The history suggests that such a semi-classical situation is only an expedient which should be replaced by a much more fundamental and satisfactory theory.
2 Loop Quantum Gravity: A Background Independent Canonical Quantization Programme

2.1 Define Loop Quantum Gravity

The research on quantum gravity theory is rather active. Many quantization programmes for gravity are being carried out (for a summary see e.g. [122]). In these different kinds of approaches, the idea of loop quantum gravity is motivated by researchers in the community of general relativity. It follows closely the thoughts of general relativity, and hence it is a quantum theory born with background independency.

Definition 2.1: Loop quantum gravity is an attempt to construct a mathematically rigorous, non-perturbative, background independent quantum theory of four-dimensional, Lorentzian general relativity plus all known matter in the continuum. [122]

The project of loop quantum gravity inherits the basic idea of Einstein that gravity is fundamentally spacetime geometry. Here one believes in that the theory of quantum gravity is a quantum theory of spacetime geometry with diffeomorphism invariance (this legacy is discussed comprehensively in Rovelli’s book [103]). To make the theory mathematically rigorous, one casts general relativity into the Hamiltonian formalism of a diffeomorphism invariant Yang-Mills gauge field theory with a compact internal gauge group. Thus the construction of loop quantum gravity is valid to all background independent gauge field theories. So the theory can also be called as a background independent quantum gauge field theory.

All classical fields theories, other than gravitational field, are defined on a fixed spacetime, which provides a foundation to the perturbative Fock space quantization. However general relativity is only defined on a manifold and hence is the unique background independent classical field theory, since gravity itself is the background. So the situation for gravity is much different from other fields by construction [103], namely gravity is not only the background stage, but also the dynamical actress. Such a double character for gravity leads to many difficulties in the understanding of general relativity and its quantization, since we cannot analog the strategy in ordinary quantum theory of matter fields. However, an amazing result in loop quantum gravity is that the background independent programme can even enlighten us to avoid the difficulties in ordinary quantum field theory. In perturbative quantum field theory in curved spacetime, it is difficult to calculate the back reaction of quantum fields to a spacetime and the renormalization of quantum field in different curved spacetimes. The reason responsible for such a situation is that present formulations of quantum field theories are background dependent. For instance, the vacuum state of a quantum field is closed related to spacetime structure, which plays an essential role in the description of quantum field theory in curved spacetime.
and their renormalization procedures. However, if the quantization programme
is by construction background independent and non-perturbative, it is possible
to solve the problems fundamentally. In loop quantum gravity there is no as-
sumption of a priori background metric at all and the gravitational field and
matter fields are coupled and fluctuating naturally with respect to each other
on a common manifold.

In the following several sections, we will review pedagogically the basic con-
struction of a completely new, background independent quantum field theory,
which is completely different from the known quantum fields theory. For com-
pleteness and accuracy, we will not avoid mathematical terminologies. While,
for simplicity, we will skip the complicated proofs of many important statements.
One may find the missing details in the references cited. Thus our review will
not be comprehensive. We refer to Ref. [122] for a more detailed exploration,
Refs. [17] and [124] for more advanced topics. It turns out that in the frame-
work of loop quantum gravity all theoretical inconsistencies introduced in the
previous section are likely to be cured. More precisely, one will see that there
is no UV divergence in quantum fields of matter if they are coupled with grav-
ity in the background independent approach. Also resent works show that the
singularities in general relativity are likely to be smeared out in the symmetry-
reduced models [37] [85] [42]. The crucial point is that gravity and matter are
coupled and consistently quantized non-perturbatively so that the problems of
classical gravity and quantum matter inconsistency disappear.

2.2 General Programme for Background Independent Canonical
Quantization

In the strategy a canonical programme is performed to quantize general relativity,
which has been cast into a diffeomorphism invariant gauge field theory, or
more generally, a dynamical system with constraints. The following is a sum-
mary for a general procedure to quantize a dynamical system with first class
constraints 1.

- Algebra of Classical Elementary Observables

After the Hamiltonian analysis for the classical action, one obtains the
classical phase space \((M, \{\cdot, \cdot\})\) and \(R\) (\(R\) can be countable infinity 2 ) first-
class constraints \(C_r (r = 1...R)\) such that \(\{C_r, C_s\} = \Sigma_{t=1}^R f_{rs}^t C_t\), where
\(f_{rs}^t\) is generally a function on phase space, namely, structure function of
Poisson algebra. Then the algebra of classical elementary observables \(P\)
is defined as:

\[\text{Definition 2.2: The algebra of classical elementary observables } P \text{ is a}\]

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1 Thanks to the enlightening lectures given by Prof. T. Thiemann at Beijing Normal University.

2 This includes the case of field theory with infinite many degree of freedom, since one can introduce the expression \(C_{n,\mu} = \int_{\Sigma} d^3x \phi_n(x)C_{\mu}(x)\), where \(\{\phi_n(x)\}_{n=1}^\infty\) forms a system of basis in \(L^2(\Sigma, d^3x)\).
collection of functions \( f(m), m \in \mathcal{M} \) on phase space such that

1. \( f(m) \in \mathcal{P} \) should separate the point of \( \mathcal{M} \), i.e., for any \( m \neq m' \), there exists \( f(m) \in \mathcal{P} \), such that \( f(m) \neq f(m') \); (analog from the \( p \) and \( q \) in \( \mathcal{M} = T^*\mathbb{R} \))

2. \( f(m), f'(m) \in \mathcal{P} \Rightarrow \{ f(m), f'(m) \} \in \mathcal{P} \) (closed under Poisson bracket);

3. \( f(m) \in \mathcal{P} \Rightarrow \bar{f}(m) \in \mathcal{P} \) (closed under complex conjugate).

So \( \mathcal{P} \) forms a sub \( \ast \)-Poisson algebra of \( C^\infty(\mathcal{M}) \). In the case of \( \mathcal{M} = T^*\mathbb{R} \), \( \mathcal{P} \) is generated by the conjugate pair \((p, q)\) with \( \{p, q\} = 1 \).

**Quantum Algebra of Elementary Observables**

Given the algebra of classical elementary observables \( \mathcal{P} \), the quantum algebra of elementary observables can be constructed as the following:

Firstly, consider the formal finite sequences of classical observable \((f_1, ..., f_n)\) with \( f_k \in \mathcal{P} \). Then the operations of multiplication and involution are defined as

\[
(f_1, ..., f_n) \cdot (f'_1, ..., f'_m) := (f_1, ..., f_n, f'_1, ..., f'_m),
\]

\[
(f_1, ..., f_n)^\ast := (f_n, ..., f_1).
\]

One can define the direct sum of different sequences with different number of elements. Then the general element of the newly constructed algebra, i.e., universal enveloping \( \ast \)-algebra \( U(\mathcal{P}) \) of \( \mathcal{P} \), is formally expressed as \( \bigoplus_{k=1}^{N} (f_1^{(k)}, ..., f_n^{(k)}) \). Consider the elements of the form (sequences consisting of only one element)

\[
(f + f') - (f) - (f'), \quad (zf) - z(f), \quad [(f), (f')] - i\hbar \{f, f'\},
\]

where \( z \in \mathbb{C} \) and the canonical commutation bracket is defined as

\[
[(f), (f')] := (f) \cdot (f') - (f') \cdot (f).
\]

A 2-side ideal \( \mathcal{Z} \) of \( U(\mathcal{P}) \) can be constructed from these element, and is preserved by the action of involution \( \ast \). One thus obtains

**Definition 2.3**: The quantum algebra \( \mathcal{A} \) of elementary observables is defined to be the quotient \( \ast \)-algebra \( U(\mathcal{P})/\mathcal{Z} \).

Note that the motivation to construct a quantum algebra of elementary observables is to avoid the problem of operators ordering in quantum theory so that the quantum algebra \( \mathcal{A} \) can be represented on a Hilbert space without ordering ambiguity.

**Representation of Quantum Algebra**

In order to obtain a quantum theory, we need to quantize the classical observable in the dynamical system. The, so called, quantization is nothing
but a representation map $\pi$ from the quantum algebra of element observables $\mathcal{A}$ to the collection of linear operators $\mathcal{L}(\mathcal{H})$ on a Hilbert Space $\mathcal{H}$, such that the map is a $*$-homomorphism with respect to linear structure, multiplication and involution $*$, and $\pi(1) = 1$ if $\mathcal{A}$ has an identity. At the level of quantum mechanics, the well-known Stone-Von Neumann Theorem concludes that in quantum mechanics, there is only one irreducible representation of the canonical commutator relation $[(q^a), (p_b)] = i\hbar \delta_b^a$, up to unitary equivalence. However, the conclusion of Stone-Von Neumann cannot be generalized to the quantum field theory because the latter has infinite many degrees of freedom. In quantum field theory, a representation can be constructed by GNS(Gel’fand-Naimark-Segal) construction for a quantum algebra of elementary observables $\mathcal{A}$, which is a unital $*$-algebra actually. The GNS construction for the representation of quantum algebra $\mathcal{A}$ is briefly summarized as the following:

**Definition 2.4:** Given a positive linear functional (a state ) $\omega$ on $\mathcal{A}$, the null space $\mathcal{N}_\omega \in \mathcal{A}$ with respect to $\omega$ is defined as $\mathcal{N}_\omega := \{ a \in \mathcal{A} | \omega(a^* \cdot a) = 0 \}$, which is a left ideal in $\mathcal{A}$. Then a quotient map can be defined as $[\cdot] : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{N}_\omega$; $a \mapsto [a] := \{ a + b | b \in \mathcal{N}_\omega \}$. The GNS-representation for $\mathcal{A}$ with respect to $\omega$ is a representation map: $\pi_\omega : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_\omega)$, where $\mathcal{H}_\omega := \langle \mathcal{A}/\mathcal{N}_\omega \rangle$ and $\langle \cdot \rangle$ denotes the completion with respect to the naturally equipped well-defined inner product

$$< [a]|[b] >_{\mathcal{H}_\omega} := \omega(a^* \cdot b)$$
on $\mathcal{H}_\omega$. This representation map is defined by

$$\pi_\omega(a)[b] := [a \cdot b], \ \forall a \in \mathcal{A} \text{ and } [b] \in \mathcal{H}_\omega,$$non

where $\pi_\omega(a)$ is an unbounded operator in general. Moreover, GNS representation is a cyclic representation, i.e., $\exists \Omega \in \mathcal{H}_\omega$, such that $\{ \pi(a)\Omega | a \in \mathcal{A} \} = \mathcal{H}_\omega$ and $\Omega$ is called a cyclic vector in the representation space. In fact $\Omega := [1]$ is a cyclic vector in $\mathcal{H}_\omega$ and $\{ \pi_\omega(a)\Omega | a \in \mathcal{A} \} = \mathcal{H}_\omega$. As a result, the positive linear functional with which we begin can be expressed as

$$\omega(a) = \langle \Omega_{\omega} | \pi_\omega(a)\Omega_{\omega} >_{\mathcal{H}_\omega}.$$non

Thus a positive linear functional on $\mathcal{A}$ is equivalent to a cyclic representation of $\mathcal{A}$, which is a triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$. Moreover, every non-degenerate representation is an orthogonal direct sum of cyclic representations (for proof, see [49]).

So the kinematical Hilbert space $\mathcal{H}_{kin}$ for the system with constrains can be obtained by GNS-construction. In the case that there are gauge symmetries in our dynamical system, supposing that there is a group $G$ acting
on $A$ by automorphisms $\alpha_g : A \to A$, $\forall g \in G$, it is preferred to construct a gauge invariant representation of $A$. So we require the positive linear functional $\omega$ on $A$ to be gauge invariant, i.e., $\omega \circ \alpha_g = \omega$. Then the group $G$ is represented on the Hilbert space $H_\omega$ as:

$$U(g)\pi_\omega(a)\Omega_\omega = \pi_\omega(\alpha_g(a))\Omega_\omega,$$

and such a representation is a unitary representation of $G$. In loop quantum gravity, it is crucial to construct a gauge invariant and diffeomorphism invariant representation for the quantum algebra of elementary observables.

- **Implementation of the Constraints**

  In the Dirac quantization programme for a system with constraints, the constraints should be quantized as some operators in a kinematical Hilbert space $H_{\text{kin}}$. One then solves them at quantum level to get a physical Hilbert space $H_{\text{phys}} \subseteq H_{\text{kin}}$, that is, to find a quantum analogy $\hat{C}_r$ of the classical constraint formula $C_r$ and to solve the general solution of the equation $\hat{C}_r \Psi = 0$. However, there are several problems in the construction of the constraint operator $\hat{C}_r$.

  (i) $C_r$ is in general not in $\mathcal{P}$, so there is a factor ordering ambiguity in quantizing $C_r$ to be an operator $\hat{C}_r$.

  (ii) In quantum field theory, there are ultraviolet (UV) divergence problems in constructing operators. However, the UV divergence can be avoided in the background independent approach.

  (iii) Sometimes, quantum anomaly appears when there are structure functions in the Poisson algebra. Although classically we have $\{C_r, C_s\} = \sum_{t=1}^{R} f_{rs}^t C_t$, $r, s, t = 1...R$, where $f_{rs}^t$ is a function on phase space, quantum mechanically it is possible that $[\hat{C}_r, \hat{C}_s] \neq i\hbar \sum_{t=1}^{R} \hat{f}_{rs}^t \hat{C}_t$ due to the ordering ambiguity between $\hat{f}_{rs}^t$ and $\hat{C}_t$. If one sets $[\hat{C}_r, \hat{C}_s] = \frac{i\hbar}{2} \sum_{t=1}^{R} (\hat{C}_t \hat{f}_{rs}^t + \hat{f}_{rs}^t \hat{C}_t)$, for $\Psi$ satisfying $\hat{C}_r \Psi = 0$, we have

$$[\hat{C}_r, \hat{C}_s] \Psi = \frac{i\hbar}{2} \sum_{t=1}^{R} \hat{C}_t \hat{f}_{rs}^t \Psi = \frac{i\hbar}{2} \sum_{t=1}^{R} [\hat{C}_t, \hat{f}_{rs}^t] \Psi. \quad (2)$$

However, $[\hat{C}_t, \hat{f}_{rs}^t] \Psi$ are not necessary to equal to zero for all $r, s, t = 1...R$. If not, the problem of quantum anomaly comes out and the new quantum constraints $[\hat{C}_t, \hat{f}_{rs}^t] \Psi = 0$ have to be imposed on physical quantum states, since the classical Poisson brackets $\{C_r, C_s\}$ are weakly equal to zero on the constraint surface $\mathcal{M} \subseteq \mathcal{M}$. Thus too many constraints are imposed so that the physical Hilbert space $H_{\text{phys}}$ would be too small. Hence the quantum anomaly should be avoided anyway.
• **Solving the Constraints and Physical Hilbert Space**

In general the original Dirac quantization approach can not be carried out directly, since there is usually no nontrivial $\Psi \in \mathcal{H}_{\text{kin}}$ such that $\hat{C}_r \Psi = 0$. This happens when the constraint operator $\hat{C}_r$ has "generalized eigenfunctions" rather than eigenfunctions. One then develops the so-called Refined Algebra Quantization Programme, where the solutions of the quantum constraint can be found in the algebraic dual space of a dense subset in $\mathcal{H}_{\text{kin}}$ (see e.g. [69]). The quantum diffeomorphism constraint in canonical gravity is solved in this approach (see section 5.2). Another interesting way to solve the quantum constraints is the Master Constraint Approach proposed by Thiemann recently [125], which seems especially suitable to deal with the particular feature of the constraint algebra of general relativity. A master constraint is defined as

$$
\mathbf{M} := \frac{1}{2} \sum_{r,s=1}^{\mathcal{R}} K_{rs} \bar{C}_r C_s
$$

for some real positive matrix $K_{rs}$. Classically one has $\mathbf{M} = 0$ if and only if $C_r = 0$ for all $r = 1...\mathcal{R}$. So quantum mechanically one may consider solving the Master Equation: $\mathbf{M} \Psi = 0$ to obtain the physical Hilbert space $\mathcal{H}_{\text{phys}}$ instead of solving $\hat{C}_r \Psi = 0$, $\forall \ r = 1...\mathcal{R}$. Because the master constraint $\mathbf{M}$ is classically positive, one has opportunities to implement it as a self-adjoint operator on $\mathcal{H}_{\text{kin}}$. If it is indeed the case and $\mathcal{H}_{\text{kin}}$ is separable, one can use the direct integral representation of $\mathcal{H}_{\text{kin}}$ associated with the self-adjoint operator $\hat{\mathbf{M}}$ to obtain $\mathcal{H}_{\text{phys}}$:

$$
\mathcal{H}_{\text{kin}} \cong \int_{\mathbb{R}} d\mu(\lambda) \mathcal{H}^\oplus_{\lambda},
$$

$$
< \Phi | \Psi >_{\text{kin}} = \int_{\mathbb{R}} d\mu(\lambda) < \Phi | \lambda >_{\text{kin}} < \lambda | \Psi >_{\text{kin}}
$$

$$
= \int_{\mathbb{R}} d\mu(\lambda) < \Phi | \Psi >_{\mathcal{H}^\oplus_{\lambda}},
$$

where $\mu$ is a so-called spectral measure and $\mathcal{H}^\oplus_{\lambda}$ is the (generalized) eigenspace of $\hat{\mathbf{M}}$ with the eigenvalue $\lambda$. The physical Hilbert space is then formally obtained as $\mathcal{H}_{\text{phys}} = \mathcal{H}^\oplus_{\lambda=0}$ with the induced physical inner product $< | >_{\mathcal{H}_{\text{phys}}^{\oplus}}$. Now the issue of quantum anomaly is represented in terms of the size of $\mathcal{H}_{\text{phys}}$ and the existence of sufficient numbers of semi-classical states.

• **Physical Observables**

We denote $\mathcal{M}$ as the original unconstrained phase space, $\overline{\mathcal{M}}$ as the constraint surface, i.e., $\overline{\mathcal{M}} := \{ m \in \mathcal{M} | C_r(m) = 0, \forall \ r = 1...\mathcal{R} \}$, and $\overset{\_}{\mathcal{M}}$ as the reduced phase space, i.e. the space of orbits for gauge transformations generated by all $C_r$. The concept of Dirac observable is defined as the following:

**Definition 2.5**:

(1) A function $\mathcal{O}$ on $\mathcal{M}$ is called a weak Dirac observable if and only if the
function depends only on \( \hat{\mathcal{M}} \), i.e., \( \{ \mathcal{O}, C_r \}_r |_{\hat{\mathcal{M}}} = 0 \) for all \( r = 1...R \). For the quantum version, a self-adjoint operator \( \mathcal{O} \) is a weak Dirac observable if and only if the operator can be well defined on the physical Hilbert space. (2) A function \( \mathcal{O} \) on \( \mathcal{M} \) is called a strong Dirac observable if and only if \( \{ \mathcal{O}, C_r \}_r |_{\hat{\mathcal{M}}} = 0 \) for all \( r = 1...R \). For the quantum version, a self adjoint operator \( \mathcal{O} \) is a strong Dirac observable if and only if the operator can be defined on the kinematic Hilbert space \( \mathcal{H}_{\text{kin}} \) and \( [\mathcal{O}, C_r] = 0 \) in \( \mathcal{H}_{\text{kin}} \) for all \( r = 1...R \).

A physical observable is at least a weak Dirac observable. In general relativity, it is difficult to construct Dirac observables. Moreover the Hamiltonian is a linear combination of first-class constraints. So there is no dynamics in the reduced phase space, and the meaning of time evolution of the Dirac observables is unclear. However, using the concepts of partial and complete observables \([102][96][103]\), a systemic method to get Dirac observables can be developed, and the problem of time in such system with a Hamiltonian \( H = \sum_{r=1}^{R} \mathcal{A}_r C_r \) may also be solved.

Classically, let \( f(m) \) and \( \{ T_r(m) \}_{r=1}^{R} \) be gauge non-invariant functions (partial observables) on phase space \( \mathcal{M} \), such that \( A_{sr} \equiv \{ C_s, T_r \} \) is a non-degenerate matrix on \( \mathcal{M} \). A system of classical weak Dirac observables (complete observables) \( F_{f,T} \) labelled by a collection of real parameters \( \tau \equiv \{ \tau_r \}_{r=1}^{R} \) can be constructed as

\[
F_{f,T} := \sum_{\{ n_1,...,n_R \}} \frac{(\tau_1 - T_1)^{n_1} \cdots (\tau_R - T_R)^{n_R}}{n_1! \cdots n_R!} \widetilde{X}_1^{n_1} \circ \cdots \circ \widetilde{X}_R^{n_R}(f),
\]

where \( \widetilde{X}_r(f) := \{ \sum_{s=1}^{R} A_{rs}^{-1} C_s, f \} \equiv \{ C_r, f \} \). It can be verified that \( [\widetilde{X}_r, \widetilde{X}_s] |_{\hat{\mathcal{M}}} = 0 \) and \( \{ F_{f,T}, C_r \}_r |_{\hat{\mathcal{M}}} = 0 \), for all \( r = 1...R \) (for details see \([52][53]\)).

The partial observables \( \{ T_r(m) \}_{r=1}^{R} \) may be regarded as clock variables, and \( \tau_r \) is the time parameter for \( T_r \). The gauge is fixed by giving a system of functions \( \{ T_r(m) \}_{r=1}^{R} \) and corresponding parameters \( \{ \tau_r \}_{r=1}^{R} \), namely, a section in \( \hat{\mathcal{M}} \) is selected by \( T_r(m) = \tau_r \) for each \( r \), and \( F_{f,T} \) is the value of \( f \) on the section. To solve the problem of dynamics, one assumes another set of canonical coordinates \( (P_1,\cdots,P_{N-R},\Pi_1,\cdots,\Pi_R;Q_1,\cdots,Q_{N-R},T_1,\cdots,T_R) \) by canonical transformations in the phase space \( (\mathcal{M},\{\;\;\;\}) \), where \( P_s \) and \( \Pi_r \) are conjugate to \( Q_s \) and \( T_r \) respectively. After solving the complete system of constraints \( \{ C_r(P_r, Q_j, \Pi_r, T_i) = 0 \}_{r=1}^{R} \), the Hamiltonian \( H_r \) with respect to the time \( T_r \) is obtained as \( H_r := \Pi_r (P_r, Q_j, T_i) \). Given a system of constants \( \{ (\tau_0)_r \}_{r=1}^{R} \) for an observable \( f(P_i, Q_j) \) depending only on \( P_i \) and \( Q_j \), the physical dynamics is given by \([52][120]\):

\[
\left( \frac{\partial}{\partial \tau_r} \right)_{\tau = \tau_0} F_{f,T} = F_{(H_r,f),T} = \{ F_{H_r,f}, F_{f,T} \}_r |_{\hat{\mathcal{M}}},
\]

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where \( F^\beta_{H,T} \) is the physical Hamiltonian function generating the evolution with respect to \( \tau_r \). Thus one has settled up the problem of time and dynamics as a result.

- **Semi-classical Analysis**

An important issue in the quantization is to check whether the quantum constraint operators have correct classical limits. This has to be done by using the kinematical semiclassical states in \( \mathcal{H}_{\text{kin}} \). Moreover, the physical Hilbert space \( \mathcal{H}_{\text{phys}} \) must contain enough semi-classical states to guarantee that the quantum theory one obtains can return to the classical theory when \( \hbar \to 0 \). The semi-classical states in a Hilbert space \( \mathcal{H} \) should have the following properties:

**Definition 2.6:** Given a subalgebra \( \mathcal{A} \) in the algebra \( \mathcal{L}(\mathcal{H}) \) of linear operators on the Hilbert space, a family of states (positive linear functional) \( \{ \omega_m \}_{m \in \mathcal{M}} \) are said to be semi-classical with respect to \( \mathcal{A} \) if and only if:

1. The observables should have correct semi-classical limit on semi-classical states and the fluctuations should be small, i.e.,
   \[
   \lim_{\hbar \to 0} \left| \frac{\omega_m(\hat{a}) - a(m)}{a(m)} \right| = 0, \\
   \lim_{\hbar \to 0} \left| \frac{\omega_m(\hat{a}^2) - \omega_m(\hat{a})^2}{\omega_m(\hat{a})^2} \right| = 0,
   \]
   for all \( \hat{a} \in \mathcal{A} \).

2. The set of cyclic vectors \( \Omega_m \) related to \( \omega_m \) via the GNS-representation \((\pi_{\omega}, \mathcal{H}_\omega, \Omega_\omega)\) is dense in \( \mathcal{H} \).

Seeking for semiclassical states are one of open issues of current research in loop quantum gravity. Recent original works focus on the construction of coherent states of loop quantum gravity in analogy with the coherent states for harmonic oscillator system \([118][119][120][121][16][13]\).

The above is the general programme to quantize a system with constraints. In the following several sections, we will apply the programme to the theory of general relativity and restrict our view to the representation with the properties of background independence and spatial diffeomorphism invariance. To end this section, we present a very crucial uniqueness theorem for canonical quantization of gauge field theory \([80][110]\):

**Theorem 2.1:** There exists exactly one Yang-Mills gauge invariant and spatial diffeomorphism invariant state (positive linear functional) on the quantum holonomy-flux \( \ast \)-algebra, i.e., there exists a unique Yang-Mills gauge invariant and spatial diffeomorphism invariant cyclic representation for the quantum holonomy-flux \( \ast \)-algebra, which is named Ashtekar-Isham-Lewandowski representation, and this representation is irreducible.
3 Classical Framework: Generalized Palatini Formalism for Connection Dynamics

3.1 Lagrangian Formalism

In order to canonically quantize the classical system of gravity, Hamiltonian analysis has to be performed to obtain a canonical formalism of the classical theory suitable to be represented on certain Hilbert space. The first canonical formalism of general relativity is the ADM formalism (Geometric dynamics) from the Einstein-Hilbert action [13, 81], which by now has not been cast into a quantum theory rigorously. Another well-known action of general relativity is the Palatini formalism, where the tetrad and the connection are regarded as independent dynamical variables. However, unluckily the dynamics of Palatini action is the same with the Einstein-Hilbert action for the gravitational field without fermion coupling [4, 72]. In 1986, Ashtekar gave a formalism of true connection dynamics with a relatively simple Hamiltonian constraint, and thus opens the door to apply quantization techniques from gauge fields theory [2, 3, 104]. However the weakness of that formalism is that the canonical variables are complex variables, which needs a complicated real section condition. Moreover, the quantization based on the complex connection could not be carried out rigorously, since the internal gauge group is noncompact. In 1995, Barbero modified the Ashtekar new variables to give a system of real canonical variables for dynamical theory of connections [29]. Then Holst constructed a generalized Palatini action to support Barbero’s real connection dynamics [77]. Although there is a free parameter (Barbero-Immirzi parameter $\gamma$) in generalized Palatini action and the Hamiltonian constraint is more complicated than the Ashtekar one, now the generalized Palatini Hamiltonian with the real connections is widely accepted by loop theorists for the quantization programme. All the following analysis is based on the generalized Palatini formalism.

Consider an 4-manifold, $M$, on which the basic dynamical variables in the generalized Palatini framework are tetrad $e_\alpha^I$ and $so(1,3)$-valued connection $\omega^{IJ}_\alpha$ (not necessarily torsion-free), where the capital Latin indices $I, J, ...$ denote the internal $SO(1,3)$ group and the Greek indices $\alpha, \beta, ...$ denote spacetime indices. A tensor with both spacetime indices and internal indices is named as a generalized tensor. The internal space is equipped with a Minkowskian metric $\eta_{IJ}$ (of signature $-, +, +, +$) fixed once for all, such that the spacetime metric reads:

$$ g_{\alpha\beta} = \eta_{IJ} e^I_\alpha e^J_\beta. $$

The generalized Palatini action in which we are interested is given by [17]:

$$ S_p[e_K^\alpha, \omega^{IJ}_\alpha] = \frac{1}{2\kappa} \int_M d^4x (e^\alpha) e^\beta_i (\Omega^{IJ}_\alpha + \frac{1}{2\gamma} \epsilon^{IJ}_{KL} \Omega^K_L), \quad (4) $$

where $e$ is the square root of the determinant of the metric $g_{\alpha\beta}$, $\epsilon^{IJ}_{KL}$ is the internal Levi-Civita symbol, $\gamma$ is the real Barbero-Immirzi parameter, and the...
so(1, 3)-valued curvature 2-form Ω_{αβ}^{IJ} of the connection ω_{αβ}^{IJ} reads:

\[ \Omega_{αβ}^{IJ} := 2D_αω_{β}^{IJ} = D_αω_{β}^{IJ} - D_βω_{α}^{IJ} + ω_{α}^{IK} ∧ ω_{βK}^{J}, \]

here \( D_α \) denote the so(1, 3) generalized covariant derivative with respect to ω_{αβ}^{IJ} acting on both spacetime and internal indices. Note that the generalized Palatini action returns to the Palatini action when \( \frac{1}{\gamma} = 0 \) and gives the (anti)self-dual Ashtekar formalism when one sets \( \frac{1}{\gamma} = ±i \). Moreover, besides spacetime diffeomorphism transformations, the action is also invariant under internal \( SO(1, 3) \) rotations:

\[ (e, ω) \mapsto (e', ω') = (b^{-1}e, b^{-1}ωb + b^{-1}db), \]

for any \( SO(1, 3) \) valued function \( b \) on \( M \). The gravitational field equations are obtained by varying this action with respect to \( e_α^I \) and \( ω_{αβ}^{IJ} \). We first study the variation with respect to the connection \( ω_{αβ}^{IJ} \). One has

\[ δΩ_{αβ}^{IJ} = (d δω_{α}^{IJ}) - δω_{IK}^{α} ∧ ω_{Jβ}^{K} + ω_{IK}^{α} ∧ δω_{Jβ}^{K} = 2D_α(δω_{β}^{IJ}), \]

by the definition of covariant generalized derivative \( D_α \). Note that \( δω_{α}^{IJ} \) is a Lorentz covariant generalized tensor field since it is the difference between two Lorentz connections \( [89] [88] \). One thus obtains

\[ δS_p = \frac{1}{2κ} \int_M d^4x(e)e_α^I e_β^J(δΩ_{αβ}^{IJ} + \frac{1}{2γ} ϵ_{IJKL} δΩ_{αβ}^{KL}) = -\frac{1}{κ} \int_M (δω_{β}^{IJ} + \frac{1}{2γ} ϵ_{IJKL} δω_{β}^{KL}) D_α((e_α^I e_β^J), \]

where we have used the fact that \( D_α(λ) = D_α(λ) \) for all vector density \( λ \) of weight +1 and neglected the surface term. Then it gives the equation of motion:

\[ D_α((e_α^I e_β^J) = -\frac{1}{4}D_α[\tilde{η}^{αβγδ} ϵ_{IJKL} e_γ^K e_δ^L] = 0, \]

where \( \tilde{η}^{αβγδ} \) is the spacetime Levi-Civita symbol. This equation leads to the torsion-free Cartan’s first equation:

\[ D_α(e_β^I) = 0, \]

which means that the connection \( ω_{α}^{IJ} \) is the unique torsion-free Levi-Civita spin connection compatible with the tetrad \( e_β^J \). As a result, the second term in the action \( [4] \) can be calculated as:

\[ (e_β^I e_β^J) = η^{αβγδ} R_αβγδ, \]

which is exactly vanishing, because of the symmetric properties of Riemann tensor. So the generalized Palatini action returns to the Palatini action, which will certainly give the Einstein field equation.
3.2 Hamiltonian Formalism

To carry out the Hamiltonian analysis of action (4), suppose the spacetime \( M \) is topologically \( \Sigma \times \mathbb{R} \) for some 3-dimensional compact manifold \( \Sigma \) without boundary. We introduce a foliation parameterized by a smooth function \( t \) and a time-evolution vector field \( t^\alpha \) such that \( t^\alpha(dt)_\alpha = 1 \) in \( M \), where \( t^\alpha \) can be decomposed with respect to the unit normal vector \( n^\alpha \) of \( \Sigma \) as:

\[
t^\alpha = N n^\alpha + N^\alpha,
\]

where \( N \) is called the lapse function and \( N^\alpha \) the shift vector. The internal normal vector is defined as \( n^I \equiv n^i e^i_I \). It is convenient to carry out a partial gauge fixing, i.e., fix a internal constant vector field \( n^I \) with \( \eta_{ij} n^i n^j = -1 \). Note that the gauge fixing puts no restriction on the real dynamics. Then the internal vector space \( V \) is 3+1 decomposed with a 3-dimensional subspace \( W \) orthogonal to \( n^I \), which will be the internal space on \( \Sigma \). With respect to the internal normal \( n^I \) and spacetime normal \( n^\alpha \), the internal and spacetime projection maps are denoted by \( q^I_i \) and \( q^\alpha_a \) respectively, where we use \( i,j,k,... \) to denote the 3-dimensional internal space indices and \( a,b,c,... \) to denote the indices of space \( \Sigma \). Then an internal reduced metric \( \delta_{ij} \) and a reduced spatial metric on \( \Sigma \), \( q_{ab} \), are obtained by these two projection maps. The two metrics are related by:

\[
q_{ab} = \delta_{ij} q^i_a q^j_b,
\]

where the orthonormal co-triad on \( \Sigma \) is defined by \( e^i_a := e^i_I q^I_i q^\alpha_a \). Now the internal gauge group \( SO(1,3) \) is reduced to its subgroup \( SO(3) \) which leaves \( n^I \) invariant. Finally, two Levi-Civita symbols are obtained respectively as

\[
\epsilon_{ijk} := q^I_i q^J_j q^K_k n^L \epsilon_{LJK},
\]

\[
\eta_{abc} := q^a q^b q^c t^\mu \eta_{\mu abc},
\]

where the internal Levi-Civita symbol \( \epsilon_{ijk} \) is an isomorphism of Lie algebra \( so(3) \). Using the connection 1-form \( \omega_{ij} \), one can defined two \( so(3) \)-valued 1-form on \( \Sigma \):

\[
\Gamma^i_a := \frac{1}{2} q^a q^b \epsilon^{ij} \epsilon_{KLM} n^j \omega_{ai}^{KL},
\]

\[
K^i_a := \frac{1}{2} q^a q^b \eta_{abc} n^j \omega_{ai}^{J},
\]

where \( \Gamma \) is a spin connection on \( \Sigma \) and \( K \) will be related to the extrinsic curvature of \( \Sigma \) on shell. After the 3+1 decomposition and the Legendre transformation, action (4) can be expressed as (7):

\[
S_p = \int \int d^3 x [\bar{F}_i \mathcal{L}_t A_i^a - \mathcal{H}_{tot}(A_i^a, \bar{F}_j^b, A^i, N, N^c)],
\]

from which the symplectic structure on the classical phase space is obtained as

\[
\{A^i_a(x), P^j_b(y)\} := \delta^i_j \delta^a_b \delta^3(x - y),
\]
where the configuration and conjugate momentum are defined respectively by:

\[ A^i_a := \Gamma^i_a + \gamma K^i_a, \]

\[ \tilde{P}^a_i := \frac{1}{2\kappa\gamma} \eta^{abc} \epsilon_{ijk} e^j_b e^k_c = \frac{1}{\kappa\gamma} \sqrt{\det q} e^a_i, \]

here \( \det q \) is the determinant of the 3-metric \( q_{ab} \) on \( \Sigma \) and hence \( \det q = (\kappa\gamma)^3 \det P \). In the definition of the configuration variable \( A^i_a \), we should emphasize that \( \Gamma^i_a \) is restricted to be the unique torsion free so(3)-valued spin connection compatible with the triad \( e^i_a \), i.e.,

\[ \nabla_a e^b_i := \partial_a e^b_i + \Gamma^b_{ac} e^c_i + \epsilon_{ij}^k \Gamma^j_a e^b_k = 0, \]

which is obtained by solving a constraint in the Hamiltonian analysis \[77\]. Note that \( \nabla_a \) is the so(3) generalized derivative operator compatible with the triad \( e^i_a \) and \( \Gamma^b_{ac} \) is the Levi-Civita connection on \( \Sigma \). In the Hamiltonian formalism, one starts with the fields \((A^i_a, \tilde{P}^a_i)\). Then neither the basic dynamical variables nor their Poisson brackets depend on the Barbero-Immerzi parameter \( \gamma \). Hence, for the case of pure gravitational field, the dynamical theories with different \( \gamma \) are simplectic equivalent. However, as we will see, the spectrum of geometric operators are modified by different value of \( \gamma \), and the non-perturbative calculation of black hole entropy is compatible with Bekenstein-Hawking’s formula only for a specific value of \( \gamma \) \[59\]. In addition, it is argued that the Barbero-Immerzi parameter \( \gamma \) may lead to observable effects in principle when the gravitational field is coupled with fermions \[93\]. In the decomposed action \[7\], the Hamiltonian density \( \mathcal{H}_{tot} \) is a linear combination of constraints:

\[ \mathcal{H}_{tot} = \Lambda^i G_i + N^a C_a + NC, \]

where \( \Lambda^i \equiv \frac{1}{4} \epsilon_{ijk} \omega^j_l \), \( N^a \) and \( N \) are Lagrange multipliers. The three constraints in the Hamiltonian are expressed as \[17\]:

\[ G_i = D_a \tilde{P}^a_i := \partial_a \tilde{P}^a_i + \epsilon_{ij}^k A^j_a \tilde{P}^k_i, \]

\[ C_a = \tilde{P}^b_i F^i_{ab} - \frac{1 + \gamma^2}{\gamma} K^i_a G_i, \]

\[ C = \frac{\kappa^2 \gamma^2}{2\sqrt{\det q}} \tilde{P}^a_i \tilde{P}^b_j \epsilon_{ijk} F^k_{ab} - 2(1 + \gamma^2) K^i_a K^i_j + \kappa(1 + \gamma^2) \partial_a \left( \frac{\tilde{P}^a_i}{\sqrt{\det q}} \right) G^i, \]

where the configuration variable \( A^i_a \) performs as a so(3)-valued connection on \( \Sigma \) and \( F^i_{ab} \) is the so(3)-valued curvature 2-form of \( A^i_a \) with the well-known expression:

\[ F^i_{ab} := 2D_i [a A^i_b] = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{jk} A^j_a A^k_b. \]
In any dynamical system with constraints, the constraint analysis is essentially important because they reflect the gauge invariance of the system. From the above three constraints of general relativity, one can know the gauge invariance of the theory. The Gaussian constraint \( G_i = 0 \) has crucial importance in formulating the general relativity into a dynamical theory of connections. The corresponding smeared constraint function, \( \mathcal{G}(\Lambda) := \int_{\Sigma} d^3x \Lambda^i(x) G_i(x) \), generates a transformation on the phase space as:

\[
\{ A^i_a(x), \mathcal{G}(\Lambda) \} = -D_a \Lambda^i(x)
\]
\[
\{ \tilde{P}^a_i(x), \mathcal{G}(\Lambda) \} = \epsilon_{ij}^k \Lambda^j(x) \tilde{P}^a_k(x),
\]

which are just the infinitesimal versions of the following gauge transformation for the \( \text{so}(3) \)-valued connection 1-form \( A \) and internal rotation for the \( \text{so}(3) \)-valued densitized vector field \( \tilde{P} \) respectively:

\[
(A_a, \tilde{P}^b) \mapsto (g^{-1}A_a g + g^{-1}(dg)_a, g^{-1}\tilde{P}^b g).
\]

To display the meaning of the vector constraint \( C_a = 0 \), one may consider the smeared constraint function:

\[
\mathcal{V}(\vec{N}) := \int_{\Sigma} d^3x (N^a \tilde{P}^b_i F^i_{ab} - (N^a A^i_a) G_i).
\]

It generates the infinitesimal spatial diffeomorphism by the vector field \( N^a \) on \( \Sigma \) as:

\[
\{ A^i_a(x), \mathcal{V}(\vec{N}) \} = \mathcal{L}_{\vec{N}} A^i_a(x),
\]
\[
\{ \tilde{P}^a_i(x), \mathcal{V}(\vec{N}) \} = \mathcal{L}_{\vec{N}} \tilde{P}^a_i(x).
\]

The smeared scalar constraint is weakly equivalent to the following function, which is re-expressed for quantization purpose as

\[
\mathcal{H}(N) := \int_{\Sigma} d^3x N(x) \tilde{C}(x) = \frac{k \gamma^2}{2} \int_{\Sigma} d^3x \frac{\tilde{P}^a_i \tilde{P}^b_j}{\sqrt{|\det q|}} [\epsilon^{ij}_k F^k_{ab} - 2(1 + \gamma^2) K^i[a] K^j_b]. \tag{10}
\]

It generates the infinitesimal time evolution off \( \Sigma \). The constraints algebra, i.e., the Poisson brackets between these constraints, play a crucial role in the quantization programme. It can be shown that the constraints algebra of (10) has the following form:

\[
\{ \mathcal{G}(\Lambda), \mathcal{G}(\Lambda') \} = \mathcal{G}([\Lambda, \Lambda']),
\]
\[
\{ \mathcal{G}(\Lambda), \mathcal{V}(\vec{N}) \} = -\mathcal{G}(\mathcal{L}_{\vec{N}} \Lambda),
\]
\[
\{ \mathcal{G}(\Lambda), \mathcal{H}(N) \} = 0,
\]
\[
\{ \mathcal{V}(\vec{N}), \mathcal{V}(\vec{N}') \} = \mathcal{V}([\vec{N}, \vec{N}']).
\]
where \( \det q^{ab} = \kappa^2 \gamma^2 \bar{P}_a \bar{P}_b \delta_{ij} \). Hence the constraints algebra is closed under the Poisson brackets, i.e., the constraints are all of first class. Note that the evolution of constraints is consistent since the Hamiltonian \( H = \int \Sigma d^3 x H_{\text{tot}} \) is the linear combination of the constraints functions. The evolution equations of the basic canonical pair read

\[
\mathcal{L}_t A^i_a = \{ A^i_a, H \}, \quad \mathcal{L}_t \bar{P}_i^a = \{ \bar{P}_i^a, H \}.
\]

Together with the three constraint equations, they are completely equivalent to the Einstein field equations. Thus general relativity is cast as a dynamical theory of connections with a compact structure group. Before completing the discussion of this section, several remarks should be emphasized.

**Canonical Transformation Viewpoint**

The above construction can be reformulated in the language of canonical transformation, since the phase space of connection dynamics is the same as that of triad geometrodynamics. In the triad formalism the basic conjugate pair consists of densitized triad \( \bar{E}^a_i \) and ”extrinsic curvature” \( K^i_j \). The Hamiltonian and constraints read

\[
H_{\text{tot}} = N^a G^i_a + N^a C_a + NC
\]

with

\[
G^i_a = \epsilon_{ij} \bar{E}^b_j K^i_k \bar{E}^a_k, \quad C_a = \bar{E}^b_j \nabla_{[a} K^i_{b]} , \quad C = \frac{1}{\sqrt{\det q}} \left[ \frac{1}{2} \det q |R + \bar{E}^a_i \bar{E}^b_j K^i_k K^j_k| \right],
\]

where \( \nabla_a \) is the \( SO(3) \) generalized derivative operator compatible with triad \( e^a_i \) and \( R \) is the scalar curvature with respect to it. Since \( \bar{E}^a_i \) is a vector density of weight one, we have

\[
\nabla_a \bar{E}^a_i = \partial_a \bar{E}^a_i + \epsilon_{ij} \bar{E}^a_k \Gamma^i_{kj} = 0.
\]

One can construct the desired Gaussian constraint by

\[
G_i := \frac{1}{\gamma} \nabla_a \bar{E}^a_i + G^i_a, \quad G_i = \partial_a \bar{P}_i^a + \epsilon_{ij} \bar{P}_k^a (\Gamma^j_{ai} + \gamma K^j_k) \bar{P}_k^a,
\]

which is weakly zero by construction. This motivates us to define the connection \( A^a_i = \Gamma^i_a + \gamma K^i_a \). Moreover, the transformation from the
pair \((\tilde{E}^a_i, K^b_j)\) to \((\tilde{P}^a_i, A^b_j)\) can be proved to be a canonical transformation [29][122], i.e., the Poisson algebra of the basic dynamical variables is preserved under the transformation:

\[
\tilde{E}^a_i \rightarrow \tilde{P}^a_i = \tilde{E}^a_i / \gamma \\
K^b_j \rightarrow A^b_j = \Gamma^b_j + \gamma K^b_j,
\]

as

\[
\{\tilde{P}^a_i(x), A^b_j(y)\} = \{\tilde{E}^a_i(x), K^b_j(y)\} = \delta^a_i \delta^b_j \delta(x - y),
\]

\[
\{A^b_i(x), A^b_j(y)\} = \{K^b_i(x), K^b_j(y)\} = 0,
\]

\[
\{\tilde{P}^a_i(x), \tilde{P}^b_j(y)\} = \{\tilde{E}^a_i(x), \tilde{E}^b_j(y)\} = 0.
\]

• **The Preparation for Quantization**

The advantage of a dynamical theory of connections is that it is convenient to be quantized background independently. In the following procedure of quantization, the quantum algebra of the elementary observables will be generated by **Holonomy**, i.e., connection smeared on a curve, and **Electric Flux**, i.e., densitized triad smeared on a 2-surface. So no information of background would affect the definition of the quantum algebra. In the remainder of the paper, in order to incorporate also spinors, we will enlarge the internal gauge group to be \(SU(2)\). This does not damage the prior constructions because the Lie algebra of \(SU(2)\) is the same as that of \(SO(3)\). Due to the well-known nice properties of compact Lie group \(SU(2)\), such as the Haar measure and Peter-Weyl theorem, one can obtain the background independent representation of the quantum algebra and the spin-network decomposition of the kinematic Hilbert space.

• **Analysis on Constraint Algebra**

The classical constraint algebra [11] is an infinite dimensional Poisson algebra. However, it is not a Lie algebra unfortunately, because the Poisson bracket between two scalar constraints has structure function depending on dynamical variables. This character causes much trouble in solving the constraints quantum mechanically. On the other hand, one can see from Eq.(11) that the algebra generated by Gaussian constraints forms not only a subalgebra but also a 2-side ideal in the full constraint algebra. Thus one can first solve the Gaussian constraints independently. It is convenient to find the quotient algebra with respect to the Gaussian constraint subalgebra as

\[
\{\mathcal{V}(\tilde{N}), \mathcal{V}(\tilde{N}')\} = \mathcal{V}([\tilde{N}, \tilde{N}']),
\]

\[
\{\mathcal{V}(\tilde{N}), \mathcal{H}(M)\} = -\mathcal{H}(\mathcal{L}_M M),
\]

\[
\{\mathcal{H}(N), \mathcal{H}(M)\} = -\mathcal{V}((N\partial_b M - M\partial_b N)q^{ab}),
\]

which plays a crucial role in solving the constraints quantum mechanically. But the subalgebra generated by the diffeomorphism constraints
can not form an ideal. Hence the procedures of solving the diffeomorphism constraints and solving Hamiltonian constraints are entangled with each other. This leads to certain ambiguity in the construction of a Hamiltonian constraint operator. Fortunately, Master Constraint Project settles up the above two problems as a whole by introducing a new classical constraint algebra which is equivalent to the prior one. The new algebra is a Lie algebra where the diffeomorphism constraints form a 2-side ideal.

4 Quantum Kinematics

In this section, we will begin to quantize the above classical dynamics of connections as a background independent quantum field theory. The main purpose of the section is to construct a suitable kinematical Hilbert space $\mathcal{H}_{kin}$ for the representation of quantum observables. It is performed in section 2.2 that the kinematical Hilbert space can be obtained via the abstract GNS-construction. However, in this section, we would like to construct the Hilbert space firstly in a more concrete and straightforward way (constructive quantum field theory). Then we reformulate the construction in the language of GNS-construction (algebraic quantum field theory) in the section 4.5. It should be noted that both constructions are completely equivalent and can be generalized to all background independent non-perturbative Yang-Mills gauge field theories with compact gauge groups.

4.1 Quantum Configuration Space

In quantum mechanics, the kinematical Hilbert space is $L^2(\mathbb{R}^3, d^3x)$, where the simple $\mathbb{R}^3$ is the classical configuration space of free particle which has finite degrees of freedom, and $d^3x$ is the Lebesgue measure on $\mathbb{R}^3$. In quantum field theory, it is expected that the kinematical Hilbert space is also the $L^2$ space on the configuration space of the field, which is infinite dimensional, with respect to some Borel measure naturally defined. However, it is often hard to define concretely a Borel measure on the classical configuration space, since the integral theory on infinite dimensional space is involved. Thus the intuitive expectation should be modified, and the concept of quantum configuration space should be introduced as a suitable enlargement of the classical configuration space so that an infinite dimensional measure, often called cylindrical measure, can be well defined on it. The example of a scalar field can be found in the references. For quantum gravity, it should be emphasized that the construction for quantum configuration space must be background independent. Fortunately, general relativity has been reformulated as a dynamical theory of $SU(2)$ connections, which would be great helpful for our further development.

The classical configuration space for gravitational field, which is denoted by $A$, is a collection of the $su(2)$-valued connection 1-form field smoothly distributed on $\Sigma$. The idea of the construction for quantum configuration is due
to the concept of Holonomy.

**Definition 4.1.1:** Given a smooth $SU(2)$ connection field $A^a_i$ and an analytic curve $c$ with the parameter $t \in [0, 1]$ supported on a compact subset (compact support) of $\Sigma$, the corresponding holonomy is defined by the solution of the parallel transport equation

$$\frac{d}{dt} A(c, t) = -[A^i_a (\frac{\partial}{\partial t})^a \tau_i] A(c, t),$$  \hspace{1cm} (15)$$

with the initial value $A(c, 0) = 1$, where $(\partial/\partial t)^a$ is the tangent vector of the curve and $\tau_i \in su(2)$ constitute an orthonormal basis with respect to the Killing-Cartan metric $\eta(\xi, \zeta) := -2\text{Tr}(\xi\zeta)$, which satisfy $[\tau_i, \tau_j] = \epsilon_{ijk} \tau_k$ and are fixed once for all. Thus the holonomy is an element in $SU(2)$, which can be expressed as

$$A(c) = \mathcal{P} \exp \left( - \int_0^1 [A^i_a (\frac{\partial}{\partial t})^a \tau_i] \, dt \right),$$  \hspace{1cm} (16)$$

where $A(c) \in SU(2)$ and $\mathcal{P}$ is a path-ordering operator along the curve $c$ (see the footnote at p382 in [88]).

The definition can be well extended to the case of piecewise analytic curves via the relation:

$$A(c_1 \circ c_2) = A(c_1) A(c_2),$$  \hspace{1cm} (17)$$

where \( \circ \) stands for the composition of two curves. It is easy to see that a holonomy is invariant under the re-parametrization and is covariant under changing the orientation, i.e.,

$$A(c^{-1}) = A(c)^{-1}.$$  \hspace{1cm} (18)$$

So one can formulate the properties of holonomy in terms of the concept of the equivalent classes of curves.

**Definition 4.1.2:** Two analytic curves $c$ and $c'$ are said to be equivalent if and only if they have the same source $s(c)$ (beginning point) and the same target $t(c)$ (end point), and the holonomies of the two curves are equal to each other, i.e., $A(c) = A(c') \forall A \in \mathcal{A}$. An equivalent class of analytic curves is defined to be an edge, a piecewise analytic path is an composition of edges.

To summarize, the holonomy is actually defined on the set $\mathcal{P}$ of piecewise analytic paths with compact supports. The two properties (17) and (18) mean that each connection in $\mathcal{A}$ is a homomorphism from $\mathcal{P}$, which is so-called a groupoid by definition (131), to our compact gauge group $SU(2)$. Note that the internal gauge transformation and spatial diffeomorphism act covariantly on a holonomy as

$$A(e) \mapsto g(t(e))^{-1} A(e) g(s(e)) \quad \text{and} \quad A(e) \mapsto A(\varphi \circ e),$$  \hspace{1cm} (19)$$
for any $SU(2)$-valued function $g(x)$ on $\Sigma$ and spatial diffeomorphism $\varphi$. All above discussion is for classical smooth connections in $A$. The quantum configuration space for loop quantum gravity can be constructed by extending the concept of holonomy, since its definition does not depend on an extra background. One thus obtains the quantum configuration space $\mathcal{A}$ of loop quantum gravity as the following.

**Definition 4.1.3:** The quantum configuration space $\mathcal{A}$ is a collection of all quantum connections $A$, which are algebraic homomorphism maps without any continuity assumption from the collection of piecewise analytic paths with compact supports, $\mathcal{P}$, on $\Sigma$ to the gauge group $SU(2)$, i.e., $\mathcal{A} := \text{Hom}(\mathcal{P}, SU(2))$. Thus for any $A \in \mathcal{A}$ and edge $e$ in $\mathcal{P}$,

$$A(e_1 \circ e_2) = A(e_1)A(e_2) \quad \text{and} \quad A(e^{-1}) = A(e)^{-1}.$$  

The transformations of quantum connections under internal gauge transformations and diffeomorphisms are defined by Eq. (19).

Several remarks on the properties of $\mathcal{A}$ are listed below.

- **Extension from Classical Configuration Space**
  
The above discussion on the smooth connections shows that the classical configuration space $A$ can be understood as a subset in the quantum configuration space $\mathcal{A}$. On the other hand, the Giles theorem [67] shows precisely that a smooth connection can be recovered from its holonomies by varying the length and location of the paths.

- **The Discrete Nature of Quantum Geometry**
  
  Since there is no continuity assumption on quantum connections, there are such elements in $\mathcal{A}$, which only take values on some finite graph $\alpha$ consisting of finite number of edges and map any other paths to the identity in $SU(2)$. This presents the discrete and distributional structure of quantum geometry, namely, the geometry exists only on finite and discrete edges on $\Sigma$. Fix a finite graph $\alpha$ with $N_{\alpha}$ edges and let the quantum connections $A_{\alpha}$ be trivial elsewhere. Then one can identify the space $\mathcal{A}_{\alpha}$ of quantum connections on $\alpha$ with the direct product group $SU(2)^{N_{\alpha}} = \bigotimes_{\alpha} SU(2)$ by the natural 1-1 map:

$$A_{\alpha} \mapsto (A_{\alpha}(e_1), ..., A_{\alpha}(e_{N_{\alpha}})),$$

where $e_i$ is the edge in $\alpha$.

- **Projection Technique Viewpoint and Topology of $\mathcal{A}$**
  
  Given the set $\mathcal{L}$ of all graphs with finite number of edges, one can equip a partial order relation $\prec$ on $\mathcal{L}^4$, defined by $\alpha \prec \alpha'$ if and only if $\alpha$ is a partial order on $\mathcal{L}$ is a relation, which is reflexive ($\alpha \prec \alpha$), symmetric ($\alpha \prec \alpha'$, $\alpha' \prec \alpha \Rightarrow \alpha' = \alpha$) and transitive ($\alpha \prec \alpha'$, $\alpha' \prec \alpha'' \Rightarrow \alpha' \prec \alpha''$). Note that not all pairs in $\mathcal{L}$ need to have a relation.
subgraph in $\alpha'$. Obviously, for any two graphs $\alpha$ and $\alpha'$ in $L$, there exists $\alpha'' \in L$ such that $\alpha, \alpha' \prec \alpha''$. In addition, for any pair $\alpha \prec \alpha'$, one can define a surjective projection map $P_{\alpha''} \alpha$ from $\mathfrak{A}_{\alpha'}$ to $\mathfrak{A}_{\alpha}$ by restricting the nontrivial domain of the map $\mathcal{A}_{\alpha'}$ from $\alpha'$ to the subgraph $\alpha$, and these projections satisfy the consistency condition $P_{\alpha''} \alpha \circ P_{\alpha''} \alpha' = P_{\alpha''} \alpha'$. Thus a projective family $\{\mathfrak{A}_{\alpha}, P_{\alpha''} \alpha\}_{\alpha \prec \alpha'}$ is obtained by above constructions. Then the projective limit $\text{lim}_\alpha(\mathfrak{A}_{\alpha})$ is naturally obtained.

**Definition 4.1.4:** The projective limit $\text{lim}_\alpha(\mathfrak{A}_{\alpha})$ of the projective family $\{\mathfrak{A}_{\alpha}, P_{\alpha''} \alpha\}_{\alpha \prec \alpha'}$ is a subset of the direct product space $\mathfrak{A}_\infty := \prod_{\alpha \in L} \mathfrak{A}_{\alpha}$ defined by

$$\text{lim}_\alpha(\mathfrak{A}_{\alpha}) := \{\{A_{\alpha}\}_{\alpha \in L} | P_{\alpha''} \alpha A_{\alpha'} = A_{\alpha}, \forall \alpha \prec \alpha'\}.$$  

Note that the projection $P_{\alpha''} \alpha$ is surjective and continuous with respect to the topology of $SU(2)^N$. One can equip the direct product space $\mathfrak{A}_\infty := \prod_{\alpha \in L} \mathfrak{A}_{\alpha}$ with the so-called Tychonov topology. Since any $\mathfrak{A}_{\alpha}$ is a compact Hausdorff space, by Tychonov theorem $\mathfrak{A}_\infty$ is also a compact Hausdorff space. One then can prove that the projective limit, $\text{lim}_\alpha(\mathfrak{A}_{\alpha})$, is a closed subset in $\mathfrak{A}_\infty$ and hence a compact Hausdorff space with respect to the topology induced from $\mathfrak{A}_\infty$. At last, one can find the relation between the projective limit and the prior constructed quantum configuration space $\mathfrak{A}$. As one might expect, there is a bijection $\Phi$ between $\mathfrak{A}$ and $\text{lim}_\alpha(\mathfrak{A}_{\alpha})$ \[122\]:

$$\Phi : \mathfrak{A} \rightarrow \text{lim}_\alpha(\mathfrak{A}_{\alpha});$$

$$A \mapsto \{A_{\alpha}\}_{\alpha \in L},$$

where $A_{\alpha}$ means the restriction of the nontrivial domain of the map $A \in \mathfrak{A} = \text{Hom}(\mathcal{P}, SU(2))$. As a result, the quantum configuration space is identified with the projective limit space and hence can be equipped with the topology. In conclusion, the quantum configuration space $\mathfrak{A}$ is constructed to be a compact Hausdorff topological space.

### 4.2 Cylindrical Functions on Quantum Configuration Space

Given the projective family $\{\mathfrak{A}_{\alpha}, P_{\alpha''} \alpha\}_{\alpha \prec \alpha'}$, the cylindrical function on its projective limit $\mathfrak{A}$ is well defined as the following:

**Definition 4.2.1:** Let $C(\mathfrak{A}_{\alpha})$ be the set of all continuous complex functions on $\mathfrak{A}_{\alpha} = SU(2)^{N_{\alpha}}$, two functions $f_{\alpha} \in C(\mathfrak{A}_{\alpha})$ and $f_{\alpha'} \in C(\mathfrak{A}_{\alpha'})$ are said to be equivalent or cylindrical consistent, denoted by $f_{\alpha} \sim f_{\alpha'}$, if and only if $P_{\alpha''} \alpha f_{\alpha} = P_{\alpha''} \alpha f_{\alpha'}, \forall \alpha'' \gtrsim \alpha, \alpha'$, where $P_{\alpha''} \alpha$ denotes the pullback map induced from $P_{\alpha''} \alpha$. Then the space $\text{Cyl}(\mathfrak{A})$ of cylindrical functions on the projective limit $\mathfrak{A}$ is defined to be the space of equivalent classes $[f]$, i.e.,

$$\text{Cyl}(\mathfrak{A}) := [\cup_{\alpha} C(\mathfrak{A}_{\alpha})]/\sim.$$
One then can easily prove the following proposition by definition.

**Proposition 4.1:**

All continuous functions $f_\alpha$ on $\overline{\mathcal{A}}_\alpha$ are automatically cylindrical since each of them can generate an equivalent class $[f_\alpha]$ via the pullback map $P^*_\alpha\alpha'$ for all $\alpha' \succ \alpha$, and the dependence of $P^*_\alpha\alpha'$ on the groups associated to the edges in $\alpha'$ but not in $\alpha$ is trivial, i.e., by the definition of the pull back map,

\[(P^*_\alpha\alpha'f_\alpha)(A(e_1),\ldots,A(e_{N_\alpha}),\ldots,A(e_{N_{\alpha'}})) = f_\alpha(A(e_1),\ldots,A(e_{N_\alpha})).\] (20)

On the other hand, by definition, given a cylindrical function $f \in \text{Cyl}(\overline{\mathcal{A}})$ there exists a suitable graph $\alpha$ such that $f = [f_\alpha]$, so one can identify $f$ with $f_\alpha$. Moreover, given two cylindrical functions $f, f' \in \text{Cyl}(\overline{\mathcal{A}})$, by definition of cylindrical functions and the property of projection map, there exists a common graph $\alpha$ and $f_\alpha, f'_\alpha \in \text{C}(\overline{\mathcal{A}}_\alpha)$ such that $f = [f_\alpha]$ and $f' = [f'_\alpha]$. The space of cylindrical functions and the space of continuous functions on $\overline{\mathcal{A}}$ are identical up to a completion with respect to a sup-norm by the following theorem.

**Theorem 4.2.1:**

(1) The space $\text{Cyl}(\overline{\mathcal{A}})$ can be constructed as a unital Abelian $C^*$-algebra after completion with respect to the sup-norm.

(2) Quantum configuration space $\overline{\mathcal{A}}$ is the spectrum space of completed $\text{Cyl}(\overline{\mathcal{A}})$ such that $\text{Cyl}(\overline{\mathcal{A}})$ is identical to the space $\text{C}(\overline{\mathcal{A}})$ of continuous functions on $\overline{\mathcal{A}}$.

**Proof:**

(1) Let $f, f' \in \text{Cyl}(\overline{\mathcal{A}})$, there exists graph $\alpha$ such that $f = [f_\alpha]$, and $f' = [f'_\alpha]$, then the following operations are well defined

\[f + f' := [f_\alpha + f'_\alpha], \quad ff' := [f_\alpha f'_\alpha], \quad zf := [zf_\alpha], \quad \bar{f} := [\bar{f}_\alpha],\]

where $z \in \mathbb{C}$ and $\bar{f}$ denotes complex conjugate. So we construct $\text{Cyl}(\overline{\mathcal{A}})$ as an Abelian $*$-algebra. In addition, there is a unital element in the algebra because $\text{Cyl}(\overline{\mathcal{A}})$ contains constant functions. Moreover, we can well define the sup-norm for $f = [f_\alpha]$ by

\[\|f\| := \sup_{A_\alpha \in \overline{\mathcal{A}}_\alpha} |f_\alpha(A_\alpha)|,\] (21)

which satisfies the $C^*$ property $\|ff\| = \|f\|^2$. After the completion with respect to the norm, $\text{Cyl}(\overline{\mathcal{A}})$ is a unital Abelian $C^*$-algebra.

(2) A unital Abelian $C^*$-algebra is identical to the space of continuous functions on its spectrum space via an isometric isomorphism, the so-called Gel’fand transformation (see e.g. [122]), which completes the proof.
4.3 Kinematical Hilbert Space

The main purpose of this subsection is to construct a kinematical Hilbert space $H_{\text{kin}}$ for loop quantum gravity, which is a $L^2$ space on the quantum configuration space $\mathcal{A}$ with respect to some measure $d\mu$. There is a well-defined probability measure on $\mathcal{A}$ originated from the Haar measure on the compact group $SU(2)$, which is named as the Ashtekar-Isham-Lewandowski Measure for loop quantum gravity. Consider the simplest case where a graph $e$ has only one edge. Then the corresponding quantum configuration space $A_e$ is identical to the group $SU(2)$. The continuous functions on $A_e$ is certainly contained in $Cyl(\mathcal{A})$. Due to the compactness of $SU(2)$, there exists a unique probability measure, namely the Haar measure, on it, which is invariant under right and left translations and inverse of the group elements.

**Theorem 4.3.1** [10]:

Given a compact group $G$ and an automorphism $\varphi : G \to G$ on it, there exist a unique measure $d\mu_H$ on $G$, named as Haar measure, such that:

\[ \int_G d\mu_H = 1, \]  
\[ \int_G f(g)d\mu_H = \int_G f(\varphi(g))d\mu_H, \]  
\[ \int_G f(g^{-1})d\mu_H = \int_G f(\varphi(g))d\mu_H, \]  
\[ \text{for all continuous functions } f \text{ on } G \text{ and for all } h \in G. \]

Thus one equips $A_e$ with the measure $\mu_e \equiv \mu_H$. Similarly, a probability measure can be defined on any graph with finite number of edges by the direct product of Haar measure, since $\mathcal{A}_\alpha = SU(2)^{N_e}$. Then for any graph $\alpha$, a Hilbert space is defined on $\mathcal{A}_\alpha$ as $H_{\text{kin}} = L^2(\mathcal{A}_\alpha, d\mu_\alpha) = \bigotimes_{e \in \alpha} L^2(\mathcal{A}_e, d\mu_e)$. Moreover, the family of measures $\{\mu_\alpha\}_{\alpha \in \mathcal{L}}$ defined on the projective family $\{\mathcal{A}_\alpha, P_{\alpha'\alpha}\}_{\alpha < \alpha'}$ are cylindrically consistent, since

\[ \int_{\mathcal{A}_{\alpha'}} (P_{\alpha'\alpha} f_{\alpha}) d\mu_{\alpha'} = \int_{\mathcal{A}_{\alpha'}} (P_{\alpha'\alpha} f_{\alpha})(A(e_1), ..., A(e_{N_{\alpha'}})) d\mu_{e_1} ... d\mu_{e_{N_{\alpha'}}}, \]
\[ = \int_{\mathcal{A}_{\alpha}} f_{\alpha}(A(e_1), ..., A(e_{N_{\alpha}})) d\mu_{e_1} ... d\mu_{e_{N_{\alpha}}}, \]
\[ = \int_{\mathcal{A}_{\alpha}} f_{\alpha} d\mu_{\alpha}, \]

due to Eqs. (20) and (22). Given such a cylindrically consistent family of measures $\{\mu_\alpha\}_{\alpha \in \mathcal{L}}$, a probability measure $d\mu$ is uniquely well defined on the quantum configuration space $\mathcal{A}$ [15], which is described precisely in the theorem.
Theorem 4.3.2: \[122\]
Given the projective family \( \{ \mathcal{A}_\alpha, P_{\alpha/\alpha'} \}_{\alpha < \alpha'} \), whose projective limit is \( \mathcal{A} \), and the cylindrically consistent family of measures \( \{ \mu_\alpha \}_{\alpha \in \mathcal{L}} \) constructed from the Haar measure on the compact group, there exists a unique regular Borel probability measure \( d\mu \) on the projective limit \( \mathcal{A} \) such that
\[
\int_{\mathcal{A}} f d\mu = \int_{\mathcal{A}_\alpha} f_\alpha d\mu_\alpha, \forall f = [f_\alpha] \in Cyl(\mathcal{A}),
\]
which is guaranteed by proposition 4.1.

Then \( \mathcal{A} \) is equipped with the Ashtekar-Isham-Lewandowski measure \( d\mu \) and becomes a topological measure space \[13][15\]. The unique measure essentially defines a state on the quantum holonomy-flux algebra for Yang-Mills gauge theory, which is called Ashtekar-Isham-Lewandowski state in the language of GNS-construction. Moreover the two important gauge invariant properties of Ashtekar-Isham-Lewandowski measure make it well suitable for the diffeomorphism invariant gauge field theory.

Theorem 4.3.3:
The Ashtekar-Isham-Lewandowski measure is invariant under Yang-Mills internal gauge transformations \( g(x) \) and spatial diffeomorphisms \( \varphi \), i.e.,
\[
\int_{\mathcal{A}} g \circ f d\mu = \int_{\mathcal{A}} f d\mu \quad \text{and} \quad \int_{\mathcal{A}} \varphi \circ f d\mu = \int_{\mathcal{A}} f d\mu,
\]
\( \forall f \in Cyl(\mathcal{A}) \).

Proof:
(1) (Internal gauge invariance)
\[
\int_{\mathcal{A}} g \circ f d\mu = \int_{\mathcal{A}_\alpha} g \circ f_\alpha d\mu_\alpha = \int_{\mathcal{A}_\alpha} f_\alpha d\mu_\alpha = \int_{\mathcal{A}} f d\mu
\]
\( \forall f = [f_\alpha] \in Cyl(\mathcal{A}) \), where we used
\[
g \circ f_\alpha (A(e_1), ..., A(e_{N_\alpha})) = f_\alpha (g(t(e_1))^{-1} A(e_1) g(s(e_1)), ..., g(t(e_{N_\alpha}))^{-1} A(e_{N_\alpha}) g(s(e_{N_\alpha}))),
\]
since Haar measure is invariant under right and left translations.
(2) (Diffeomorphism invariance)
\[
\int_{\mathcal{A}} \varphi \circ f d\mu = \int_{\mathcal{A}_{\varphi \circ \alpha}} f_{\varphi \circ \alpha} d\mu_{\varphi \circ \alpha} = \int_{\mathcal{A}_\alpha} f_\alpha d\mu_\alpha = \int_{\mathcal{A}} f d\mu,
\]
where \( f_{\varphi \circ \alpha} \equiv f_\alpha (A(\varphi \circ e_1), ..., A(\varphi \circ e_{N_\alpha})) \) and we relabel \( A(\varphi \circ e_i) \mapsto A(e_i) \) in the second step.
With the above constructed measure on $\mathcal{A}$, the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ is obtained straightforwardly as

$$\mathcal{H}_{\text{kin}} := L^2(\mathcal{A}, d\mu).$$  \hfill (24)

Several remarks on the properties of $\mathcal{H}_{\text{kin}}$ are listed below.

- **Kinematical Inner Product**
  Given any $f = [f_\alpha], f' = [f'_\alpha] \in \text{Cyl}(\mathcal{A})$, the $L^2$ inner product of them is defined as

$$\langle f|f' \rangle_{\text{kin}} := \int_{\mathcal{A}} \left( P_{\alpha''\alpha}^\ast f_{\alpha''} \right) \left( P_{\alpha''\alpha'}^\ast f'_{\alpha'} \right) d\mu_{\alpha''},$$  \hfill (25)

for any graph $\alpha''$ containing both $\alpha$ and $\alpha'$. This expression is consistent with the $L^2$ inner product defined in Eq. (24) by proposition 4.1. It should be noted that the cylindrical functions in $\mathcal{H}_{\text{kin}}$ is dense with respect to the $L^2$ inner product, as they are dense in $C(\mathcal{A})$ with respect to the sup-norm. As a result, the kinematical Hilbert space can be viewed as the completion of $\text{Cyl}(\mathcal{A})$ with respect to the inner product (25), i.e.,

$$\mathcal{H}_{\text{kin}} = \left\langle \text{Cyl}(\mathcal{A}) \right\rangle = \left\langle \bigcup_{\alpha \in \mathcal{L}} \mathcal{H}_\alpha \right\rangle,$$  \hfill (26)

here the $\langle \cdot \rangle$ means the completion with respect to the inner product (25). Later we will show that $\mathcal{H}_{\text{kin}}$ is a non-separable Hilbert space.

- **Background Independence and Uniqueness**
  It is important to note that all above constructions are background independent. Moreover, theorem 2.1 means that Ashtekar-Isham-Lewandowski measure on $\mathcal{A}$ is the unique one with internal Yang-Mills gauge invariance and diffeomorphism invariance. Hence the Ashtekar-Isham-Lewandowski representation space $\mathcal{H}_{\text{kin}}$ is unique such that the Yang-Mills gauge transformation and diffeomorphism are represented as unitary operators on it.

### 4.4 Spin-network Decomposition of Kinematical Hilbert Space

Up to now, the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ for loop quantum gravity has been well defined. In this subsection, it will be shown that $\mathcal{H}_{\text{kin}}$ can be decomposed into the orthogonal direct sum of 1-dimensional subspaces. One can thus find a system of basis, named as spin-network basis, in the Hilbert space, which consists of uncountably infinite elements. So the kinematic Hilbert space is non-separable. In the following, we will do the decomposition in three steps.
Spin-network Decomposition on Single Edge

Given a graph of one edge $e$, which naturally associates with a group $SU(2) = A_e$, the elements of $A_e$ are the quantum connections only taking nontrivial values on $e$. Then we consider the decomposition of the Hilbert space $H_e = L^2(A_e, d\mu_e) = L^2(SU(2), d\mu_H)$, which is nothing but the space of square integrable functions on the compact group $SU(2)$ with the natural $L^2$ inner product. It is natural to define several operators on $H_e$. Firstly, the so-called configuration operator $\hat{f}(A(e))$ whose operation on any $\psi \in L^2(SU(2), d\mu_H)$ is nothing but multiplication by the function $f(A(e)) \in L^2(SU(2), d\mu_H)$, i.e.,
\[
\hat{f}(A(e))\psi_e(A(e)) := f(A(e))\psi_e(A(e)),
\]
where $A(e) \in SU(2)$. Secondly, given any vector $\xi \in su(2)$, it generates left invariant vector field $L(\xi)$ and right invariant vector field $R(\xi)$ on $SU(2)$ by
\[
L(\xi)\psi_e(A(e)) := \frac{d}{dt}|_{t=0}\psi_e(e^{t\xi}A(e)),
\]
\[
R(\xi)\psi_e(A(e)) := \frac{d}{dt}|_{t=0}\psi_e(e^{-t\xi}A(e)),
\]
for any function $\psi_e \in C^1(SU(2))$. Then one can define the so-called momentum operators on $H_e$ by
\[
\hat{J}^{(L)}_i = iL(\tau_i) \quad \text{and} \quad \hat{J}^{(R)}_i = iR(\tau_i),
\]
where the generators $\tau_i \in su(2)$ constitute an orthonormal basis with respect to the Killing-Cartan metric. The momentum operators have the well-known commutation relation of the angular momentum operators in quantum mechanics:
\[
[\hat{J}^{(L)}_i, \hat{J}^{(L)}_j] = i\epsilon^{ijk} \hat{J}^{(L)}_k, \quad [\hat{J}^{(R)}_i, \hat{J}^{(R)}_j] = i\epsilon^{ijk} \hat{J}^{(R)}_k, \quad [\hat{J}^{(L)}_i, \hat{J}^{(R)}_j] = 0.
\]
Thirdly, the Casimir operator on $H_e$ can be expressed as
\[
\hat{J}_i^2 := \delta^{ij} \hat{J}^{(L)}_i \hat{J}^{(L)}_j = \delta^{ij} \hat{J}^{(R)}_i \hat{J}^{(R)}_j.
\]

The decomposition of $H_e = L^2(SU(2), d\mu_H)$ is provided by the Peter-Weyl Theorem:

**Theorem 4.4.1** [46]:

*Given a compact group $G$, the function space $L^2(G, d\mu_H)$ can be decomposed as an orthogonal direct sum of finite dimensional Hilbert space, and the matrix elements of the equivalent classes of finite dimensional irreducible representations of $G$ form a complete orthogonal basis in $L^2(G, d\mu_H)$.***
Note that a finite dimensional irreducible representation of $G$ can be regarded as a matrix valued function on $G$, so the matrix elements are functions on $G$. Using this theorem, one can find the decomposition of the Hilbert space:

$$L^2(SU(2), d\mu_H) = \oplus_j [H_j \otimes H^*_j],$$

where $j$, labelling irreducible representations of $SU(2)$, are the half-integers, $H_j$ denotes the carrier space of the $j$-representation of dimension $2j + 1$, and $H^*_j$ is its dual space. The basis $\{e^j_m \otimes e^j_\ast_n\}$ in $H_j \otimes H^*_j$ maps a group element $g \in SU(2)$ to a matrix $\{\pi^j_{mn}(g)\}$, where $m, n = -j, ..., j$, which means that the space $H_j \otimes H^*_j$ is spanned by the matrix element functions $\pi^j_{mn}$ of equivalent $j$-representations. Moreover, the spin-network basis can be defined.

**Proposition 4.2**[47]

The system of spin-network functions on $H_e$, consisting of matrix elements $\{\pi^j_{mn}\}$ in finite dimensional irreducible representations labelled by half-integers $\{j\}$, satisfies

$$\hat{J}^2 \pi^j_{mn} = j(j+1)\pi^j_{mn}, \quad \hat{J}^{(L)}_3 \pi^j_{mn} = m\pi^j_{mn}, \quad \hat{J}^{(R)}_3 \pi^j_{mn} = n\pi^j_{mn},$$

where $j$ is called angular momentum quantum number and $m, n = -j, ..., j$ magnetic quantum number. The normalized functions $\{\sqrt{2j+1} \pi^j_{mn}\}$ form a system of complete orthonormal basis in $H_e$ since

$$\int_{A_e} \overline{\pi^{j'}_{m'n'}} \pi^j_{mn} d\mu_e = \frac{1}{2j+1} \delta^{j'j} \delta_{m'm} \delta_{n'n},$$

which is called the spin-network basis on $H_e$. So the Hilbert space on a single edge has been decomposed into one dimensional subspaces.

Note that the system of operators $\{\hat{J}^2, \hat{J}^{(R)}_3, \hat{J}^{(L)}_3\}$ forms a complete set of commutable operators in $H_e$. There is a cyclic "vacuum state" in the Hilbert space, which is the $(j = 0)$-representation $\Omega_e = \pi^{j=0} = 1$, representing that there is no geometry on the edge.

- **Spin-network Decomposition on Finite Graph**

For a graph $\alpha$ with $N$ oriented edges $e_i$ and $M$ vertexes, one can define the configuration operators on the corresponding Hilbert space $H_\alpha$ by

$$\hat{f}(A(e_i))\psi_\alpha (A(e_1), ..., A(e_N)) := f(A(e_i))\psi_\alpha (A(e_1), ..., A(e_N)).$$

The momentum operators $\hat{J}^{(e,v)}_i$ associated with an edge $e$ connecting a vertex $v$ are defined as

$$\hat{J}^{(e,v)}_i := (1 \otimes \ldots \otimes \hat{J}_i \otimes \ldots \otimes 1),$$

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where we set $\hat{J}_i = \hat{J}_i^{(L)}$ if $v = s(e)$ and $\hat{J}_i = \hat{J}_i^{(R)}$ if $v = t(e)$. Note that the choice is based on the definition of gauge transformations \[19\]. Note also that $\hat{J}_i^{(e,v)}$ only act nontrivially on the Hilbert space associated with the edge $e$. Then one can define a vertex operator associated with vertex $v$ in analogy with the total angular momentum operator via

$$[\hat{J}_v]^2 := \delta^{ij} \hat{J}_i \hat{J}_j,$$

where

$$\hat{J}_i := \sum_{e' \text{ at } v} \hat{J}_i^{(e',v)}.$$

Obviously, $\mathcal{H}_\alpha$ can be firstly decomposed by the representations on each edge $e$ of $\alpha$ as:

$$\mathcal{H}_\alpha = \otimes_e \mathcal{H}_e = \otimes_e [\oplus_j (\mathcal{H}_j^e \otimes \mathcal{H}_j^{e'})] = \oplus_j [\otimes_e (\mathcal{H}_j^e \otimes \mathcal{H}_j^{e'})],$$

where $j := (j_1, ..., j_N)$ assigns to each edge a irreducible representation of $SU(2)$, in the fourth step the Hilbert spaces associated with the edges are allocated to the vertexes where these edges meet so that for each vertex $v$,

$$\mathcal{H}_j^{v=s(e)} \equiv \otimes_{e(s(e)=v)} \mathcal{H}_j^e \quad \text{and} \quad \mathcal{H}_j^{v=t(e)} \equiv \otimes_{e(t(e)=v)} \mathcal{H}_j^e.$$ 

The group of gauge transformations $g(v) \in SU(2)$ at each vertex is reducibly represented on the Hilbert space $\mathcal{H}_j^{v=s(e)} \otimes \mathcal{H}_j^{v=t(e)}$ in a natural way. So this Hilbert space can be decomposed as a direct sum of irreducible representation space via Clebsch-Gordon decomposition:

$$\mathcal{H}_j^{v=s(e)} \otimes \mathcal{H}_j^{v=t(e)} = \oplus_l \mathcal{H}_j^{v,e,l}.$$ 

As a result, $\mathcal{H}_\alpha$ can be further decomposed as:

$$\mathcal{H}_\alpha = \oplus_j [\otimes_v (\oplus_l \mathcal{H}_j^{v,e,l})] = \oplus_j [\oplus_l (\otimes_v \mathcal{H}_j^{v,e,l})] \equiv \oplus_j [\oplus_l \mathcal{H}_{\alpha,j,l}]. \quad (27)$$

It can also be viewed as the eigenvector space decomposition of the commuting operators $[\hat{J}_v]^2$ (with eigenvalues $l(l+1)$) and $[\hat{J}_e]^2 \equiv \delta^{ij} \hat{J}_i \hat{J}_j$. Note that $l := (l_1, ..., l_M)$ assigns to each vertex of $\alpha$ a irreducible representation of $SU(2)$. One may also enlarge the set of commuting operators to further refine the decomposition of the Hilbert space. Note that the subspace of $\mathcal{H}_\alpha$ with $l = 0$ is Yang-Mills gauge invariant, since the representation of gauge transformations is trivial.

• **Spin-network Decomposition of $\mathcal{H}_{\text{kin}}$**

Since $\mathcal{H}_{\text{kin}}$ has the structure $\mathcal{H}_{\text{kin}} = \bigcup_{\alpha \in \mathcal{L}} \mathcal{H}_\alpha$, one may consider to construct it as a direct sum of $\mathcal{H}_\alpha$. The construction is precisely described.
as a theorem below.

**Theorem 4.4.2:**
Consider assignments $j = (j_1, ..., j_N)$ to the edges of any graph $\alpha \in \mathcal{L}$, which are all non-trivial representations, and assignments $l = (l_1, ..., l_M)$ to the vertexes, which are non-trivial at spurious \(^5\) vertexes of $\alpha$. Let $\mathcal{H}_\alpha'$ be the Hilbert space composed by the subspaces $\mathcal{H}_{\alpha,j}$ assigned the above conditions according to Eq. (27). Then $\mathcal{H}_{\text{kin}}$ can be decomposed as the direct sum of the Hilbert spaces $\mathcal{H}_\alpha'$, i.e.,

$$\mathcal{H}_{\text{kin}} = \bigoplus_{\alpha \in \mathcal{L}} \mathcal{H}_\alpha'.$$

**Proof:**
Since the representation on each edge is non-trivial, by definition of the inner product, it is easy to see that $\mathcal{H}_\alpha'$ and $\mathcal{H}_{\alpha'}$ are mutual orthogonal if one of the graphs $\alpha$ and $\alpha'$ has at least an edge $e$ outside the other graph due to

$$\int_{\mathcal{A}_e} \pi^{j}_{m,n} d\mu_e = \int_{\mathcal{A}_e} 1 \cdot \pi^{j}_{m,n} d\mu_e = 0$$

for any $j \neq 0$. Now consider the case of the spurious vertex. An edge $e$ with $j$-representation in a graph is assigned the Hilbert space $\mathcal{H}_e^j \otimes \mathcal{H}_e^{j*}$. Inserting a vertex $v$ into the edge, one obtains two edges $e_1$ and $e_2$ split by $v$ both with $j$-representations, which belong to a different graph. By the decomposition of the corresponding Hilbert space,

$$\mathcal{H}_j^{e_1} \otimes \mathcal{H}_j^{e_1*} \otimes \mathcal{H}_j^{e_2} \otimes \mathcal{H}_j^{e_2*} = \mathcal{H}_j^{e_1} \otimes (\oplus_{l=0...2j} \mathcal{H}_j^l) \otimes \mathcal{H}_j^{e_2*},$$

the subspace for all $l \neq 0$ are orthogonal to the space $\mathcal{H}_j^l \otimes \mathcal{H}_j^{e_*}$, while the subspace for $l = 0$ coincides with $\mathcal{H}_j^0 \otimes \mathcal{H}_j^{e_*}$ since $\mathcal{H}_j^0 = \mathbb{C}$ and $\mathcal{H}_j^l = \mathcal{A}(e) = \mathcal{A}(e_1)\mathcal{A}(e_2)$.

Since there are uncountable many graphs on $\Sigma$, the kinematical Hilbert $\mathcal{H}_{\text{kin}}$ is non-separable. We denote the spin-network basis in $\mathcal{H}_{\text{kin}}$, by $\Pi_s$, $s = (\alpha, j, m, n)$, i.e.,

$$\Pi_s := \prod_{e \in \alpha} \sqrt{\frac{2j_e}{2j_e + 1}} \pi^{j_e}_{m_e n_e},$$

which form a system of complete orthonormal basis with the relation

$$\langle \Pi_s | \Pi_{s'} \rangle_{\text{kin}} = \delta_{ss'}.$$ This basis plays an important role in the following discussion about quantum geometry and quantum spin dynamics.

---

\(^5\)A vertex $v$ is spurious if it is bivalent and $e \circ e'$ is itself analytic edge with $e$, $e'$ meeting at $v$. 

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The spin network basis can be used to construct the so-called spin network representation of loop quantum gravity.

**Definition 4.4.1:** The spin network representation is a vector space $\tilde{H}$ of complex valued functions

$$\tilde{\Psi} : S \to \mathbb{C}; \ s \mapsto \tilde{\Psi}(s),$$

where $S$ is the set of the labels $s$ for the spin network states. $\tilde{H}$ is equipped with the scalar product

$$<\tilde{\Psi}, \tilde{\Psi}'> := \sum_{s \in S} \tilde{\Psi}(s)\tilde{\Psi}'(s)$$

between square summable functions.

The relation between the Hilbert spaces $\tilde{H}$ and $\mathcal{H}_{kin}$ is clarified by the following proposition [122].

**Proposition 4.3:**

The spin network transformation

$$T : \mathcal{H}_{kin} \to \tilde{H}; \ \Psi \mapsto \tilde{\Psi}(s) := \langle \Pi_s, \Psi \rangle_{kin}$$

is a unitary transformation with inverse

$$T^{-1}\Psi = \sum_{s \in S} \tilde{\Psi}(s)\Pi_s.$$

Thus the connection representation and the spin network representation are "Fourier transforms" of each other, where the role of the kernel of the transform is played by the spin network basis. Note that, in the gauge invariant Hilbert space of loop quantum gravity (see section 5.1), the Fourier transform with respect to the gauge invariant spin network basis is the so-called loop transform, which leads to the unitary equivalent loop representation of the theory [99] [103].

### 4.5 Holonomy-Flux Algebra and Quantum Operators

The central aim of quantum kinematics for loop quantum gravity is looking for a proper representation of quantum algebra of elementary observables. In the classical theory, the basic dynamic variables are $su(2)$-valued connection field $A_i^a$ and densitized triad field $\tilde{P}_i^a$ on $\Sigma$. However, these two basic variables are not in the algebra of elementary classical observables which will be represented in the quantum theory, whence they do not have direct quantum analogs in loop quantum gravity. The elementary classical observables in our representation theory are the complex valued functions (cylindrical functions) $f_c$ of holonomies
$A(e)$ along paths $e$ in $\Sigma$, and fluxes $P_\iota(S)$ of triad field across 2-surfaces $S$, which is defined as

$$P_\iota(S) := \int_S n_{abc} \bar{F}_\iota^c.$$  

In the simplest case where a single edge $e$ intersects a 2-surface $S$ at a point $p$, one can calculate the Poisson bracket between two functions $f(A(e))$ and $P_\iota(S)$ on classical phase space $\mathcal{M}$ as [122],

$$\{P_\iota(S), f(A(e))\} = \left[ \frac{\partial}{\partial A(e)_{mn}} f(A(e)) \right] \cdot \{P_\iota(S), A(e)_{mn}\}$$

where $\{A(e)_{mn}\}_{m,n=1,2}$ is the matrix elements of $A(e) \in SU(2)$ and

$$\kappa(S, e) = \begin{cases} 0, & \text{if } e \cap S = \emptyset, \text{or } e \text{ lies in } S; \\ 1, & \text{if } e \text{ lies above } S \text{ and } e \cap S = p; \\ -1, & \text{if } e \text{ lies below } S \text{ and } e \cap S = p. \end{cases}$$

Since the surface $S$ is oriented with normal $n_a$, "above" means $n_a(\partial/\partial t)^a|_p > 0$, and "below" means $n_a(\partial/\partial t)^a|_p < 0$, where $(\partial/\partial t)^a|_p$ is the tangent vector of $e$ at $p$. Then as one might expect, each flux $P_\iota(S)$ is associated with a flux vector field $Y_\iota(S)$ on the quantum configuration space $\mathfrak{A}$, algebraically introduced by the cylindrically consistent action on cylindrical functions $\psi_\alpha \in \mathcal{H}_{kin}$ as:

$$Y_\iota(S) \circ \psi_\alpha(\{A(e)\}_{e \in E(\alpha)}) = \{P_\iota(S), \psi_\alpha\}(\{A(e)\}_{e \in E(\alpha)}),$$

where $E(\alpha)$ is the collection of all edges of the graph $\alpha$. The corresponding momentum operator associated with $S$ is defined by

$$\hat{P}_\iota(S) := i\hbar Y_\iota(S) = i\hbar \{P_\iota(S), \cdot \},$$

which is essentially self-adjoint on $\mathcal{H}_{kin}$ [122]. Its action on cylindrical functions can be expressed explicitly as

$$\hat{P}_\iota(S) \psi_\alpha(\{A(e)\}_{e \in E(\alpha)}) = \frac{\hbar}{2} \sum_{v \in V(\alpha)} \left[ \sum_{e \atop \text{at } v} \kappa(S, e) \hat{j}^{(v,e)}_\iota \right] \psi_\alpha(\{A(e)\}_{e \in E(\alpha)})$$

where $V(\alpha)$ is the collection of all vertices of $\alpha$, and

\begin{align*}
\hat{j}^{(S,v)}_{i(u)} & = \hat{j}^{(v,e_1)}_i + \ldots + \hat{j}^{(v,e_n)}_i, \\
\hat{j}^{(S,v)}_{i(d)} & = \hat{j}^{(v,e_{n+1})}_i + \ldots + \hat{j}^{(v,e_{n+d})}_i,
\end{align*} \tag{28}
for the edges \(e_1, \ldots, e_u\) lying above \(S\) and \(e_{u+1}, \ldots, e_{u+d}\) lying below \(S\). It is obvious to construct configuration operators by cylindrical functions on \(A\) as:

\[
\hat{f}_\beta \psi_\alpha(\{A(e)\}_{e \in E(\alpha)}) := \hat{f}_\beta(\{A(e)\}_{e \in E(\beta)}) \psi_\alpha(\{A(e)\}_{e \in E(\alpha)}).
\]

Note that \(\hat{f}_\beta\) may change the graph, i.e., \(\hat{f}_\beta : \text{Cyl}_\alpha \to \text{Cyl}_{\alpha'}\), where \(\text{Cyl}_\alpha\) denotes the collection of cylindrical functions \(f\) such that \(f = f_\alpha\) for graph \(\alpha\). So far, the elementary operators of quantum kinematics have been well defined on \(\mathcal{H}_{\text{kin}}\). One can calculate the elementary canonical commutation relations between these operators as:

\[
\begin{align*}
[\hat{f}_e(A(e)), \hat{f}_{e'}(A'(e'))] &= 0, \\
[\hat{P}_i(S), \hat{f}_e(A(e))] &= i\hbar \left[ \frac{\partial}{\partial A(e)_{mn}} f_e(A(e)) \cdot \frac{\kappa(S, e)}{2} \right] \left\{ \begin{array}{ll}
\sum_k A(e)_{mk}(\tau_i)_{kn} & \text{if } p = s(e), \\
-\sum_k (\tau_i)_{mk} A(e)_{kn} & \text{if } p = t(e),
\end{array} \right. \\
[\hat{P}_i(S), \hat{P}_j(S')] &= i\hbar \left[ \frac{\kappa(S', e)}{2} \epsilon_{ij} \hat{P}_k(S) \right] f_e(A(e)),
\end{align*}
\]

where we assume the simplest case of one edge graphs. From the commutation relations, one can see that the commutators between momentum operators do not necessarily vanish if \(S \cap S' \neq \emptyset\). This unusual property reflects the non-commutativity of quantum Riemannian structures \([20]\). We conclude that the quantum algebra of elementary observables (holonomy-flux algebra) has been well represented on \(\mathcal{H}_{\text{kin}}\) background-independently. So the construction of quantum kinematics is finished. Several remarks on the quantum kinematics are listed below.

- **Algebraic Quantum Field Theory Viewpoint and GNS-Construction**

  All prior constructions bear analogy with constructive quantum field theory. This item performs the background-independent construction of algebraic quantum field theory for loop quantum gravity. Firstly we construct the algebra of classical observables. Taking account of the future quantum analogs, we define the algebra of classical observables \(\mathcal{P}\) as the Poisson \(\ast\)-subalgebra generated by the functions of holonomies (cylindrical functions) and the fluxes of triad fields on some 2-surface by Definition 2.2. While, one can equivalently define the classical algebra in analogy with geometric quantization in finite dimensional phase space case by the so-called classical Ashtekar-Corichi-Zapata holonomy-flux \(\ast\)-algebra as the following \([80]\).

**Definition 4.5.1**

The classical Ashtekar-Corichi-Zapata holonomy-flux \(\ast\)-algebra is defined to be a vector space \(\mathcal{P}_{\text{ACZ}} := \text{Cyl}(\overline{A}) \times \mathcal{V}^C(\overline{A})\), where \(\mathcal{V}^C(\overline{A})\) is the vector space of cylindrically consistent vector fields spanned by the vector fields.
ψY_i(S) and their commutators, here ψ are cylindrical functions on \( \mathcal{A} \). We equip \( \mathcal{P}_{ACZ} \) with the structure of an \( * \)-Lie algebra by:

1. Lie bracket \( \{ \cdot, \cdot \} : \mathcal{P}_{ACZ} \times \mathcal{P}_{ACZ} \to \mathcal{P}_{ACZ} \) is defined by
   \[
   \{(\psi, Y), (\psi', Y')\} := (Y \circ \psi' - Y' \circ \psi, [Y, Y']),
   \]
   for all \( (\psi, Y), (\psi', Y') \in \mathcal{P}_{ACZ} \) with \( \psi, \psi' \in Cyl(\mathcal{A}) \) and \( Y, Y' \in V^C(\mathcal{A}) \).

2. Involution: \( a \mapsto \bar{a} \forall a \in \mathcal{P}_{ACZ} \) is defined by complex conjugate of cylindrical functions and vector fields, i.e., \( \bar{a} := (\bar{\psi}, \bar{Y}) \forall a = (\psi, Y) \in \mathcal{P}_{ACZ} \), where \( Y \circ \psi := Y \circ \psi \).

3. \( \mathcal{P}_{ACZ} \) admits a natural action of \( Cyl(\mathcal{A}) \) by
   \[
   \psi' \circ (\psi, Y) := (\psi' \psi, \psi' Y),
   \]
   which gives \( \mathcal{P}_{ACZ} \) a module structure.

The classical Ashtekar-Corichi-Zapata holonomy-flux \( * \)-algebra serves as an elementary algebra in our dynamic system of gauge field. Then one can construct the quantum algebra of elementary observables from \( \mathcal{P}_{ACZ} \) in analogy with Definition 2.3.

**Definition 4.5.2**

The universal enveloping algebra \( U(\mathcal{P}_{ACZ}) \) of the classical \( * \)-algebra is defined by the formal direct sum of finite sequences of classical observables \( (a_1, ..., a_n) \) with \( a_k \in \mathcal{P}_{ACZ} \), where the operations of multiplication and involution are defined as

\[
(a_1, ..., a_n) \cdot (a'_1, ..., a'_m) := (a_1, ..., a_n, a'_1, ..., a'_m),
\]

\[
(a_1, ..., a_n)^* := (\bar{a}_n, ..., \bar{a}_1).
\]

A 2-sided ideal \( Z \) can be generated by the following elements,

\[
(a + a') - (a) - (a'), \quad (za) - z(a), \quad [(a), (a')] - i\hbar \{a, a'\},
\]

\[
\frac{1}{2}((\psi, 0), a) + \frac{1}{2}(a, (\psi, 0)) - (\psi \circ a),
\]

where the canonical commutation bracket is defined by

\[
[(a), (a')] := (a) \cdot (a') - (a') \cdot (a).
\]

Note that the ideal \( Z \) is preserved by the involution \( * \).

The quantum holonomy-flux \( * \)-algebra is defined by the quotient \( * \)-algebra \( \mathcal{A} = U(\mathcal{P}_{ACZ})/Z \), which contains the unital element \( 1 := ((1, 0)) \). Note that a sup-norm has been defined by Eq. (21) for the Abelian sub-\( * \)-algebra.
For simplicity, we denote the one element sequences \( (\psi, 0) \) and \((0, Y)) \forall \psi \in Cyl(\mathcal{A}), Y \in \mathcal{V}C(\mathcal{A}) \) in \( A \) by \((\psi)\) and \((Y)\) respectively. In particular, for all cylindrical functions \((\psi)\) and flux vector fields \((Y(S))\),

\[
(\psi)^* = (\bar{\psi}) \quad \text{and} \quad (Y(S))^* = (Y(S)).
\]

Note that every element of the algebra \( A \) is a finite linear combination of elements of the form

\[
(\psi),
(\psi_1) \cdot (Y_{11}(S_{11})),
(\psi_2) \cdot (Y_{21}(S_{21})) \cdot (Y_{22}(S_{22})),
...
(\psi_k) \cdot (Y_{k1}(S_{k1})) \cdot (Y_{k2}(S_{k2})) \cdot ... \cdot (Y_{kk}(S_{kk})),
...
\]

Moreover, given a cylindrical function \( \psi \) and a flux vector field \( Y_i(S) \), one has the relation from the commutation relation:

\[
(Y_i(S) \cdot (\psi)) = i\hbar (Y_i(S) \circ \psi) + (\psi) \cdot (Y_i(S)).
\]

Then the kinematical Hilbert space \( H_{kin} \) can be obtained properly via the GNS-construction for unital *-algebra \( A \) in the same way as in Definition 2.4. By GNS-construction, a positive linear functional, i.e. a state \( \omega \), on \( A \) defines a cyclic representation \((H_\omega, \pi_\omega, \Omega_\omega)\) for \( A \). In our case of quantum holonomy-flux *-algebra, the state with both Yang-Mills gauge invariance and diffeomorphism invariance is defined for any \( \psi = [\psi_\alpha] \in Cyl(\mathcal{A}) \) and non-vanishing flux vector field \( Y_i(S) \in \mathcal{V}C(\mathcal{A}) \) as [33]:

\[
\omega\left((\psi)\right) := \int_{SU(2)^N} d\mu_H(A(e_1))...d\mu_H(A(e_N))\psi_\alpha(A(e_1), ..., A(e_N)),
\]

\[
\omega\left(a \cdot (Y_i(S))\right) := 0, \quad \forall a \in \mathcal{A},
\]

where we assume that \( \alpha \) contains \( N \) edges. This \( \omega \) is called Ashtekar-Isham-Lewandowski state. The null space \( N_\omega \in \mathcal{A} \) with respect to \( \omega \) is defined as \( N_\omega := \{ a \in \mathcal{A} | \omega(a^* \cdot a) = 0 \} \), which is a left ideal. Then a quotient map can be defined as:

\[
[.] : \mathcal{A} \rightarrow \mathcal{A}/N_\omega;
\]

\[
a \mapsto [a] := \{ a + b | b \in N_\omega \}.
\]

The GNS-representation for \( \mathcal{A} \) with respect to \( \omega \) is a representation map:

\[
\pi_\omega : \mathcal{A} \rightarrow \mathcal{L}(H_\omega) \quad \text{such that} \quad \pi_\omega(a \cdot b) = \pi_\omega(a)\pi_\omega(b), \quad \text{where} \quad H_\omega := \langle \mathcal{A}/N_\omega \rangle =
\]
\[ \langle \text{Cyl}(\mathcal{A}) \rangle = \mathcal{H}_{\text{kin}} \] by straightforward verification and the \( \langle \cdot \rangle \) denotes the completion with respect to the natural equipped inner product on \( \mathcal{H}_{\omega} \),

\[ \langle [a]|b \rangle_{\mathcal{H}_{\omega}} := \omega(a^* \cdot b), \]

which is equivalent to Eq. (25). The representation map \( \pi_{\omega} \) is defined by

\[ \pi_{\omega}(a)|b \rangle := [a \cdot b], \quad \forall a \in \mathcal{A}, \text{ and } |b \rangle \in \mathcal{H}_{\omega}. \]

Note that \( \pi_{\omega}(a) \) is an unbounded operator in general. It is easy to verify that

\[ \pi_{\omega}((Y_i(S)))(\psi) = i\hbar[(Y_i(S) \circ \psi)] \]

via Eq. (32). Hence \( \pi_{\omega}((Y_i(S)))(\psi) \) is identical with \( \hat{P}_i(S) \) on \( \mathcal{H}_{\text{kin}} \). Moreover, since \( \Omega_{\omega} := [1] \) is a cyclic vector in \( \mathcal{H}_{\omega} \), the positive linear functional with which we begin can be expressed as

\[ \omega(a) = \langle \Omega_{\omega} | \pi_{\omega}(a) \Omega_{\omega} \rangle_{\mathcal{H}_{\omega}}. \]

Thus the Ashtekar-Isham-Lewandowski state \( \omega \) on \( \mathcal{A} \) is equivalent to a cyclic representation \( (\mathcal{H}_{\omega}, \pi_{\omega}, \Omega_{\omega}) \) for \( \mathcal{A} \), which is the Ashtekar-Isham-Lewandowski representation for quantum holonomy-flux \( * \)-algebra of background independent gauge field theory. One thus obtains the kinematical representation of loop quantum gravity via the construction of algebraic quantum field theory. It is important to note that the Ashtekar-Isham-Lewandowski state is the unique state on quantum holonomy-flux \( * \)-algebra \( \mathcal{A} \) invariant under Yang-Mills gauge transformations and spatial diffeomorphisms, which are both automorphisms \( \alpha_g \) and \( \alpha_\phi \) on \( \mathcal{A} \) such that \( \alpha_g \circ \omega = \omega \) and \( \alpha_\phi \circ \omega = \omega \) (Theorem 1.2.1 in [80]). So these gauge transformations are represented as unitary transformations on \( \mathcal{H}_{\text{kin}} \), while the cyclic vector \( \Omega_{\omega} \) is the unique state in \( \mathcal{H}_{\text{kin}} \) invariant under the gauge transformations.

• Kinematical Vacuum and Polymer Representation

The cyclic vector \( \Omega_{\omega} = 1 \) has the physical meaning of a kinematical vacuum state in Hilbert space \( \mathcal{H}_{\text{kin}} \) due to its following characters. Firstly, \( \Omega_{\omega} \) is the unique state in \( \mathcal{H}_{\text{kin}} \) with maximal gauge symmetry under Yang-Mills gauge transformations and spatial diffeomorphisms. Secondly, \( \Omega_{\omega} = 1 \) means that there is completely no geometry on the 3-manifold \( \Sigma \), since the elementary operators \( \hat{A}(e)_{mn} \) and \( \hat{P}_i(S) \), corresponding to connections and triad fields in classical sense, have vanishing expectation values on \( \Omega_{\omega} \). Hence it implies that the vacuum of quantum geometry is no geometry but a bare manifold. While the cyclic vector serves as a ground state in the kinematical Hilbert space, the low excited states (cylindrical functions) are only excited on graphs with finite edges. There is only 1-dimensional geometry living on these graphs, so the quantum geometry is polymer-like object. When one increase the amount of edges and
graphs such that the graphs are densely distributed in $\Sigma$, the quantum state is highly excited and the quantum geometry can weave the classical smooth one. Because of this picture, the quantum kinematical representation which we obtain is also called polymer representation for background-independent quantum geometry.

- **Quantum Geometric Operator and Quantum Riemannian Geometry**

The well-established quantum kinematics of loop quantum gravity is now in the status just like the Riemannian geometry before the appearance of general relativity and Einstein’s equation, which gives general relativity mathematical foundation and offers living place to the Einstein equation. Instead of classical geometric quantities, such as scalar, vector, tensor etc., the quantities in quantum geometry are operators on the kinematical Hilbert space $H_{kin}$, and their spectrum serve as the possible values of the quantities in measurements. So far, the kinematical quantum geometric operators constructed properly in loop quantum gravity include area operators [106], volume operators [106], length operator [117], $\tilde{Q}$ operator [84] etc. Recently there are discussions on the consistency check to different regularization approaches for volume operators with the triad operator [65]. We thus will only introduce the volume operator defined by Ashtekar and Lewandowski [19], which is shown to be correct in the consistency check.

Firstly, we define the area operator with respect to a 2-surface $S$ by the elementary operators. Given a closed 2-surface or a surface $S$ with boundary, we can divide it into a large number $N$ of small area cells $S_I$. By directly analog to classical expression of an area, we set the area of the 2-surface to be the limit of the Riemannian sum

$$A_S := \lim_{N \to \infty} [A_S]_N = \lim_{N \to \infty} \kappa \gamma \sum_{I=1}^{N} \sqrt{\hat{P}_i(S_I) \hat{P}_j(S_I) \delta^{ij}}.$$ 

Then one can unambiguously obtain a quantum operator of area from the momentum operators $\hat{P}_i(S_I)$ associated with a graph $\alpha$, the action of the area operator on $\psi = [\psi_\alpha]$ is defined in the limit by requiring that each area cell contains at most only one intersecting point $v$ of $\alpha$ and $S$ as

$$\hat{A}_S \circ \psi_\alpha := \lim_{N \to \infty} [\hat{A}_S]_N \circ \psi_\alpha = \lim_{N \to \infty} \kappa \gamma \sum_{I=1}^{N} \sqrt{\hat{P}_i(S_I) \hat{P}_j(S_I) \delta^{ij}} \circ \psi_\alpha.$$

The regulator $N$ is easy to be removed, since the result of the operation of the operator $\hat{P}_i(S_I)$ does not change when $S_I$ shrinks to a point. Since the refinement of the partition does not affect the result of action of $[\hat{A}_S]_N$ on $\psi$, the limit area operator $\hat{A}_S$, which is shown to be self-adjoint [18], is
well defined on $H_{\text{kin}}$ and takes the explicit expression as:

$$\hat{A}_S \psi_\alpha = 4\pi\gamma \ell_p^2 \sum_{v \in V(\alpha \cap S)} \sqrt{(\hat{J}^{(S,v)}_{i(u)} - \hat{J}^{(S,v)}_{i(d)}) (\hat{J}^{(S,v)}_{j(u)} - \hat{J}^{(S,v)}_{j(d)})} \delta^{ij} \psi_\alpha,$$

where $\hat{J}^{(S,v)}_{i(u)}$ and $\hat{J}^{(S,v)}_{i(d)}$ have been defined in Eq. (28). It turns out that the spin network basis in $H_{\text{kin}}$ diagonalizes $\hat{A}_S$ with eigenvalues given by finite sums,

$$a_S = 4\pi\gamma \ell_p^2 \sum_I \sqrt{2j^{(u)}(j^{(u)} + 1) + 2j^{(d)}(j^{(d)} + 1) - j^{(u+d)}(j^{(u+d)} + 1)},$$

where $j^{(u)}, j^{(d)}$ and $j^{(u+d)}$ are arbitrary half-integers subject to the standard condition

$$j^{(u+d)} \in \{ |j^{(u)} - j^{(d)}|, |j^{(u)} - j^{(d)}| + 1, ..., j^{(u)} + j^{(d)} \}.$$

(34)

Hence the spectrum of the area operator is fundamentally pure discrete, while its continuum approximation becomes excellent exponentially rapidly for large eigenvalues. However, in fundamental level, the area is discrete and so is the quantum geometry. One has seen that the eigenvalue of $\hat{A}_S$ does not vanish even in the case where only one edge intersects the surface at a single point, whence the quantum geometry is distributional.

Secondly, we introduce the volume operator. Given a region $R$ with a fixed coordinate system $\{x^a\}_{a=1,2,3}$ in it, one can introduce a partition of $R$ in the following way. Divide $R$ into small volume cells $C$ such that, each cell $C$ is a cube with coordinate volume less than $\epsilon$ and two different cells at most share the points on their boundaries. In each cell $C$, we introduce three 2-surfaces $s = (S^1, S^2, S^3)$ such that $x^a$ is constant on the surface $S^a$. We denote this partition $(C, s)$ as $P_\epsilon$. Then the volume of the region $R$ can be expressed classically as

$$V_R^\epsilon = \lim_{\epsilon \to 0} \sum_C \sqrt{|q_C, s|},$$

where

$$q_C, s = \frac{(\kappa \gamma)^3}{3!} \epsilon^{ijk} \eta_{abc} P_i(S^a) P_j(S^b) P_k(S^c).$$

This motivates us to define the volume operator by naively changing $P_i(S^a)$ to $\hat{P}_i(S^a)$:

$$\hat{V}_R^\epsilon = \lim_{\epsilon \to 0} \sum_C \sqrt{|\hat{q}_C, s|},$$

$$\hat{q}_C, s = \frac{(\kappa \gamma)^3}{3!} \epsilon^{ijk} \eta_{abc} \hat{P}_i(S^a) \hat{P}_j(S^b) \hat{P}_k(S^c).$$

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Note that, given any cylindrical function \( \psi = [\psi_\alpha] \), we require the vertices of the graph \( \alpha \) to be at the intersecting points of the triples of 2-surfaces \( s = (S^1, S^2, S^3) \) in corresponding cells. Thus the limit operator will trivially exist due to the same reason in the case of the area operator. However, the volume operator defined here depends on the choice of orientations for the triples of surfaces \( s = (S^1, S^2, S^3) \), or essentially, the choice of coordinate systems. So it is not uniquely defined. Since, for all choice of \( s = (S^1, S^2, S^3) \), the resulting operators have correct semi-classical limit, one settles up the problem by averaging different operators labelled by different \( s \) [19]. The process of averaging removes the freedom in defining the volume operator up to an overall constant \( \kappa_0 \). The resulting self-adjoint operator acts on any cylindrical function \( \psi_\alpha \) as

\[
\hat{V}_R \circ \psi_\alpha = \kappa_0 \sum_{v \in V(\alpha)} \sqrt{|\hat{q}_{v,\alpha}|} \circ \psi_\alpha,
\]

where

\[
\hat{q}_{v,\alpha} = (8\pi\gamma\ell_p^2)^{\frac{3}{4}} \sum_{e,e',e''} \epsilon^{ijk} \epsilon(e,e',e'') \hat{J}^{(v,e)}_i \hat{J}^{(v,e')}_j \hat{J}^{(v,e'')}_k,
\]

here \( \epsilon(e,e',e'') \equiv sgn(\epsilon_{abc} \hat{e}^a \hat{e}^b \hat{e}^c) |_v \) with \( \hat{e}^a \) as the tangent vector of edge \( e \) and \( \epsilon_{abc} \) as the orientation of \( \Sigma \). The only unsatisfactory point in the present volume operator is the choice ambiguity of \( \kappa_0 \). However, fortunately, the most recent discussion shows that the overall undetermined constant \( \kappa_0 \) can be fixed to be \( \sqrt{6} \) by the consistency check between the volume operator and the triad operator [65][66].

5 Quantum Gaussian Constraint and Quantum Diffeomorphism Constraint

After constructing the kinematical Hilbert space \( \mathcal{H}_{\text{kin}} \) of loop quantum gravity, one should implement the constraints on it to obtain physical Hilbert space. Recall the constraints [11] in generalized Palatini Hamiltonian for general relativity and the Poission algebra [11] among them. The subalgebra generated by the Gaussian constraints \( \mathcal{G}(\Lambda) \) forms a Lie algebra and a 2-sided ideal in the constraints algebra. So in this section, we firstly solve the Gaussian constraints independently of the other two kinds of constraints and find the solution space \( \mathcal{H}^G \), which is constituted by Yang-Mills gauge invariant quantum states. Secondly, although the subalgebra generated by the diffeomorphism constraints is not an ideal in the constraints algebra, we still would like to solve them independently of the scalar constraints for the technical convenience. At the end of this section, we will obtain the Hilbert space \( \mathcal{H}^G_{\text{Diff}} \) free of Gaussian constraints and diffeomorphism constraints.
5.1 Implementation of Quantum Gaussian Constraint

Recall the classical expression of Gaussian constraints:

\[ G(\Lambda) = \int_{\Sigma} d^3x \Lambda^i D_a \tilde{P}_i = -\int_{\Sigma} d^3x \tilde{P}_i D_a \Lambda^i \equiv -P(DA), \]

where \( D_a \Lambda^i = \partial_a \Lambda^i + \epsilon^i_{jk} A^j_a \Lambda^k \). As the situation of triad flux, the Gaussian constraints can be defined as cylindrically consistent vector fields \( Y_{DA} \) on \( \mathcal{A} \), which act on any cylindrical function \( f = [f_\alpha] \) by

\[ Y_{DA} \circ f(\{A(e)\}_{e \in E(\alpha)}) := \{-P(DA), f(\{A(e)\}_{e \in E(\alpha)})\}. \]

Then the Gaussian constraint operator can be defined in analogy with the momentum operator, which acts on \( f = [f_\alpha] \) as:

\[ \hat{G}(\Lambda) f(\{A(e)\}_{e \in E(\alpha)}) := i\hbar Y_{DA} \circ f(\{A(e)\}_{e \in E(\alpha)}) = \hbar \sum_{v \in V(\alpha)} [\Lambda^i(v) \tilde{J}_i] f(\{A(e)\}_{e \in E(\alpha)}), \]

which is the generator of Yang-Mills gauge transformations on \( \mathcal{H}_\alpha \). The kernel of the operator is easily obtained in terms of spin-network decomposition, which is the Yang-Mills gauge invariant Hilbert space:

\[ \mathcal{H}^G = \bigoplus_{\alpha,j} \mathcal{H}_{\alpha,j,l=0}. \]

One then naturally gets the gauge invariant spin network basis in \( \mathcal{H}_G \) [107][22][27]. All Yang-Mills gauge invariant operators are well defined on \( \mathcal{H}_G \). However, the condition of acting on gauge invariant states often changes the structure of the spectrum of quantum geometric operators. For the area operator, the spectrum depends on certain global properties of the surface \( S \) (see [17][18] for details). For the volume operators, non-zero spectrum arises from at least 4-valent vertices.

5.2 Implementation of Quantum Diffeomorphism Constraint

Unlike the strategy in solving Gaussian constraint, one cannot define an operator for quantum diffeomorphism constraint as the infinitesimal generator of finite diffeomorphism transformations (unitary operators since the measure is diffeomorphism invariant) represented on \( \mathcal{H}_{kin} \). The representation of finite diffeomorphisms is a family of unitary operators \( \hat{U}_\varphi \) acting on cylindrical functions \( \psi_\alpha \) by

\[ \hat{U}_\varphi \psi_\alpha := \psi_{\varphi \alpha}, \quad (35) \]

for any spatial diffeomorphism \( \varphi \) on \( \Sigma \). An 1-parameter subgroup \( \varphi_t \) in the group of spatial diffeomorphisms is then represented as an 1-parameter unitary group \( \hat{U}_{\varphi_t} \) on \( \mathcal{H}_{kin} \). However, \( \hat{U}_{\varphi_t} \) is not weakly continuous, since the subspaces
\( H'_\alpha \) and \( H'_{\varphi \alpha} \) are orthogonal to each other no matter how small the parameter \( t \) is. So one always has

\[
| \langle \psi_\alpha | \hat{U}_\varphi | \psi_\alpha \rangle_{\text{kin}} - | \langle \psi_\alpha | \psi_\alpha \rangle_{\text{kin}} | = < \psi_\alpha | \psi_\alpha >_{\text{kin}} \neq 0,
\]

(36) even in the limit when \( t \) goes to zero. Therefore, the infinitesimal generator of \( \hat{U}_\varphi \) does not exist. In the strategy to solve the diffeomorphism constraint, due to the Lie algebra structure of diffeomorphism constraints subalgebra, the so-called group averaging technique is employed. We now outline the procedure. Firstly, given a colored graph (a graph and a cylindrical function on it), one can define the group of graph symmetries \( GS_\alpha \) by

\[
GS_\alpha := \text{Diff}(\alpha) / TDiff(\alpha),
\]

where \( \text{Diff}(\alpha) \) is the group of all diffeomorphisms preserving the colored \( \alpha \), and \( TDiff(\alpha) \) is the group of diffeomorphisms which trivially acts on \( \alpha \). We define a projection map by averaging with respect to \( GS_\alpha \) to obtain the subspace in \( H'_\alpha \) which is invariant under the transformation of \( GS_\alpha \):

\[
\hat{P}_{\text{Diff},\alpha} \psi_\alpha := \frac{1}{n_\alpha} \sum_{\varphi \in GS_\alpha} \hat{U}_\varphi \psi_\alpha,
\]

for all cylindrical functions \( \psi_\alpha \in H'_\alpha \), where \( n_\alpha \) is the number of the finite elements of \( GS_\alpha \). Secondly, we average with respect to all remaining diffeomorphisms which move the graph \( \alpha \). For each cylindrical function \( \psi_\alpha \in H'_\alpha \), there is an element \( \eta(\psi_\alpha) \) associated to it in the algebraic dual space \( \text{Cyl}^* \) of \( \text{Cyl}(\mathcal{A}) \), which acts on any cylindrical function \( \phi_\beta \) as:

\[
\eta(\psi_\alpha)[\phi_\beta] := \sum_{\varphi \in Dif f(\Sigma) / Dif f_\alpha} \langle \hat{U}_\varphi \hat{P}_{\text{Diff},\alpha} \psi_\alpha | \phi_\beta \rangle_{\text{kin}},
\]

It is well defined since, for any given graph \( \beta \), only finite terms are non-zero in the summation. It is easy to verify that \( \eta(\psi_\alpha) \) is invariant under the group action of \( Dif f(\Sigma) \), since

\[
\eta(\psi_\alpha)[\hat{U}_\varphi \phi_\beta] = \eta(\psi_\alpha)[\phi_\beta].
\]

Thus we have defined a rigging map \( \eta : \text{Cyl}(\mathcal{A}) \rightarrow Cyl^*_{\text{Diff}} \), which maps every cylindrical function to a diffeomorphism invariant one. So \( Cyl^*_{\text{Diff}} \) is spanned by rigged spin-network functions \( \{ \eta(\Pi[s]) \}_{[s]=(\alpha),j,m,n} \) associated with diffeomorphism classes \([\alpha]\) of graphs \( \alpha \). Moreover a Hermitian inner product can be defined on \( Cyl^*_{\text{Diff}} \) by the natural action of the algebraic functional:

\[
< \eta(\psi_\alpha) | \eta(\phi_\beta) >_{\text{Diff}} := \eta(\psi_\alpha)[\phi_\beta].
\]

The diffeomorphism invariant Hilbert space \( \mathcal{H}_{\text{Diff}} \) is defined by the completion of \( Cyl^*_{\text{Diff}} \) with respect to the above inner product \(< | >_{\text{Diff}} \). The diffeomorphism invariant spin-network functions \( \{ \eta(\Pi[s]) \}_{[s]=(\alpha),j,m,n} \) form an
orthonormal basis in $\mathcal{H}_{\text{Diff}}$. Finally, we can obtain the general solutions invariant under both Yang-Mills gauge transformations and spatial diffeomorphisms in the same way by averaging the cylindrical functions in $\mathcal{H}^G$. The resulting Hilbert space is denoted by $\mathcal{H}^G_{\text{Diff}}$.

In general relativity, the problem of observables is a subtle issue due to the diffeomorphism invariance [97][100][101]. Now we discuss the operators of diffeomorphism invariant observables on $\mathcal{H}_{\text{Diff}}$. We call an operator $\hat{O} \in L(\mathcal{H}_{\text{kin}})$ a strong observable if and only if $\hat{O}\hat{U}_\varphi^{\dagger} \hat{O}\hat{U}_\varphi = \hat{O}$, $\forall \varphi \in \text{Diff}(\Sigma)$. We call it a weak observable if and only if $\hat{O}$ leaves $\mathcal{H}_{\text{Diff}}$ invariant. Then it is easy to see that a strong observable $\hat{O}$ must be a weak one. One notices that a strong observable $\hat{O}$ can first be defined on $\mathcal{H}_{\text{Diff}}$ by its dual operator $\hat{O}^\dagger$ as $$(\hat{O}^\dagger \Phi_{\text{Diff}})[\psi] := \Phi_{\text{Diff}}[\hat{O}\psi],$$ then one gets $$(\hat{O}^\dagger \Phi_{\text{Diff}})[\hat{U}_\varphi \psi] = \Phi_{\text{Diff}}[\hat{O}\hat{U}_\varphi \psi] = \Phi_{\text{Diff}}[\hat{U}_\varphi^{\dagger} \hat{O}\hat{U}_\varphi \psi] = (\hat{O}^\dagger \Phi_{\text{Diff}})[\psi],$$ for any $\Phi_{\text{Diff}} \in \mathcal{H}_{\text{Diff}}$ and $\psi \in \mathcal{H}_{\text{kin}}$. Hence $\hat{O}^\dagger \Phi_{\text{Diff}}$ is also diffeomorphism invariant. In addition, a strong observable also has the property of $\hat{O}^\dagger \eta(\psi_\alpha) = \eta(\hat{O}^\dagger \psi_\alpha)$ since, $\forall \phi_\beta, \psi_\alpha \in \mathcal{H}_{\text{kin}}$, $$<\psi_\alpha|\hat{O}_{\text{Diff}}^{\dagger} \phi_\beta>_{\text{Diff}} = <\hat{O}^\dagger \eta(\psi_\alpha)|\eta(\phi_\beta)>_{\text{Diff}} = \sum_{\varphi \in \text{Diff}/\text{Diff}} <\hat{U}_\varphi \hat{P}_{\text{Diff},\alpha} \psi_\alpha|\hat{O}\phi_\beta>_{\text{kin}}$$ $$= \frac{1}{n_\alpha} \sum_{\varphi \in \text{Diff}/\text{Diff}, \varphi' \in \text{GS}_\alpha} <\hat{U}_\varphi \hat{U}_{\varphi'} \psi_\alpha|\hat{O}\phi_{\beta}>_{\text{kin}}$$ $$= \frac{1}{n_\alpha} \sum_{\varphi \in \text{Diff}/\text{Diff}, \varphi' \in \text{GS}_\alpha} <\hat{U}_\varphi \hat{U}_{\varphi'} \hat{O}^{\dagger} \psi_\alpha|\phi_{\beta}>_{\text{kin}}$$ $$= <\eta(\hat{O}^{\dagger} \psi_\alpha)|\eta(\phi_\beta)>_{\text{Diff}}.$$ Note that the Hilbert space $\mathcal{H}_{\text{Diff}}$ is still non-separable if one considers the $C^n$ diffeomorphisms with $n > 0$. However, if one enlarges the diffeomorphisms to be $C^0$ homomorphisms (which can be viewed as an extension of the classical concept to the quantum case), the Hilbert space $\mathcal{H}_{\text{Diff}}$ would be separable [17].

6 Quantum Dynamics

In this section, we go to the discussion on the quantum dynamics of loop quantum gravity. One may first consider to construct a Hamiltonian constraint (scalar constraint) operator in $\mathcal{H}_{\text{kin}}$ or $\mathcal{H}_{\text{Diff}}$, then attempt to find the physical Hilbert space $\mathcal{H}_{\text{phys}}$ by solving the quantum Hamiltonian constraint. However, difficulties arise here due to the special role played by the scalar constraints in
the constraint algebra \[ \mathcal{H}_{\text{kin}} \]. Firstly, the scalar constraints do not form a Lie subalgebra. Hence the strategy of group average cannot be used directly on \( \mathcal{H}_{\text{kin}} \) for them. Secondly, modulo the Gauss constraint, there is still a structure function in the Poisson bracket between two scalar constraints:

\[
\{ \mathcal{H}(N), \mathcal{H}(M) \} = -\mathcal{V}((N \partial_b M - M \partial_b N) q^{ab}),
\]

which raises the danger of quantum anomaly in quantization. Moreover, the diffeomorphism constraints do not form an ideal in the quotient constraint algebra modulo the Gaussian constraints. This fact results in that the scalar constraint operator cannot be well defined on \( \mathcal{H}_{\text{Diff}} \), as it does not commute with the diffeomorphism transformations \( \hat{U}_\phi \). Thus the previous construction of \( \mathcal{H}_{\text{Diff}} \) seems not much meaningful for the final construction of \( \mathcal{H}_{\text{phys}} \), for which we are seeking. However, one may still first try to construct a Hamiltonian constraint operator in \( \mathcal{H}_{\text{kin}} \).

### 6.1 Hamiltonian Constraint Operator

The aim in this subsection is to define a quantum operator for the Hamiltonian constraint. Its classical expression reads:

\[
\mathcal{H}(N) := \frac{\kappa \gamma^2}{2} \int_{\Sigma} d^3 x N \tilde{P}_i^a \tilde{P}_j^b \left[ e^{ij} k^a F_{ab}^k - 2(1 + \gamma^2) K_i^a K_i^b \right]
\]

\[
= \mathcal{H}_E(N) - 2(1 + \gamma^2) T(N).
\]

The main idea of the construction is to first express \( \mathcal{H}(N) \) in terms of the combination of Poisson brackets between the variables which have been represented as operators on \( \mathcal{H}_{\text{kin}} \), then replace the Poisson brackets by canonical commutators between the operators. We will use the volume functional for a region \( R \subset \Sigma \) and the extrinsic curvature functional defined by:

\[
K := \kappa \gamma \int_{\Sigma} d^3 x \tilde{P}^a K_a^i.
\]

A key trick here is to consider the following classical identity of the co-triad \( e_i^a(x) \):

\[
e_i^a(x) = \frac{(\kappa \gamma)^2}{2} \eta^{abc} e^{ijk} \frac{1}{\sqrt{\det q}} \tilde{P}^b k_k(x) = \frac{2}{\kappa \gamma} \{ A_i^a(x), V_R \},
\]

where \( x \in R \), and the expression of the extrinsic curvature 1-form \( K_a^i(x) \):

\[
K_a^i(x) = \frac{1}{\kappa \gamma} \{ A_a^i(x), K \}.
\]

Note that \( K \) can be expressed by a Poisson bracket as

\[
K = \gamma^{-2} \{ \mathcal{H}_E(1), V_\Sigma \}.
\]

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Thus one can obtain the equivalent classical expressions of $\mathcal{H}_E(N)$ and $T(N)$ as:

\[
\mathcal{H}_E(N) = \frac{\kappa \gamma^2}{2} \int_{\Sigma} d^3x \frac{\bar{P}_a \bar{P}_b}{\sqrt{\det q}} e^{ij} F_{ab}^k.
\]

\[
T(N) = \frac{\kappa \gamma^2}{2} \int_{\Sigma} d^3x \frac{\bar{P}_a \bar{P}_b}{\sqrt{\det q}} K^i_{[a} K^j_{b]}.
\]

where $A_a = A^i_{a} \tau_i$, $F_{ab} = F^i_{ab} \tau_i$, $Tr$ represents the trace of the Lie algebra matrix, and $R_x \subset \Sigma$ denotes an arbitrary neighborhood of $x \in \Sigma$. In order to quantize the Hamiltonian constraint as a well-defined operator on $\mathcal{H}_{\text{kin}}$, one have to express the classical formula of $\mathcal{H}(N)$ in terms of holonomies $A(\epsilon)$ and other variables with clear quantum analogs. This can be realized by introducing a triangulation $T(\epsilon)$, where the parameter $\epsilon$ describes how fine the triangulation is, and the triangulation will fill out the spatial manifold $\Sigma$ when $\epsilon \to 0$. Given an tetrahedron $\Delta \in T(\epsilon)$, we use $\{e_i(\Delta)\}_{i=1,2,3}$ to denote the three outgoing oriented edges in $\Delta$ with a common beginning point $v(\Delta) = s(e_i(\Delta))$, and use $e_{ij}(\Delta)$ to denote the edge connecting the end points of $e_i(\Delta)$ and $e_j(\Delta)$. Then several loops $\alpha_{ij}(\Delta)$ are formed by $\alpha_{ij}(\Delta) := e_i(\Delta) \circ e_{ij}(\Delta) \circ e_j(\Delta)^{-1}$. Thus we have the identities:

\[
\left\{ \int_{e_i(\Delta)} A_a e^{a}_{i}(\Delta), V_{R_{\epsilon}(\Delta)} \right\} = -A(e_i(\Delta))^{-1} \{ A(e_i(\Delta)), V_{R_{\epsilon}(\Delta)} \} + o(\epsilon),
\]

\[
\left\{ \int_{e_i(\Delta)} A_a e^{a}_{i}(\Delta), K \right\} = -A(e_i(\Delta))^{-1} \{ A(e_i(\Delta)), K \} + o(\epsilon),
\]

\[
\int_{P_{ij}} F_{ab}(x) = A(\alpha_{ij}(\Delta)) + o(\epsilon^2),
\]

where $P_{ij}$ is the plane with boundary $\alpha_{ij}$. Note that the above identities are constructed by taking account of internal gauge invariance of the final formula of Hamiltonian constraint operator. So we have the regularized expression of $\mathcal{H}(N)$ by the Riemannian sum:

\[
\mathcal{H}_E(N) = \frac{2}{3 \kappa^2 \gamma} \sum_{\Delta \in T(\epsilon)} N(v(\Delta)) \epsilon^{ijk} \times \text{Tr}(A(\alpha_{ij}(\Delta)) A(e_k(\Delta))^{-1} \{ A(e_k(\Delta)), V_{R_{\epsilon}(\Delta)} \}),
\]

\[
T^\epsilon(N) = \frac{2}{3 \kappa^4 \gamma^3} \sum_{\Delta \in T(\epsilon)} N(v(\Delta)) \epsilon^{ijk} \times \text{Tr}(A(e_i(\Delta))^{-1} \{ A(e_i(\Delta)), K \} A(e_j(\Delta))^{-1} \{ A(e_j(\Delta)), K \} \times
\]

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\[ A(e_k(\Delta))^{-1}\{A(e_k(\Delta)), V_{Rv(\Delta)}\}) \]
\[ \mathcal{H}'(N) = \mathcal{H}'_E(N) - 2(1 + \gamma^2)T'(N), \]

such that \( \lim_{\epsilon \to 0} \mathcal{H}'(N) = \mathcal{H}(N) \). It is clear that the above regulated formula of \( \mathcal{H}(N) \) is invariant under internal gauge transformations. Since all constituents in the expression have clear quantum analogs, one can quantize the regulated Hamiltonian constraint as an operator on \( \mathcal{H}_{kin} \) (or \( \mathcal{H}^G \)) by replacing them by the corresponding operators and Poisson brackets by canonical commutators, i.e.,

\[ A(e) \mapsto \hat{A}(e), \quad V_R \mapsto \hat{V}_R, \quad \{ , \} \mapsto \frac{[ , ]}{i\hbar}, \]

and \( K \mapsto \hat{K} = \frac{\gamma^{-2}}{i\hbar}[\hat{H}'_E(1), \hat{V}_2] \).

Then remove the regulator by \( \epsilon \to 0 \). It turns out that one can obtain a well-defined limit operator on \( \mathcal{H}_{kin} \) (or \( \mathcal{H}^G \)) with respect to a natural operator topology.

Now we begin to construct the Hamiltonian constraint operator in analogy with the classical expression (38). All we should do is to define the corresponding regulated operators on different \( \mathcal{H}'_{\alpha} \) separately, then remove the regulator \( \epsilon \) so that the limit operator is defined on \( \mathcal{H}_{kin} \) (or \( \mathcal{H}^G \)) cylindrically consistently. Moreover, to ensure that the final operator is diffeomorphism covariant, one can make the triangulation \( T(\epsilon) \) adapted to the graph \( \alpha \) of \( \psi_{\alpha} \in \mathcal{H}'_{\alpha} \) with the following properties.

- The graph \( \alpha \) is embedded in \( T(\epsilon) \) for all \( \epsilon \), so that every vertex \( v \) of \( \alpha \) coincides with a vertex \( v(\Delta) \) in \( T(\epsilon) \).
- For every triple of edges \( (e_1, e_2, e_3) \) of \( \alpha \) such that \( v = s(e_1) = s(e_2) = s(e_3) \), there is a tetrahedra \( \Delta \in T(\epsilon) \) such that \( v = v(\Delta) \) and \( e_i = e_i(\Delta), \forall i = 1, 2, 3 \). We denote such a tetrahedra as \( \Delta^0_{e_1, e_2, e_3} \).
- For each tetrahedra \( \Delta^0_{e_1, e_2, e_3} \) one can construct seven additional tetrahedron \( \Delta^\varphi_{e_1, e_2, e_3}, \varphi = 1, \ldots, 7 \), by backward analytic extensions of \( e_i(\Delta) \) so that \( U_{e_1, e_2, e_3} := \bigcup_{\varphi=0}^7 \Delta^\varphi_{e_1, e_2, e_3} \) is a neighborhood of \( v \).
- The triangulation must be fine enough so that the neighborhoods \( U(v) := \cup_{e_1, e_2, e_3} U_{e_1, e_2, e_3}(v) \) are disjoint for different vertices \( v \) and \( v' \) of \( \alpha \). Thus for any open neighborhood \( U_{\alpha} \) of the graph \( \alpha \), there exists a triangulation \( T(\epsilon) \) such that \( \cup_{v \in V(\alpha)} U(v) \subseteq U_{\alpha} \).
- The distance between a vertex \( v(\Delta) \) and the corresponding edges \( e_{ij}(\Delta) \) is described by the parameter \( \epsilon \). For any two different \( \epsilon \) and \( \epsilon' \), the edges \( e_{ij}(\Delta') \) and \( e_{ij}(\Delta'') \) with respect to one vertex \( v(\Delta) \) are analytically diffeomorphic with each other.
• With the triangulation $T(\epsilon)$, the integral over $\Sigma$ is replaced by the Riemannian sum:

$$\int_{\Sigma} = \int_{U_{\alpha}} + \int_{\Sigma - U_{\alpha}},$$

$$\int_{U_{\alpha}} = \sum_{v \in V(\alpha)} \int_{U(v)} + \int_{U_{\alpha} - \cup_{v \in U(v)}},$$

$$\int_{U(v)} = \frac{1}{C_{n(v)}} \sum_{e_{1},e_{2},e_{3}} \left[ \int_{U_{e_{1},e_{2},e_{3}}(v)} + \int_{U(v) - U_{e_{1},e_{2},e_{3}}(v)} \right],$$

where $n(v)$ is the valence of the vertex $v = s(e_{1}) = s(e_{2}) = s(e_{3})$. One then observes that

$$\int_{U_{e_{1},e_{2},e_{3}}(v)} = 8 \int_{\Delta_{e_{1},e_{2},e_{3}}(v)}$$

in the limit $\epsilon \to 0$.

• The triangulation for the regions $U(v) - U_{e_{1},e_{2},e_{3}}(v)$, $U_{\alpha} - \cup_{v \in V(\alpha)} U(v)$, $\Sigma - U_{\alpha}$, (41)

are arbitrary. These regions do not contribute to the construction of the operator, since the commutator term $[A(e_{i}(\Delta)), V_{R_{e_{i}(\Delta)}}]\psi_{\alpha}$ vanishes for all tetrahedron $\Delta$ in the regions (41).

Thus we find the regulated expression of Hamiltonian constraint operator with respect to the triangulation $T(\epsilon)$ as

$$\hat{H}_{\epsilon}^{E}(N)\psi_{\alpha} = \frac{16}{3i\hbar\kappa^{2}\gamma} \sum_{v \in V(\alpha)} \frac{N(v)}{C_{n(v)}} \sum_{e(\Delta)=v} \epsilon^{ijk} \times$$

$$\text{Tr}(\hat{A}(e_{i}(\Delta)))\hat{A}(e_{k}(\Delta))^{-1}[\hat{A}(e_{k}(\Delta)), \hat{V}_{U_{\epsilon}}])\psi_{\alpha},$$

$$\hat{T}^{\epsilon}(N)\psi_{\alpha} = -\frac{16}{3i\hbar^{3}\kappa^{4}\gamma^{3}} \sum_{v \in V(\alpha)} \frac{N(v)}{C_{n(v)}} \sum_{e(\Delta)=v} \epsilon^{ijk} \times$$

$$\text{Tr}(\hat{A}(e_{i}(\Delta))^{-1}[\hat{A}(e_{i}(\Delta)), \hat{K}^{\epsilon}][\hat{A}(e_{j}(\Delta))^{-1}[\hat{A}(e_{j}(\Delta)), \hat{K}^{\epsilon}] \times$$

$$\hat{A}(e_{k}(\Delta))^{-1}[\hat{A}(e_{k}(\Delta)), \hat{V}_{U_{\epsilon}}])\psi_{\alpha},$$

$$\hat{\mathcal{H}}^{\epsilon}(N)\psi_{\alpha} = (\hat{H}_{\epsilon}^{E}(N) - 2(1 + \gamma^{2})\hat{T}^{\epsilon}(N))\psi_{\alpha} = \sum_{v \in V(\alpha)} N(v)\hat{H}_{\epsilon}^{E}\psi_{\alpha},$$

where, by construction, the operation of $\hat{\mathcal{H}}^{\epsilon}(N)$ on any $\psi_{\alpha} \in Cyl(\overline{A})$ is reduced to a finite combination of that of $\hat{H}_{\epsilon}^{E}$ with respect to different vertices of $\alpha$. 48
Hence, for each $\epsilon > 0$, $\hat{\mathcal{H}}^\epsilon(N)$ is a well-defined Yang-Mills gauge invariant and diffeomorphism covariant operator on $Cyl(\mathcal{A})$.

The last step is to remove the regulator by taking the limit $\epsilon \to 0$. However, the action of the Hamiltonian constraint operator on $\psi_\alpha$ adds edges $e_{ij}(\Delta)$ with $\frac{1}{2}$-representation with respect to each $v(\Delta)$ of $\alpha$, i.e., the action of $\hat{\mathcal{H}}^\epsilon(N)$ on cylindrical functions is graph-changing. Hence the operator does not converge with respect to the weak operator topology in $\mathcal{H}_{\text{kin}}$ when $\epsilon \to 0$, since different $\mathcal{H}^\epsilon_0$ with different graphs $\alpha$ are mutually orthogonal. Thus one has to define a weaker operator topology to make the operator limit meaningful.

By physical motivation and the naturally available Hilbert space $\mathcal{H}_{\text{Diff}}$, the convergence of $\hat{\mathcal{H}}^\epsilon(N)$ holds with respect to the so-called Uniform Rovelli-Smolin Topology \cite{105}, where one defines $\hat{\mathcal{H}}^\epsilon(N)$ to converge if and only if $\Psi_{\text{Diff}}[\hat{\mathcal{H}}^\epsilon(N)\phi]$ converge for all $\Psi_{\text{Diff}} \in Cyl^*_{\text{Diff}}$ and $\phi \in Cyl(\mathcal{A})$. Since the value of $\Psi_{\text{Diff}}[\hat{\mathcal{H}}^\epsilon(N)\phi]$ is actually independent of $\epsilon$, the sequence converges to a nontrivial result $\Psi_{\text{Diff}}[\hat{\mathcal{H}}^0(N)\phi]$ with arbitrary fixed $\epsilon_0 > 0$. Thus we have defined a diffeomorphism covariant, densely defined, closed but non-symmetric operator $\hat{\mathcal{H}}(N) = \lim_{\epsilon \to 0} \hat{\mathcal{H}}^\epsilon(N)$, on $\mathcal{H}_{\text{kin}}$ (or $\mathcal{H}^G$) representing the Hamiltonian constraint. Moreover, a dual Hamiltonian constraint operator is also defined on $Cyl^*$ as

$$(\hat{\mathcal{H}}^\epsilon(N)\Psi)[\phi] := \Psi[\hat{\mathcal{H}}(N)\phi],$$

for all $\Psi \in Cyl^*$ and $\phi \in Cyl(\mathcal{A})$. For $\Psi_{\text{Diff}} \in Cyl^*_{\text{Diff}} \subset Cyl^*$, one gets

$$(\hat{\mathcal{H}}^\epsilon(N)\Psi_{\text{Diff}})[\phi] = \Psi_{\text{Diff}}[\hat{\mathcal{H}}^\epsilon(N)\phi].$$

Several remarks on the Hamiltonian constraint operator are listed in the following.

- **Finiteness of $\hat{\mathcal{H}}(N)$ on $\mathcal{H}_{\text{kin}}$**

In ordinary quantum field theory, the continuous quantum field is only recovered when one lets lattices spacing to approach zero, i.e., takes the continuous cut-off parameter to its continuous limit. However, this will produce the well-known infinity in quantum field theory and make the Hamiltonian operator ill-defined on the Fock space. So it seems surprising that our operator $\hat{\mathcal{H}}(N)$ is still well defined, when one takes the limit $\epsilon \to 0$ with respect to the Uniform Rovelli-Smolin Topology so that the triangulation goes to the continuum. The reason behind it is that the cut-off parameter is essentially noneffective due to the diffeomorphism invariance of our quantum field theory. This is why there is no UV divergence in the background independent quantum gauge field theory with diffeomorphism invariance. On the other hand, from a convenient viewpoint, one may think the Hamiltonian constraint operator as an operator usually defined on a dense domain in $\mathcal{H}_{\text{Diff}}$. However, we will see that the dual Hamiltonian constraint operator cannot leave $\mathcal{H}_{\text{Diff}}$ invariant.
• Implementation of Dual Quantum Constraint Algebra

One important task is to check whether the commutator algebra (quantum constraint algebra) among the corresponding quantum operators of constraints both physically and mathematically coincides with the classical constraint algebra by substituting quantum constraint operators to classical constraint functionals and commutators to Poisson brackets:

\[
\{ \cdot, \cdot \} \rightarrow \frac{1}{\hbar} \{ \cdot, \cdot \}.
\]

Here the quantum anomaly has to be avoided in the construction of constraint operators (see the discussion for Eq. (42)). Firstly, the subalgebra of the quantum diffeomorphism constraint algebra is free of anomaly by construction:

\[
\hat{U}_\varphi \hat{U}_{\varphi'}^{-1} \hat{U}_{\varphi'}^{-1} = \hat{U}_{\varphi \circ \varphi' \circ \varphi^{-1} \circ \varphi'^{-1}},
\]

which coincides with the exponentiated version of the Poisson bracket between two diffeomorphism constraints generating the transformations \( \varphi, \varphi' \in Diff(\Sigma) \).

Secondly, the quantum constraint algebra between the dual Hamiltonian constraint operator \( \hat{\mathcal{H}}'(\mathcal{N}) \) and the finite diffeomorphism transformation \( \hat{U}_\varphi \) on diffeomorphism-invariant states coincides with the classical Poisson algebra between \( \mathcal{V}(\bar{\mathcal{N}}) \) and \( \mathcal{H}(M) \). Given a cylindrical function \( \phi_\alpha \) associated with a graph \( \alpha \) and a triangulation \( T(\epsilon) \) adapted to the graph \( \alpha \), the triangulation \( T(\varphi \circ \epsilon) \equiv \varphi \circ T(\epsilon) \) is compatible with the graph \( \varphi \circ \alpha \). Then we have by definition:

\[
(\{ [\hat{\mathcal{H}}'(\mathcal{N}), \hat{U}_\varphi], \Psi_{Diff}[\hat{\mathcal{H}}_\varphi]\} \Psi_{Diff})[\phi_\alpha] = \sum_{v \in V(\alpha)} \{ \mathcal{N}(\varphi \circ \epsilon) \Psi_{Diff}[\hat{\mathcal{H}}_{\varphi \circ \epsilon}^{\varphi \circ \epsilon}] - \mathcal{N}(\varphi \circ \epsilon) \Psi_{Diff}[\hat{\mathcal{H}}_{\varphi \circ \epsilon}^{\varphi \circ \epsilon}] \}
\]

Thus there is no anomaly. However, Eq. (42) also explains why the Hamiltonian constraint operator \( \hat{\mathcal{H}}(\mathcal{N}) \) cannot leave \( \mathcal{H}_{Diff} \) invariant.

Thirdly, we compute the commutator between two Hamiltonian constraint operators. Notice that

\[
[\hat{\mathcal{H}}(\mathcal{N}), \hat{\mathcal{H}}(M)] \phi_\alpha = \sum_{v \in V(\alpha)} [M(v)\hat{\mathcal{H}}(\mathcal{N}) - \mathcal{N}(\varphi \circ \epsilon) \hat{\mathcal{H}}(M)] \hat{\mathcal{H}}_{\varphi} \phi_\alpha
\]
In this subsection, we consider the situation of background-independent quantum dynamics of a real massless scale field coupled to gravity. The coupled generalized Palatini action reads [72]

\[ S[\phi, \alpha, \beta] = S_p[\phi, \alpha, \beta] + S_{KG}[\phi, \alpha], \]

where \( \alpha' \) is the graph changed from \( \alpha \) by the action of \( \hat{H}(N) \) or \( \hat{H}(M) \), which adds the edges \( e_{ij}(\Delta) \) on \( \alpha \), \( T(\epsilon) \) is the triangulation adapted to \( \alpha \) and \( T(\epsilon') \) adapted to \( \alpha' \). Since the newly added vertices by \( \hat{H}_v^\epsilon \) is planar, they will never contribute the final result. So one has

\[
\begin{align*}
\langle [\hat{H}(N), \hat{H}(M)] \phi \rangle & = \sum_{v, v' \in V(\alpha), v \neq v'} [M(v)N(v') - N(v)M(v')] \hat{H}_v^\epsilon \hat{H}_v^\epsilon \phi, \\
\langle [\hat{H}(N), \hat{H}(M)] \phi \rangle & = \frac{1}{2} \sum_{v, v' \in V(\alpha), v \neq v'} [M(v)N(v') - N(v)M(v')] [\hat{H}_v^\epsilon \hat{H}_v^\epsilon - \hat{H}_v^\epsilon \hat{H}_v^\epsilon] \phi,
\end{align*}
\]

(43)

where we have used the facts that \([\hat{H}_v^\epsilon, \hat{H}_v^\epsilon] = 0\) for \(v \neq v'\) and there exists a diffeomorphism \(\varphi_{v, v'}\) such that \(\hat{H}_v^\epsilon \hat{H}_v^\epsilon = \hat{U}_{\varphi_{v, v'}} \hat{H}_v^\epsilon \hat{H}_v^\epsilon\). Obviously, we have in the Uniform Rovelli-Smolin Topology

\[ ([\hat{H}(N), \hat{H}(M)])' \Psi_{Diff} = 0 \]

for all \(\Psi_{Diff} \in Cyl_{Diff}^*\), which proves the absence of a strong anomaly in the quantization of the constraint algebra. However, it is unclear whether the result of Eq. (43) resembles the classical formula [37]. So there still exists the danger of physical anomaly. So more works on the semi-classical analysis are needed to check if the right hand side of Eq. (43) has correct classical limit, which coincides with Eq. (37). The semi-classical analysis for the Hamiltonian constraint operator and the quantum constraint algebra is still an open issue today.

### 6.2 Inclusion of Matter Field

In this subsection, we consider the situation of background independent quantum dynamics of a real massless scale field coupled to gravity. The coupled generalized Palatini action reads [72]

\[ S[\phi, \alpha, J] = S_p[\phi, \alpha, J] + S_{KG}[\phi, \alpha], \]

where

\[
\begin{align*}
S_p[\phi, \alpha, J] & = \frac{1}{2\kappa} \int_M d^4x(e) e^\gamma_i e_\gamma^j (\Omega_{\gamma i} J^j + \frac{1}{2\gamma} J^K L \Omega_{\alpha \beta}) , \\
S_{KG}[\phi, \alpha] & = \frac{\alpha_M}{2} \int_M d^4x(e) \eta^I e^\gamma_i e_\gamma^j (\partial_\alpha \phi) \partial_\beta \phi ,
\end{align*}
\]

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here the real number $\alpha_M$ is the coupling constant. After 3+1 decomposition and Legendre transformation, one obtains the total Hamiltonian of the coupling system on the 3-manifold $\Sigma$ as:

$$H_{tot} = \Lambda^i G_i + N^a C_a + NC,$$

where $\Lambda^i$, $N^a$ and $N$ are Lagrange multipliers, and the three constraints in the Hamiltonian are expressed as:

$$G_i = D_a \pi^a_i := \partial_a \pi^a_i + \epsilon_{ij}^k A^i_a \pi^a_k,$$

$$C_a = \dot{\Pi}^b_i F_{ab} - A^i_a G_i + \bar{\pi} \partial_a \phi,$$

$$C = \frac{\kappa^2}{2 \sqrt{|\det q|}} \left( \frac{\kappa^2}{2 \alpha_M} \right) \left[ \delta^{ij} \dot{\Pi}^a_i \Pi^b_j (\partial_a \phi) \partial_b \phi + \frac{1}{2 \alpha_M} \bar{\pi}^2 \right],$$

where $\bar{\pi}$ denotes the momentum conjugate to $\phi$:

$$\bar{\pi} := \frac{\partial L}{\partial \dot{\phi}} = \frac{\alpha_M}{N} \sqrt{|\det q|} (\dot{\phi} - N^a \partial_a \phi).$$

Thus one has the elementary Poisson brackets

$$\{ A^i_a(x), \pi^b_j(y) \} = \delta^i_j \delta^b_a \delta(x - y),$$

$$\{ \phi(x), \bar{\pi}(y) \} = \delta(x - y).$$

Note that the second term of the Hamiltonian constraint is just the Hamiltonian of the real scalar field. Then we look for the background independent representation for the real scalar field coupled to gravity, following the polymer representation of the scalar field. The classical configuration space, $\mathcal{U}$, consists of all real-valued smooth functions $\phi$ on $\Sigma$. Given a set of finite number of points $X = \{x_1, ..., x_N\}$ in $\Sigma$, denote $Cyl_X$ the vector space generated by finite linear combinations of the following functions of $\phi$:

$$\Pi_{X, \lambda}(\phi) := \prod_{x_j \in X} \exp[i \lambda_j \phi(x_j)],$$

where $\lambda \equiv (\lambda_1, \lambda_2, \cdots, \lambda_N)$ are arbitrary real numbers. It is obvious that $Cyl_X$ has the structure of a $*$-algebra. The space $Cyl$ of all cylindrical functions on $\mathcal{U}$ is defined by

$$Cyl := \cup_X Cyl_X.$$  

Completing $Cyl$ with respect to the sup norm, one obtains a unital Abelian $C^*$-algebra $Cyl$. Thus one can use the GNS structure to construct its cyclic representations. A preferred positive linear functional $\omega_0$ on $Cyl$ is defined by

$$\omega_0(\Pi_{X, \lambda}) = \begin{cases} 1 & \text{if } \lambda_j = 0 \forall j \\ 0 & \text{otherwise} \end{cases}.$$
which defines a diffeomorphism-invariant faithful Borel measure \( \mu \) on \( \mathcal{U} \) as

\[
\int_{\mathcal{U}} d\mu(\Pi_{X,\lambda}) = \begin{cases} 1 & \text{if } \lambda_j = 0 \forall j \\ 0 & \text{otherwise} \end{cases}
\]

Thus one obtains the Hilbert space, \( \mathcal{H}^{KG}_{\text{kin}} \equiv L^2(\mathcal{U}, d\mu) \), of square integrable functions on a compact topological space \( \mathcal{U} \) with respect to \( \mu \), where \( Cyl \) acts by multiplication. The quantum configuration space \( \mathcal{U} \) is called the Gel’fand spectrum of \( Cyl \). More concretely, for a single point set \( X_0 \equiv \{ x_0 \} \), \( Cyl_{X_0} \) is the space of all almost periodic functions on a real line \( \mathbb{R} \). The Gel’fand spectrum of the corresponding \( C^* \)-algebra \( Cyl_{X_0} \) is the Bohr completion \( \mathbb{R}^x_{x_0} \) of \( \mathbb{R} \), which is a compact topological space such that \( Cyl_{X_0} \) is the \( C^* \)-algebra of all continuous functions on \( \mathbb{R}^x_{x_0} \). Since \( \mathbb{R} \) is densely embedded in \( \mathbb{R}^x_{x_0} \), \( \mathbb{R}^x_{x_0} \) can be regarded as a completion of \( \mathbb{R} \).

It is clear from Eq.(48) that an orthonomal basis in \( \mathcal{H}^{KG}_{\text{kin}} \) is given by the so-called scalar network functions \( \Pi_{c}(\phi) \), where \( c \) denotes \( (X, \lambda) \) and \( \lambda \equiv (\lambda_1, \lambda_2, \cdots , \lambda_N) \) are non-zero real numbers now. So the total kinematical Hilbert space \( \mathcal{H}^{\text{kin}} \) is the direct product of the kinematical Hilbert space \( \mathcal{H}^{GR}_{\text{kin}} \) for gravity and the kinematical Hilbert space for real scalar field, i.e., \( \mathcal{H}^{\text{kin}} :\equiv \mathcal{H}^{GR}_{\text{kin}} \otimes \mathcal{H}^{KG}_{\text{kin}} \), where \( \Pi_{s}(A) \otimes \Pi_{c}(\phi) \) constitutes a complete set of orthonormal basis as

\[
< \Pi_{s'}(A) \otimes \Pi_{c'}(\phi) | \Pi_{s}(A) \otimes \Pi_{c}(\phi) >_{\text{kin}} = \delta_{s's} \delta_{c'c}.
\]

Note that none of \( \mathcal{H}^{\text{kin}} \), \( \mathcal{H}^{GR}_{\text{kin}} \) and \( \mathcal{H}^{KG}_{\text{kin}} \) is a separable Hilbert space.

Given a a pair \( (x_0, \lambda_0) \), there is an elementary configuration for the scalar field, the so-called point holonomy,

\[
U(x_0, \lambda_0) := \exp[i\lambda_0\phi(x_0)].
\]

It corresponds to a configuration operator \( \hat{U}(x_0, \lambda_0) \), which acts on any cylindrical function \( \psi(\phi) \in \mathcal{H}^{KG}_{\text{kin}} \) by

\[
\hat{U}(x_0, \lambda_0)\psi(\phi) = U(x_0, \lambda_0)\psi(\phi).
\]

All these operators are unitary. But since the family of operators \( \hat{U}(x_0, \lambda) \) fails to be weakly continuous in \( \lambda \), there is no operator \( \hat{\phi}(x) \) on \( \mathcal{H}^{KG}_{\text{kin}} \). The momentum functional smeared on a 3-dimensional region \( R \subset \Sigma \) is expressed by

\[
\pi(R) := \int_{R} d^3x\tilde{\pi}(x).
\]

The Poisson bracket between the momentum functional and a point holonomy can be easily calculated to be

\[
\{ \pi(R), U(x, \lambda) \} = -i\lambda\chi_R(x)U(x, \lambda),
\]

where \( \chi_R(x) \) is the characteristic function for the region \( R \). So the momentum operator is defined by the action on scalar network functions \( \Pi_{c}(\phi) \) as

\[
\hat{\pi}(R)\Pi_{c}(\phi) := i\hbar\{ \pi(R), \Pi_{c}(\phi) \} = \hbar\sum_{x_j \in X} \lambda_j\chi(x_j)\Pi_{c}(\phi).
\]
Now one can consider the quantum dynamics and impose the quantum constraints on $H_{\text{kin}}$. Firstly, the Gaussian constraint can be solved independently of $H^{KG}_{\text{kin}}$, since it only involves gravitational field. The diffeomorphism constraint can be implemented by the group averaging strategy for both $Cyl(A)$ and $Cyl$ in the same way as that in Section 5.2. Thus one obtains the diffeomorphism-invariant Hilbert space $\mathcal{H}^{\text{Diff}} = \mathcal{H}^{GR}_{\text{Diff}} \otimes \mathcal{H}^{KG}_{\text{Diff}}$. Then the only nontrivial task is the implementation of the Hamiltonian constraint $H(N)$. One thus needs to define a corresponding Hamiltonian constraint operator on $H_{\text{kin}}$. While the gravitational part of

$$H(N) := \int_\Sigma d^3 x N$$

is a well-defined operator $\hat{H}^{GR}(N)$ by the Uniform Rovelli-Smolin Topology, the crucial point in this subsection is to define an operator corresponding to the Hamiltonian functional $H^{KG}(N)$ of the scalar field, which can be decomposed into two parts

$$H^{KG}(N) = H^{KG,\phi}(N) + H^{KG,\text{Kin}}(N),$$

where

$$H^{KG,\phi}(N) = \frac{\kappa_2 \gamma^2 \alpha M}{2} \int_\Sigma d^3 x N \frac{1}{\sqrt{|\det q|}} \delta^{ij} \tilde{P}_i^a \tilde{P}_j^b (\partial_a \phi) \partial_b \phi,$$

$$H^{KG,\text{Kin}}(N) = \frac{1}{2 \alpha M} \int_\Sigma d^3 x N \frac{1}{\sqrt{|\det q|}} \tilde{p}^2.$$

For the first part $H^{KG,\phi}(N)$, we use the identities, for $x \in R$

$$\tilde{P}_i^a = \frac{1}{2 \kappa \gamma} \eta^{abc} \epsilon_{ijk} \epsilon_i^j \epsilon_j^k \text{ and } \epsilon_i^j(x) = \frac{2}{\kappa \gamma} \{ A_i^j(x), V_R \}.$$

Hence

$$\tilde{P}_i^a(x) = \frac{2}{\kappa \gamma^3} \eta^{abc} \epsilon_{ijk} \{ A_i^j(x), V_R \} \{ A_k^c(x), V_R \}.$$

Then the expression of $H^{KG,\phi}(N)$ can be regulated as

$$H^{KG,\phi}(N) = \frac{\kappa_2 \gamma^2 \alpha M}{2} \int_\Sigma d^3 y \int_\Sigma d^3 x N^{1/2}(x) N^{1/2}(y) \chi_e(x - y) \times \frac{1}{\sqrt{V_{U_x}}} \tilde{P}_i^a(x) (\partial_a \phi(x)) \frac{1}{\sqrt{V_{U_y}}} \tilde{P}_j^b(y) \partial_b \phi(y)$$

$$= \frac{32 \alpha M}{\kappa^4 \gamma^4} \int_\Sigma d^3 y \int_\Sigma d^3 x N^{1/2}(x) N^{1/2}(y) \chi_e(x - y) \delta^{ij} \times$$

$$\eta^{ace}(\partial_c \phi(x)) \text{Tr} \left( \tau_i \{ A_e(x), V_{U_x}^{3/4} \} \{ A_c(x), V_{U_x}^{3/4} \} \right) \times \frac{1}{\sqrt{V_{U_y}}} \tilde{P}_i^a(\partial_c \phi(y)) \text{Tr} \left( \tau_j \{ A_f(y), V_{U_y}^{3/4} \} \{ A_c(y), V_{U_y}^{3/4} \} \right),$$

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where the matrices $A_n \equiv A_n (x-y)$ is the characteristic function of a box containing $x$ with scale $\epsilon$ such that $\lim_{\epsilon \to 0} \chi_\epsilon (x-y)/\epsilon^3 = \delta (x-y)$, and $V_{U^c}$ is the volume of the box. Notice the following useful identities

$$\begin{align*}
\{ \int_\epsilon dt \ A_n \hat{e}^{x}(t), V_{U^c(\epsilon)}^{3/4} \} &= -A(e) \cdot \{ A(e), V_{U^c(e)}^{3/4} \} + o(\epsilon),
\int_\epsilon dt \ \hat{e}^{x} \partial_\epsilon \phi(t) &= -iU(s(\epsilon))^{-1} [U(t(\epsilon)) - U(s(\epsilon))] + o(\epsilon),
\end{align*}$$

where we set $U(x) \equiv U(x,1)$.

So $\mathcal{H}_{KG,\phi} (N)$ can be expressed as

$$\begin{align*}
\mathcal{H}_{KG,\phi} (N) &= \sum_{\Delta \in T(\epsilon)} \sum_{\Delta \subset \Delta (\epsilon)} N^{1/2} (v(\Delta)) N^{1/2} (v(\Delta')) \chi_\epsilon (v(\Delta) - v(\Delta')) \delta^{ij} \times \nonumber \\
&= \sum_{\Delta \subset \Delta (\epsilon)} N^{1/2} (v(\Delta)) N^{1/2} (v(\Delta')) \chi_\epsilon (v(\Delta) - v(\Delta')) \delta^{ij} \times \nonumber \\
&= \sum_{\Delta \subset \Delta (\epsilon)} N^{1/2} (v(\Delta)) N^{1/2} (v(\Delta')) \chi_\epsilon (v(\Delta) - v(\Delta')) \delta^{ij} \times 
\end{align*}$$

with respect to the triangulation $T(\epsilon)$ of $\Sigma$. It is easy to verify that $\lim_{\epsilon \to 0} \mathcal{H}_{KG,\phi} (N) = \mathcal{H}_{KG,\phi} (N)$. Then we can define a positive quadratic form on (a suitable form domain of) $\mathcal{H}_{Diff}$ by

$$\begin{align*}
Q_{KG,\phi} (N) (\Psi_{Diff}, \Psi_{Diff}) := \lim_{\epsilon \to 0} \sum_{[s,c]} 64 \times \frac{32 N_M}{9 \epsilon^4} \times 
\sum_{\varepsilon \in V(\alpha)} \sum_{\varepsilon' \in V(\alpha')} \frac{N^{1/2} (v) N^{1/2} (v')}{C^3 \ v(v) C^3 \ v(v')} \sum_{v(\Delta) = v} \sum_{v(\Delta') = v'} \chi_\epsilon (v - v') \delta^{ij} \times 
\Psi_{Diff} [h^{\epsilon, \Delta \phi}_{\psi, x, \gamma}] \Psi_{Diff} [h^{\epsilon, \Delta \phi}_{\psi, x, \gamma}],
\end{align*}$$

where $\Pi_{s,c}[A] \otimes \Pi_{c}[\phi]$ with $s[s] = (\alpha, j, m, n)$, $c[c] = (X, \lambda)$ in the diffeomorphism equivalent class $[s]$ and $[c]$, and

$$\begin{align*}
h^{\epsilon, \Delta \phi}_{\psi, x, \gamma} := i \frac{\epsilon}{h} \xi \nu \times 
\end{align*}$$

is a densely defined operator on $\mathcal{H}_{kin}$. In the same way, one can find the regulated expression of $\mathcal{H}_{KG,Kin} (N)$ as

$$\begin{align*}
\mathcal{H}_{KG,Kin} (N)
\end{align*}$$
\begin{equation}
\frac{1}{2\alpha_M} \int d^3x N^{1/2}(x) \overline{\pi}(x) \int d^3x N^{1/2}(y) \overline{\pi}(y) \times \\
\int d^3u \frac{\det q(u)}{(V_{\mathbb{C}})^{3/2}} \int d^3u \frac{\det q(v)}{(V_{\mathbb{C}})^{3/2}} \chi_\epsilon(x - y) \chi_\epsilon(u - x) \chi_\epsilon(w - y) \\
= \frac{1}{2\alpha_M} 2^8 \int d^3x N^{1/2}(x) \overline{\pi}(x) \int d^3x N^{1/2}(y) \overline{\pi}(y) \times \\
\int d^3w \tilde{\eta} \eta^{abc} \text{Tr}\{ \{ A_u(u), \sqrt{V_{\mathbb{C}}} \} \{ A_u(u), \sqrt{V_{\mathbb{C}}} \} \} \times \\
\int d^3w \tilde{\eta} \eta^{def} \text{Tr}\{ \{ A_d(w), \sqrt{V_{\mathbb{C}}} \} \{ A_c(w), \sqrt{V_{\mathbb{C}}} \} \} \times \\
\chi_\epsilon(x - y) \chi_\epsilon(u - x) \chi_\epsilon(w - y) \\
= \frac{1}{2\alpha_M} 2^8 \sum_{\Delta \in T(e)} N^{1/2}(v(\Delta)) \pi(\Delta) \sum_{\Delta' \in T(e)} N^{1/2}(v(\Delta')) \pi(\Delta') \times \\
\sum_{\Delta'' \in T(e)} \epsilon^{imn} \text{Tr}\{ A(e_i(\Delta''))^{-1} A(e_i(\Delta'')) , \sqrt{V_{\mathbb{C}}^{e_i(\Delta'')}} \} \times \\
A(e_m(\Delta''))^{-1} A(e_m(\Delta'')) , \sqrt{V_{\mathbb{C}}^{e_m(\Delta'')}} \} \times \\
A(e_n(\Delta''))^{-1} A(e_n(\Delta'')) , \sqrt{V_{\mathbb{C}}^{e_n(\Delta'')}} \} \times \\
\sum_{\Delta''' \in T(e)} \epsilon^{kl} \text{Tr}\{ A(e_j(\Delta'''))^{-1} A(e_j(\Delta''')) , \sqrt{V_{\mathbb{C}}^{e_j(\Delta''')}} \} \times \\
A(e_k(\Delta'''))^{-1} A(e_k(\Delta''')) , \sqrt{V_{\mathbb{C}}^{e_k(\Delta''')}} \} \times \\
A(e_l(\Delta'''))^{-1} A(e_l(\Delta''')) , \sqrt{V_{\mathbb{C}}^{e_l(\Delta''')}} \} \times \\
\chi_\epsilon(v(\Delta) - v(\Delta')) \chi_\epsilon(v(\Delta'') - v(\Delta')) \chi_\epsilon(v(\Delta''') - v(\Delta')), \\
\end{equation}

which satisfies \( \lim_{\epsilon \to 0} \mathcal{H}_{KG, Kin}^{\epsilon}(N) = \mathcal{H}_{KG, Kin}(N) \). Then a positive quadratic form \( Q_{KG, Kin} \) on (a dense form domain of) \( \mathcal{H}_{Diff} \) can be defined by

\begin{equation}
Q_{KG, Kin}(N) (\Psi_{Diff} , \Psi'_{Diff}) := \lim_{\epsilon \to 0} \sum_{[\alpha, \epsilon]} 8^4 \times \frac{1}{2\alpha_M} 2^8 \sum_{\Delta \in T(e)} N^{1/2}(v(\Delta)) \pi(\Delta) \sum_{\Delta' \in T(e)} N^{1/2}(v(\Delta')) \pi(\Delta') \times \\
\sum_{\Delta'' \in T(e)} \sum_{\Delta''' \in T(e)} \frac{N^{1/2}(v(\Delta'''))}{C^3_n(\Delta''')} \frac{N^{1/2}(v(\Delta'''))}{C^3_n(\Delta''')} \sum_{v(\Delta)=v(\Delta')} \sum_{v(\Delta'')=v(\Delta''')} \chi_\epsilon(v - v') \times \\
\psi_{Diff} [ \hat{h}_{t,\epsilon, u}^{\Delta'''} \Pi_{s,c[s,c]} ] \psi_{Diff} [ \hat{h}_{t,\epsilon, u}^{\Delta'} \Pi_{s,c[s,c]} ], \\
\end{equation}

where

\begin{equation}
\hat{h}_{t, \epsilon, \Delta, u} := \frac{1}{(\hbar)^3} \frac{\pi(\Delta)}{\pi(\Delta')} \sum_{v'' \in V(\alpha)} \frac{1}{C^3_n(v'')} \sum_{v(\Delta)=v'} \chi_\epsilon(v - v') \times \\
\epsilon^{imn} \text{Tr}\{ A(e_i(\Delta''))^{-1} A(e_i(\Delta'')) , \sqrt{V_{\mathbb{C}}^{e_i(\Delta'')}} \} \times \\
\end{equation}
\[
\hat{A}(e_m(\Delta''))^\dagger \hat{A}(e_m(\Delta'')), \sqrt{V_U'} \times \\
\hat{A}(e_n(\Delta''))^\dagger \hat{A}(e_n(\Delta'')), \sqrt{V_U''}) \chi_e (v'' - v)
\]
is also a densely defined operator on \( \mathcal{H}_{kin} \). As a result, we have two positive quadratic forms \( Q_{KG,\phi}(N) \) and \( Q_{KG,kin}(N) \) on \( \mathcal{H}_{Diff} \), which are both symmetric. So it is not difficult to prove the following theorem \([74]\):

**Theorem 6.2.1:** Both \( Q_{KG,\phi}(N) \) and \( Q_{KG,kin}(N) \) are densely defined, positive and closed quadratic forms on \( \mathcal{H}_{Diff} \), which are both symmetric. So it is not difficult to prove the following theorem \([74]\):

Theorem 6.2.1: Both \( Q_{KG,\phi}(N) \) and \( Q_{KG,kin}(N) \) are densely defined, positive and closed quadratic forms on \( \mathcal{H}_{Diff} \), which are both symmetric. So it is not difficult to prove the following theorem \([74]\):

\[
Q_{KG,\phi}(N)\langle \Psi_{Diff}, \Psi'_{Diff} \rangle = \langle \Psi_{Diff} | \hat{H}_{KG,\phi}(N) | \Psi'_{Diff} \rangle_{Diff}.
\]

Therefore the Hamiltonian operator of real scalar field,

\[
\hat{H}_{KG}(N) := \hat{H}_{KG,\phi}(N) + \hat{H}_{KG,kin}(N),
\]
is a positive and self-adjoint operator on \( \mathcal{H}_{Diff} \) without UV divergence. The reason for it is that we used the background-independent representation for the quantum field theory of matter, so that the Hamiltonian operator are represented on diffeomorphism invariant states in \( \mathcal{H}_{Diff} \). Hence we present how the quantum gravity can be a natural regulator for matter quantum field theory \([115]\).

Thus we have obtained the desired matter-coupled quantum Hamiltonian constraint equation

\[
- \left( \hat{H}_{KG}(N) \Psi_{Diff} \right)[f_\alpha] = \left( \hat{H}_{GR}'(N) \Psi_{Diff} \right)[f_\alpha], \tag{49}
\]

for all \( f_\alpha \in \text{Cyl}(\mathcal{A}) \otimes \text{Cyl} \). Compare it with the well-known Schrödinger equation for a particle:

\[
i\hbar \frac{\partial}{\partial t} \psi(x,t) = H(\hat{x}, -i\hbar \frac{\partial}{\partial x}) \psi(x,t),
\]

where \( \psi(x,t) \in \mathcal{H}_x = L^2(\mathbb{R}, dx) \) and \( t \) is a parameter labelling time evolution. One may take the viewpoint that the matter field \( \phi \) plays the role of time in the time evolution of general relativity, then \( \phi \) is a parameter labelling the evolution of the gravitational field state \( \Psi_{Diff}(\mathcal{A}, \phi) \in \mathcal{H}_{Diff}^{GR} \). However, one may also take another point of view to regard Eq. (49) as a simple constraint equation, while write Schrödinger equation as

\[
\hat{H}_{Sch} \psi(x,t) = [i\hbar \frac{\partial}{\partial t} - H(\hat{x}, -i\hbar \frac{\partial}{\partial x})] \psi(x,t) = 0,
\]

where the state \( \psi(x,t) \) is regarded as a vector in the direct product Hilbert space \( \mathcal{H}_{kin} = \mathcal{H}_x \otimes \mathcal{H}_t \). Then there is no evolution for the solution state \( \psi_{phys}(x,t) \),
since it represents the quantum state of a four dimensional configuration of a particle so that time evolution is not needed. Thus one can accept the viewpoint that there is no evolution in matter-coupled general relativity, as $\Psi_{Diff}(A, \phi)$ is in the direct product space $\mathcal{H}_{Diff}^{GR} \otimes \mathcal{H}_{Diff}^{KG}$. Then a solution of constraint equation (49), $\Psi_{phys}(A, \phi) \in \mathcal{H}_{phys}$, stands for the quantum state of a four dimensional field configuration of matter-coupled gravity. This viewpoint just reflects the general covariance in Einstein’s theory of relativity [103], since general relativity is a four-dimensional covariant theory by construction.

6.3 Master Constraint Programme

Although the Hamiltonian constraint operator introduced in Section 6.1 is densely defined on $\mathcal{H}_{kin}$ and diffeomorphism covariant, there are still several problems unsettled which are listed below.

- It is unclear whether the commutator between two Hamiltonian constraint operators resembles the classical Poisson bracket between two Hamiltonian constraints. Hence it is doubtful whether the quantum Hamiltonian constraint produces the correct quantum dynamics with correct classical limit [61][79].

- The dual Hamiltonian constraint operator does not leave the Hilbert space $\mathcal{H}_{Diff}$ invariant. Thus the inner product structure of $\mathcal{H}_{Diff}$ cannot be employed in the construction of physical inner product.

- Classically the collection of Hamiltonian constraints do not form a Lie algebra. So one cannot employ group average strategy in solving the Hamiltonian constraint quantum mechanically, since the strategy depends on group structure crucially.

However, if one could construct an alternative classical constraint algebra, giving the same constraint phase space, which is a Lie algebra (no structure function), where the subalgebra of diffeomorphism constraints forms an ideal, then the programme of solving constraints would be much improved at a basic level. Such a constraint Lie algebra was first introduced by Thieman in [125]. The central idea is to introduce the master constraint:

$$M := \frac{1}{2} \int_{\Sigma} d^3x \frac{|\tilde{C}(x)|^2}{\sqrt{|\det q(x)|}},$$

where $\tilde{C}(x)$ is the scalar constraint in Eq. (10). One then gets the master constraint algebra:

$$\{\mathcal{V}(\vec{N}), \mathcal{V}(\vec{N}')\} = \mathcal{V}([\vec{N}, \vec{N}']),$$
$$\{\mathcal{V}(\vec{N}), M\} = 0,$$
$$\{M, M\} = 0.$$
The master constraint programme has been well tested in various examples \[54\]–\[57\]. In the following, we extend the diffeomorphism transformations (to e.g. homomorphisms) such that the Hilbert space \(\mathcal{H}_{\text{Diff}}\) is separable. This separability of \(\mathcal{H}_{\text{Diff}}\) and the positivity and the diffeomorphism invariance of \(\mathcal{M}\) will be working together properly and provide us with powerful functional analytic tools in the programme to solve the constraint algebra quantum mechanically. The regularized version of the master constraint can be expressed as

\[
\mathcal{M}^\epsilon := \frac{1}{2} \int \int d^3 x \chi_\epsilon(x - y) \frac{\tilde{C}(y) \tilde{C}(x)}{\sqrt{V_{U_\epsilon^n}} \sqrt{V_{U_\epsilon^m}}},
\]

where \(\chi_\epsilon(x - y)\) is any 1-parameter family of functions such that \(\lim_{\epsilon \to 0} \chi_\epsilon(x - y)/\epsilon^3 = \delta(x - y)\) and \(\chi_\epsilon(0) = 1\). Then repeating the procedure outlined in section 6.1, one can find a positive quadratic form on (a dense form domain of) \(\mathcal{H}_{\text{Diff}}\) for the master constraint as

\[
Q_{\mathcal{M}}(\Psi_{\text{Diff}}, \Psi_{\text{Diff}}') := \lim_{\epsilon \to 0} \sum_{[s]} \times
\]

\[
\frac{1}{2} \sum_{v \in V(\alpha)} \sum_{v' \in V(\alpha)} \frac{1}{C_{n(v)}^3 C_{n(v')}^3} \sum_{v(\Delta) = v, v'(\Delta) = v'} \chi_\epsilon(v - v') \times
\]

\[
\Psi_{\text{Diff}}[\hat{h}_{\nu}^{\epsilon, \Delta}] \Pi_{s \in [s]} \Psi_{\text{Diff}}'[\hat{h}_{\nu'}^{\epsilon, \Delta'}] \Pi_{s \in [s]},
\]

(50)

where \(\Pi_{s \in [s]}\) is the spin-network function with \(s \in [s] = (\alpha, j, m, n)\) in diffeomorphism equivalent class \([s]\), and the expression of \(\hat{h}_{\nu}^{\epsilon, \Delta}\) can be found from that of the Hamiltonian constraint operator as

\[
\hat{h}_{\nu}^{\epsilon, \Delta} = \frac{16}{3 \hbar^2 \gamma} e^{ijk} \text{Tr}(\hat{A}(\alpha_{ij}(\Delta))\hat{A}(\epsilon_k(\Delta))^{-1}) \sqrt{V_{U_\epsilon}}
\]

\[
+ 2(1 + \gamma^2) \frac{16}{3 \hbar^2 \gamma^3} e^{ijk} \text{Tr}(\hat{A}(\epsilon_i(\Delta))^{-1}) \sqrt{V_{U_\epsilon}}
\]

\[
- \hat{A}(e_j(\Delta))^{-1} \text{Tr}(\hat{A}(\epsilon_j(\Delta))^{-1}) \sqrt{V_{U_\epsilon}}.
\]

The quadratic form \(Q_{\mathcal{M}}(\ , \ )\) is positive and hence semi-bounded. One can finally obtain the following theorem \[73\].

**Theorem 6.3.1:** The quadratic form \(Q_{\mathcal{M}}(\ , \ )\) is a closed quadratic form on \(\mathcal{H}_{\text{Diff}}\). Hence there exists a unique densely defined, positive self-adjoint operator \(\hat{M}\) on \(\mathcal{H}_{\text{Diff}}\), leaving \(\mathcal{H}_{\text{Diff}}\) invariant, such that:

\[
Q_{\mathcal{M}}(\Psi_{\text{Diff}}, \Psi_{\text{Diff}}') = \langle \Psi_{\text{Diff}} | \hat{M} | \Psi_{\text{Diff}}' \rangle_{\text{Diff}}.
\]

**Proof:** The strategy of our proof is first to introduce a symmetric master constraint operator \(\mathcal{M}\), which is densely defined on \(\mathcal{H}_{\text{Diff}}\), then prove that the
quadratic form associated with it coincides with Eq. (50).

(1) Define the master constraint operator

Introducing a partition \( P \) of the 3-manifolds \( \Sigma \) into cells \( C \), we have the operator

\[
\hat{H}_C f_\alpha = \sum_{v \in V(\alpha)} \frac{\chi_C(v)}{C_n(v)} \sum_{v(\Delta) = v} \hat{h}_v^{\alpha \Delta} f_\alpha, \tag{51}
\]

via a state-dependent triangulation \( T(\epsilon) \) on \( \Sigma \), where \( \chi_C(v) \) is the characteristic function of the cell \( C \). Then we define a Master Constraint Operator, \( \hat{M} \), in \( \mathcal{H}_{Diff} \) as

\[
\hat{M} := \lim_{P \to \Sigma} \sum_{C \in P} \frac{1}{2} \hat{H}_C^\dagger \hat{H}_C, \tag{52}
\]

where \( \hat{H}_C^\dagger \) and \( \hat{H}_C \), depending on a cell \( C \), are well defined by

\[
(\hat{H}_C^\dagger \Psi)[f_\alpha] := \lim_{\epsilon \to 0} \Psi[\hat{H}_C f_\alpha],
\]

\[
(\hat{H}_C^\dagger \Psi)[f_\alpha] := \lim_{\epsilon \to 0} \Psi[\hat{H}_C^\dagger f_\alpha],
\]

for any cylindrical function \( f_\alpha \in Cyl(\mathcal{A}) \), and any \( \Psi \in Cyl^* \), which is the algebraic dual of \( Cyl(\mathcal{A}) \). Note that the second dagger in the second equation is the adjoint operation with respect to the inner product on \( \mathcal{H}_{Kin} \). Since the actions of \( \hat{H}_C \) and \( \hat{H}_C^\dagger \) on any \( f_\alpha \) only add finite edges with \( \frac{1}{2} \)-representations to the graph \( \alpha \), one has \( \lim_{P \to \sigma} \sum_{C \in P} \frac{1}{2} \hat{H}_C^\dagger \hat{H}_C f_\alpha \in Cyl(\mathcal{A}) \), and hence given \( \Psi_{Diff} \in \mathcal{H}_{Diff} \), the value of

\[
(\hat{M}\Psi_{Diff})[f_\alpha] := \lim_{P \to \sigma, \epsilon \to 0} \Psi_{Diff}[\sum_{C \in P} \frac{1}{2} \hat{H}_C^\dagger \hat{H}_C f_\alpha] \tag{53}
\]

is finite. Moreover, We can show that \( \hat{M} \) leaves \( \mathcal{H}_{Diff} \) invariant. For any diffeomorphism transformation \( \varphi \),

\[
(\hat{U}_\varphi^\dagger \hat{M}\Psi_{Diff})[f_\alpha] = \lim_{P \to \sigma, \epsilon \to 0} \Psi_{Diff}[\sum_{C \in P} \frac{1}{2} \hat{H}_C^\dagger \hat{H}_C^\dagger \hat{U}_\varphi f_\alpha]
\]

\[
= \lim_{P \to \sigma, \epsilon \to 0} \Psi_{Diff}[\hat{U}_\varphi \sum_{C \in P} \frac{1}{2} \hat{H}_{\varphi^{-1}(C)}^\dagger \hat{H}_{\varphi^{-1}(C)} f_\alpha]
\]

\[
= \lim_{P \to \sigma, \epsilon \to 0} \Psi_{Diff}[\sum_{C \in P} \frac{1}{2} \hat{H}_{\varphi(C)}^\dagger \hat{H}_{\varphi(C)} f_\alpha],
\]

where in the last step, we used the fact that the diffeomorphism transformation \( \varphi \) leaves the partition invariant in the limit \( P \to \sigma \) and relabel \( \varphi(C) \) to be \( C \). So we have the result

\[
(\hat{U}_\varphi^\dagger \hat{M}\Psi_{Diff})[f_\alpha] = (\hat{M}\Psi_{Diff})[f_\alpha].
\]
In conclusion, the master constraint operator $\hat{M}$ defined by Eq. (52) is densely defined in $\mathcal{H}_{\text{Diff}}$.

(2) The self-adjointness of $\hat{M}$

Given two diffeomorphism invariant cylindrical functions $\eta(f_\beta)$ and $\eta(g_\alpha)$ associated with the cylindrical functions $f_\beta$ and $g_\alpha$, the quadratic form associated with $\hat{M}$ is calculated as

$$<\eta(f_\beta)|\hat{M}|\eta(g_\alpha)>_{\text{Diff}}$$

$$= \lim_{\mathcal{P} \to \sigma, \epsilon', \epsilon' \to 0} \sum_{C \in \mathcal{P}} \frac{1}{2} \frac{\eta(g_\alpha)}{\eta(f_\beta)} |\hat{H}_C^\dagger \hat{H}_C^\epsilon f_\beta\rangle$$

$$= \lim_{\mathcal{P} \to \sigma, \epsilon', \epsilon' \to 0} \sum_{C \in \mathcal{P}} \frac{1}{2} \frac{1}{n_\alpha} \frac{1}{n_\beta} \sum_{\varphi \in \text{Diff}/\text{Diff}_{s}} \sum_{\varphi' \in \text{GS}_s} <U_\varphi U_{\varphi'} g_\alpha |\hat{H}_C^\dagger \hat{H}_C^\epsilon f_\beta\rangle_{\text{Kin}}$$

$$= \sum_{[s]} \sum_{v \in \text{V}(\gamma(s \in [s]))} \frac{1}{2} \lim_{\epsilon', \epsilon' \to 0} <\eta(g_\alpha)|\hat{H}_v^\dagger \hat{H}_v^\epsilon s\rangle >_{\text{Diff}} \sum_{s \in [s]} <\hat{H}_v^\dagger \hat{H}_v^\epsilon s\rangle_{\text{Kin}}$$

$$= \sum_{[s]} \sum_{v \in \text{V}(\gamma(s \in [s]))} \frac{1}{2} \lim_{\epsilon', \epsilon' \to 0} <\eta(g_\alpha)|\hat{H}_v^\dagger \hat{H}_v^\epsilon s\rangle >_{\text{Diff}} <\eta(g_\alpha)|\hat{H}_v^\dagger \hat{H}_v^\epsilon s\rangle >_{\text{Diff}}$$

$$= \sum_{[s]} \sum_{v \in \text{V}(\gamma(s \in [s]))} \frac{1}{2} \hat{H}_v^\dagger \eta(g_\alpha) \hat{H}_v^\epsilon s\rangle [\hat{H}_v^\dagger \eta(f_\beta) \hat{H}_v^\epsilon s\rangle]$$

where $n_\alpha$ is the number of the elements of the group, $\text{GS}_s$, of colored graph symmetries of $\alpha$, $\text{Diff}_{s}$ denotes the subgroup of $\text{Diff}$ which maps $\alpha$ to itself, and $\gamma(s)$ is the graph associated with the spin-network function $\Pi_s$. Note that we have used the resolution of identity trick in the fourth step. In the fifth step, we exchange $\lim_{\mathcal{P} \to \sigma, \epsilon', \epsilon' \to 0} \sum_{C \in \mathcal{P}}$ and $\sum_{s}$, then take the limit $C \to v$ and split the sum $\sum_{s}$ into $\sum_{[s]} \sum_{s \in [s]}$, where $[s]$ denotes the diffeomorphism equivalent class associated with $s$. Here we also use the fact that, given $\gamma(s)$ and $\gamma(s')$, which are different up to a diffeomorphism transformation, there is always a diffeomorphism $\varphi$ transforming the graph associated with $\hat{H}_v^\dagger \hat{H}_v^\epsilon s\rangle$ to that of $\hat{H}_v^\dagger \hat{H}_v^\epsilon s\rangle$ with $\varphi(v) = v'$, hence $<\eta(g_\alpha)|\hat{H}_v^\dagger \hat{H}_v^\epsilon s\rangle >_{\text{Diff}}$ is constant for different $s \in [s]$. In the sixth step, we use the fact that the sums $\sum_{s \in [s]}$ and $\sum_{v \in \text{V}(\gamma(s \in [s]))}$, where $a(v)$ is the loop with scale $\epsilon'$ added at the vertex $v$ by the operator $\hat{H}_v^\dagger$, are different up to the diffeomorphism class of loops with different scale; however, there is only one term surviving in $\sum_{a(v) \in [a(v)]} <\hat{H}_v^\dagger \hat{H}_v^\epsilon s\rangle$ since the graph $\beta$ is fixed.
From Eq. (54), it is obvious that the master constraint operator $\hat{M}$ is a positive and symmetric operator in $\mathcal{H}_{\text{Diff}}$. Since the result of Eq. (54) coincides with the quadratic form $Q_M(\eta(f, f)\eta(g, g))$ defined by Eq. (50), $Q_M$ is closable [95], whose closure is the quadratic form of a unique self-adjoint operator $\hat{M}$, called the Friedrichs extension of $\hat{M}$. This completes our proof.

In the following discussion, we relabel $\hat{M}$ to be $\hat{M}$ for simplicity. To obtain the physical Hilbert space $\mathcal{H}_{\text{phys}}$, we can use the direct integral decomposition (DID) of separable $\mathcal{H}_{\text{Diff}}$ associated with the self-adjoint operator $\hat{M}$ just as Eq. (3). Since zero is in the spectrum of $\hat{M}$ [127], the physical Hilbert space is just the (generalized) eigenspace of $\hat{M}$ with the eigenvalue zero, i.e., $\mathcal{H}_{\text{phys}} = \mathcal{H}_{\lambda=0}^\oplus$, where the physical inner product is $\langle \cdot | \cdot \rangle_{\text{phys}}$. The issue of quantum anomaly is represented in terms of the size of $\mathcal{H}_{\text{phys}}$ and the existence of semi-classical states. We list some open problems in master constraint programme for further research.

- **Kernel of Master Constraint Operator**
  
  Since the master constraint operator $\hat{M}$ is self-adjoint, it is a practical problem to find the DID of $\mathcal{H}_{\text{Diff}}$ and physical Hilbert space $\mathcal{H}_{\text{phys}} = \mathcal{H}_{\lambda=0}^\oplus$. However, the expression of master constraint operator is surely so complicated that it is very hard to obtain the DID representation of $\mathcal{H}_{\text{Diff}}$. Fortunately, the subalgebra generated by master constraints is an Abelian Lie algebra in the master constraint algebra. So one can employ group averaging strategy to solve the master constraint. Since $\hat{M}$ is self-adjoint, by Stone’s theorem there exists a strong continuous one-parameter unitary group,

$$\hat{U}(t) := \exp[it\hat{M}],$$

on $\mathcal{H}_{\text{Diff}}$. Then, given any diffeomorphism invariant cylindrical functions $\Psi_{\text{Diff}} \in \text{Cyl}^\star_{\text{Diff}}$, one can obtain algebraic distributions of $\mathcal{H}_{\text{Diff}}$ by a rigging map $\eta_{\text{phys}}$ from $\text{Cyl}^\star_{\text{Diff}}$ to $\text{Cyl}^\star_{\text{phys}}$, which are invariant under the action of $\hat{U}(t)$ and constitute a subset of the algebraic dual of $\text{Cyl}^\star_{\text{Diff}}$.

The rigging map is formally defined as

$$\eta_{\text{phys}}(\Psi_{\text{Diff}})[\Phi_{\text{Diff}}] := \int_{\mathbb{R}} \frac{dt}{2\pi} \langle \hat{U}(t)\Psi_{\text{Diff}}|\Phi_{\text{Diff}} \rangle_{\text{Diff}}.$$

The physical inner product is then defined formally as

$$\langle \eta_{\text{phys}}(\Psi_{\text{Diff}})|\eta_{\text{phys}}(\Phi_{\text{Diff}}) \rangle_{\text{phys}} := \eta_{\text{phys}}(\Psi_{\text{Diff}})[\Phi_{\text{Diff}}] = \int_{\mathbb{R}} \frac{dt}{2\pi} \langle \hat{U}(t)\Psi_{\text{Diff}}|\Phi_{\text{Diff}} \rangle_{\text{Diff}}.$$

Now the central task is to calculate the integrand$^6$

$$\langle \hat{U}(t)\Psi_{\text{Diff}}|\Phi_{\text{Diff}} \rangle_{\text{Diff}}$$

$^6$T. Thiemann’s idea of seeking for physical inner product and physical Hilbert space.
\[ = <\Psi_{Diff}|\exp(-it\hat{M})|\Phi_{Diff}>_{Diff} \]
\[ = \lim_{N \to \infty} <\Psi_{Diff}|(\exp(-\frac{it}{N}\hat{M}))^N|\Phi_{Diff}>_{Diff} \]
\[ = \lim_{N \to \infty} \sum_{[s_1]\ldots[s_{N-1}]} <\Psi_{Diff}|\exp[-\frac{it}{N}\hat{M}]|\eta(\Pi_{[s_1]})>_{Diff} \times \]
\[ <\eta(\Pi_{[s_1]})|\exp[-\frac{it}{N}\hat{M}]|\eta(\Pi_{[s_2]})>_{Diff} \times \]
\[ \ldots <\eta(\Pi_{[s_{N-2}]})|\exp[-\frac{it}{N}\hat{M}]|\eta(\Pi_{[s_{N-1}]})>_{Diff} \times \]
\[ <\eta(\Pi_{[s_{N-1}]})|\exp[-\frac{it}{N}\hat{M}]|\Phi_{Diff}>_{Diff}, \]

where \{\eta(\Pi_{[s]}))\}_{[s]=(\alpha,j,m,n)} \text{ associated with diffeomorphism class } [\alpha] \text{ of a graph are diffeomorphism invariant spin-network functions which form a system of complete orthonormal basis in } \mathcal{H}_{Diff}. \text{ Then one may consider the strategy of an approximate calculation:} \\
\[ <\eta(\Pi_{[s]}))|\exp[-\frac{it}{N}\hat{M}]|\eta(\Pi_{[s']}))>_{Diff} \]
\[ = <\eta(\Pi_{[s]}))|1-\frac{it}{N}\hat{M}|\eta(\Pi_{[s']}))>_{Diff} + O(\frac{1}{N^2}) \]
\[ = \delta_{[s][s']} - \frac{it}{N} Q M(\eta(\Pi_{[s]}),\eta(\Pi_{[s']}))) + O(\frac{1}{N^2}). \]

which might provide a possible method to calculate the physical inner product serving for the physical Hilbert space.

---

**Dirac Observables**

Classically, one can prove that a function \( \mathcal{O} \in \mathcal{C}^\infty(\mathcal{M}) \) is a weak observable with respect to the scalar constraint if and only if
\[ \{\mathcal{O}, \mathcal{O}, \mathcal{M}\}|_{\mathcal{M}} = 0. \]

We define \( \mathcal{O} \) to be a strong observable with respect to the scalar constraint if and only if
\[ \{\mathcal{O}, \mathcal{M}\}|_{\mathcal{M}} = 0, \]
and to be a ultra-strong observable if and only if
\[ \{\mathcal{O}, \mathcal{H}(N)\}|_{\mathcal{M}} = 0. \]

In quantum version, an observable \( \hat{\mathcal{O}} \) is a weak Dirac observable if and only if \( \hat{\mathcal{O}} \) leaves \( \mathcal{H}_{phys} \) invariant, while \( \hat{\mathcal{O}} \) is now called a strong Dirac observable.
if and only if ˆ\(O\) commutes with the master constraint operator ˆ\(M\). Given a bounded self-adjoint operator ˆ\(O\) defined on \(\mathcal{H}_{Diff}\), for instance, a spectral projection of some observables leaving \(\mathcal{H}_{Diff}\) invariant, if the uniform limit exists, the bounded self-adjoint operator defined by group averaging

\[
[\widehat{O}] := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \, \hat{U}(t)^{-1} \hat{O} \hat{U}(t)
\]

commutes with ˆ\(M\) and hence becomes a strong Dirac observable on the physical Hilbert space.

- Testing the Classical Limit of Master Constraint Operator

One needs to construct spatial diffeomorphism invariant semiclassical states to calculate the expectation value and fluctuation of the master constraint operator. If the results coincide with the classical values up to \(\hbar\) corrections, one can go ahead to finish our quantization programme with confidence.

6.4 ADM Energy Operator of Loop Quantum Gravity

To solve the dynamical problem in loop quantum gravity, one may consider to find a suitable Hamiltonian operator, in order to settle up the problem of time. A strategy for that is to seek for an operator corresponding to the ADM energy for asymptotically flat spacetime, which equivalently takes the form [116]

\[
\mathbf{H}_{ADM} = \lim_{S \to \partial \Sigma} -2\kappa \gamma^2 \int_{S} dS \frac{n_{a}}{\sqrt{|\det q|}} \bar{P}_{i}^{a} \partial_{a} \bar{P}_{j}^{b} \delta^{ij},
\]

(55)

where \(n_{a}\) is the normal co-vector of a close 2-sphere \(S\) and \(dS\) is the coordinate volume element on \(S\) induced from that of a asymptotically Cartesian coordinate system on \(\Sigma\).

In Ref. [116], Thiemann quantized the ADM energy [116] to obtain a positive semi-definite and self-adjoint operator \(\hat{E}_{ADM}\) as

\[
\hat{E}_{ADM} f_{\alpha} := 2\hbar^2 \kappa \gamma^2 \sum_{v \in V(\alpha) \cap \partial \Sigma} \frac{1}{\hat{V}_{v}} \delta^{ij} \hat{j}_{i}^{v} \hat{j}_{j}^{v} f_{\alpha},
\]

which is defined on an extension of \(\mathcal{H}_{kin}\) allowing for edges without compact support (see the infinite tensor product extension of kinematical Hilbert space [121]). Since the volume operator \(\hat{V}_{v}\) commutes with the "total angular momentum" operator \(\hat{J}^{v} = \delta^{ij} \hat{j}_{i}^{v} \hat{j}_{j}^{v}\), these two operators can be simultaneous diagonalized with respect to certain linear combinations of spin network states. The eigenvalues of \(\hat{E}_{ADM}\) are of the form \(2\hbar^2 \kappa \gamma^2 \Sigma_{v \in V(\alpha) \cap \partial \Sigma} \lambda_{v} (j_{v} + 1)/\lambda_{v}\), where \(\lambda_{v}\) is the eigenvalue of \(\hat{V}_{v}\). Thus we may think that the spin quantum numbers of spin network states are playing the role of the occupied numbers of
Fock states in quantum field theory, which provide a non-linear Fork decomposition for loop quantum gravity. This motivates us to call the future quantum dynamics of loop quantum gravity as Quantum Spin Dynamics (QSD) \[111\][112][113][114][115][116].

Moreover, \( \hat{E}_{ADM} \) trivially commutes with all constraint operators, since the gauge transformations are trivial at \( \partial \Sigma \). Hence \( \hat{E}_{ADM} \) is a true quantum Dirac observable. Then a meaningful time parameter can be selected by the continuous one-parameter unitary group generated by \( \hat{E}_{ADM} \), which leads to a ”Schrödinger equation” for QSD as:

\[
- \frac{i}{\hbar} \frac{\partial}{\partial t} f = \hat{E}_{ADM} f.
\]

7 Applications and Advances

This section is presented as a summary of the applications and some recent advances which are not discussed in the main content of the article. Some key open problems in the current research of loop quantum gravity will also be raised. However, our aim is to give a guidance to references for beginners rather than a concrete exploration.

7.1 Symmetric Models and Black Hole Entropy

It is well known that the most difficulty in general relativity is the singularity problem. The presence of singularities, such as the big bang and black holes, is widely believed to be a signal that classical general relativity has been pushed beyond the domain of its validity. Can loop quantum gravity at the present stage resolve the singularity problem? The other puzzling issue in general relativity is the thermodynamics of black holes \[36\][30][133]. Can one use loop quantum gravity to calculate the microscopic degrees of freedom which account for the black hole entropy?

As the full quantum dynamics of loop quantum gravity has not been solved completely, one then deals with the singularity problem in certain symmetric models by applying the ideas and techniques from loop quantum gravity. For simplifications, one generally freezes all but a finite number of degrees of freedom by imposing the suitable symmetry condition \[45\]. The symmetry-reduced models can also provide a mathematically simple arena to test the ideas and constructions in the full loop quantum gravity theory. The singularity problem was first considered in the so-called loop quantum cosmology models by imposing spatially homogeneity and (or) isotropy. The seminar work by Bojowald \[37\] shows that the big bang singularity is absent in loop quantum cosmology \[9\]. The result then leads to a new understanding on the initial condition problem in quantum cosmology \[38\][40]. Another remarkable result is that the loop quantum cosmological modification of Friedmann equation may cure the fine turning problem of the inflation potential, so that the inflation can arise naturally and exit gracefully due to the quantum geometry effect \[39\][51]. Loop quantum
cosmology is currently a very active research field. One may see Ref. [43] for a recent overview. For readers who want to know the fundamental structure of loop quantum cosmology, we refer to Ref. [9].

By imposing spatially spherical symmetry, one can study nonhomogeneous models, such as the Schwarzschild black hole [85], where the techniques from loop quantum gravity are also employed [11]. The treatment of these models is thus quite similar to that of loop quantum cosmology. It turns out that the interiors of the black holes may be singularity-free due to the quantum geometric properties [85] [86] [12] [8]. One may further study the “end state” of the gravitational collapse of matter fields inside a black hole [11] [7] and black hole evaporation [7]. There are still appealing issues which one may consider about the quantum black holes. The investigation in this direction has just started.

Besides the above noticeable models, there are also some other symmetric models, such as the Husain-Kuchar model [22] and static spacetimes [82], which have been studied from the constructions of loop quantum gravity.

We now turn to the black hole entropy calculation in loop quantum gravity. Recall that the definition of the event horizon of a black hole in general relativity concerns the global structure of the spacetime [132]. However, to account for black hole entropy by statistical calculations in loop quantum gravity, one needs to define locally the notion of a horizon, which can assume that the black hole itself is in equilibrium while the exterior geometry is not forced to be time independent. This is the so-called isolated horizon classically defined by Ashtekar et al (see Ref. [12] for a precise definition). It turns out that the zeroth and the first laws of black-hole mechanics can be naturally extended to type II isolated horizons [12] [6], where the horizon geometry is axi-symmetric. If one considers the spacetimes which contain an isolated horizon as an internal boundary, the action principle and the Hamiltonian description are well defined. Note that, in contrast with the symmetry-reduced models, here the phase space has an infinite number of degrees of freedom.

In quantum kinematical setting, it is natural to begin with a total Hilbert space $\mathcal{H} = \mathcal{H}_V \times \mathcal{H}_S$, where $\mathcal{H}_V$ is built from suitable functions of generalized connections in the bulk and $\mathcal{H}_S$ from suitable functions of generalized surface connections. The horizon boundary condition can then be imposed as an operator equation on $\mathcal{H}$. Taking account of the structure of the surface term in the symplectic structure, this quantum boundary condition implies that $\mathcal{H}_S$ is the Hilbert space of a $U(1)$ Chern-Simons theory on a punctured 2-sphere [5] [11]. To calculate entropy, one constructs the micro-canonical ensemble by considering only the subspace of the bulk theory with a fixed area of the horizon (a similar idea was raised in an earlier paper by Rovelli [95]). Employing the spectrum [38] of the area operator in $\mathcal{H}_V$, a detail analysis can estimates the number of Chern-Simons surface states consistent with the given area. One thus obtains the (black hole) horizon entropy, whose leading term is indeed proportional to the horizon area [5]. However, the expression of the entropy agrees with the Hawking-Bekenstein formula only if one chooses a particular Barbero-Immirzi parameter $\gamma_0$ (see Ref. [59] for a recent discussion on the choice of $\gamma_0$). The non-trivial fact is that this theory with fixed $\gamma_0$ can yield the Hawking-Bekenstein
value of entropy of all isolated horizons, irrespective of the values of charges, angular momentum and cosmology constant, the amount of distortion or hair \[17\]. The sub-leading term has also been calculated and shown to be proportional to the logarithm of the horizon area \[78\]. Note that in the entropy calculation the quantum Gauss and diffeomorphism constraints are crucially used, while the final result is insensitive to the details of how the Hamiltonian constraint is imposed.

### 7.2 Semiclassical Analysis and Quantum Dynamics

As shown in section 6, either the Hamiltonian constraint operator \( \hat{H}(N) \) or the master constraint operator \( \hat{M} \) can be well defined in the framework of loop quantum gravity. However, since the Hilbert spaces \( \mathcal{H}_{\text{kin}} \) and \( \mathcal{H}_{\text{Diff}} \), the operators \( \hat{H}(N) \) and \( \hat{M} \) are constructed in such ways that are drastically different from usual quantum field theory, one has to check whether the constraint operators and the corresponding algebras have correct classical limits with respect to suitable semiclassical states.

As the first purpose is to check the classical limit of \( \hat{H}(N) \), kinematical coherent states are constructed in two different approaches. One leads to the so-called complexifier coherent states proposed by Thiemann et al \[118\] \[119\] \[120\] \[121\]. The other is promoted by Varadarajan \[128\] \[129\] \[130\] and developed by Ashtekar et al \[16\] \[13\]. The former approach is somehow motivated by the coherent state construction for compact Lie groups \[71\]. One begins with a positive function \( C \) (complexifier) on the classical phase space and arrives at a "coherent state" \( \psi_m \), which more possibly belongs to the dual space \( \mathbb{Cyl}^* \) rather than \( \mathcal{H}_{\text{kin}} \). However, one may consider the "cut-off state" of \( \psi_m \) with respect to a finite graph as a graph-dependent coherent state in \( \mathcal{H}_{\text{kin}} \) \[122\]. By construction, the coherent state \( \psi_m \) is an eigenstate of an annihilation operator coming also from the complexifier \( C \) and hence has desired semiclassical properties \[119\] \[120\]. In the latter approach, one gains insights from the comparison between the polymer representation and Fock representation of quantum Maxwell field. A "Laplacian operator" can then be defined on \( \mathcal{H}_{\text{kin}} \) \[23\] \[16\], from which one may define a candidate coherent state \( \Phi_0 \), also in \( \mathbb{Cyl}^* \), corresponding to the Minkowski spacetime. To calculate the expectation values of kinematical operators, one only consider the so-called "shadow state" of \( \Phi_0 \), which is the restriction of \( \Phi_0 \) to a given finite graph. However, the construction of shadow states is subtly different from that of cut-off states. Both approaches have their own virtues. The complexifier approach provides a clean construction mechanism and manageable calculation method, while the Laplacian operator approach is related closely with the well-known Fock vacuum state. One may find comparisons of the two approaches from both sides \[123\] \[17\].

Although powerful tools have been developed to construct semiclassical states, the analysis of the classical limits of the Hamiltonian constraint operator and the corresponding constraints algebra is yet to be carried out. Furthermore, to do semiclassical analysis of the master constraint operator, one still needs diffeomorphism invariant coherent states in \( \mathcal{H}_{\text{Diff}} \) (see Refs. \[123\] and \[10\] for
recent progress in this aspect). Moreover, a crucial question of the semiclassical analysis is whether there are enough physical semiclassical states in certain unknown physical Hilbert space of loop quantum gravity, which may correspond to all classical solutions of the Einstein equation. This is the final theoretical criterion for any candidate theory of quantum gravity with general relativity as its classical limit. The physical semiclassical states are also relevant, if one wishes to use the full theory rather than symmetric models to analyze cosmology and black holes. In the matter coupled to gravity content, one would like to check whether the coupled quantum system approaches quantum field theory in curved spacetime in suitable semiclassical limit. This issue is being studied at the kinematical level [108][109].

In the light of the canonical quantization of loop quantum gravity, the so-called spin foams are devised as histories traced out by ”time evolution” of spin networks, which provide a path-integral approach to quantum dynamics [28]. One expects that the path integral can be used to compute ”transition amplitudes” and extract physical states, which may shed new light on the quantum Hamiltonian constraint and on the physical inner product. In the successful Barrett-Crane model and its various modifications [31][32], one regards classical general relativity as a topological field theory (the so-called BF theory), supplemented with an algebraic constraint. An interesting discovery in this approach is that a certain modified version of the Barrett-Crane model is equivalent to a manageable group field theory [90][92][50]. It then turns out that the sum over geometries for a fixed discrete topology is finite. For a detail exploration of spin foam models, we refer to the recent review article [91] and references therein. Although many developments in spin foam approach are very interesting from a mathematical physics perspective, their significance to quantum gravity is still less clear [17]. An obvious weakness in most of these works is that the discrete topology is fixed, whence the the issue of summing over all topologies remains largely unexplored. However, it is expected that a judicious combination of methods from the canonical treatment of the Hamiltonian constraint and spin foam models may lead to significant progress in both approaches. There are also other approaches to deal with the quantum dynamics such as, the Vassiliev knot invariants approach [33] and the ”consistent discretization” approach [62][63]. Here we will not introduce their concrete ideas. One may find the detail exploration of the former in Refs. [34] and [35], and a recent summary for the latter in Ref.[64].

In summary, the full treatments of the semiclassical analysis and quantum dynamics are entangled with each other and expected to be settled together. These are the core open problems in loop quantum gravity, which are now under investigations.

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