We study the strong and electromagnetic decay properties of scalar mesons above 1 GeV within a chiral approach. The scalar-isoscalar states are treated as mixed states of quarkonia and glueball configurations. A fit to the experimental mass and decay rates listed by the Particle Data group is performed to extract phenomenological constraints on the nature of the scalar resonances and to the issue of the glueball decays. A comparison to other experimental results and to other theoretical approaches in the scalar meson sector is discussed.


Keywords: Scalar and pseudoscalar mesons, glueball, effective chiral approach, strong and electromagnetic decays

I. INTRODUCTION

The unique interpretation of scalar mesons constitutes an unsolved problem of hadronic QCD. Below the mass scale of 2 GeV various scalar states are encountered: the isoscalar resonances $\sigma = f_0(400-1200)$, $f_0(980)$, $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$, the isovectors $a_0(980)$ and $a_0(1450)$ and the isodoublets $K^*_0(800)$ and $K^*_0(1430)$. The existence of the $K^*_0(800)$ is still not well established and omitted from the summary tables of [1]. From a theoretical point of view one expects the scalar quark-antiquark ground state nonet $0^{++}$, a scalar-isoscalar glueball, which lattice QCD predicts to be the lightest gluonic meson with a mass between 1.4-1.8 GeV [2], and possibly other exotic states (non $\bar{q}q$ states), like e.g. four quark states or mesonic molecules [3]. Various interpretations of and assignments for the physical scalar resonances in terms of the expected theoretical states have been proposed (see, for instance, the review papers [4, 5] and references therein).

In this work we follow the original assignment of Ref. [6], where in a minimal scenario the bare quarkonia states $N \equiv \sqrt{1/2}(\bar{u}u + \bar{d}d) = \bar{n}n$, $S \equiv \bar{s}s$ and the bare scalar glueball $G$ mix, resulting in the three scalar-isoscalar resonances $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$. Such a mixing scheme has been previously investigated by many authors, as for example in the lattice study of [7] or within the model approaches of [8]-[13].

The mesons $a_0(1450)$ and $K^*_0(1430)$ are considered as the $I = 1$ and $I = 1/2$ quarkonia $J^{PC} = 0^{++}$ states. In this way the low-lying scalar quarkonia nonet is located in the energy range of 1-2 GeV, where the other p-wave nonets of tensor $(2^{++})$ and pseudovector mesons $(1^{++})$ are also situated. Masses of selected scalar quarkonia states were also estimated on the Lattice; in Ref. [14] the $I = 1$ scalar quarkonium state is predicted to have a mass of $M_{a_0} = 1.51 \pm 0.19$. This result favours the interpretation of the state $a_0(1450)$ (and not $a_0(980)$) as the isovector ground-state scalar quarkonium. However, previous lattice studies (as in [15]; see also [14, 16] and Refs. therein) find different results. Scalar resonances below 1 GeV can possibly be interpreted as four quark states or mesonic molecules. In [17] the hypothesis of Jaffe’s four quark states was studied on the Lattice, where the masses of four-quark states are found to be lighter than 1 GeV. Recent attempts in the context of chiral perturbation theory to describe the
scalar states below 1 GeV as "dynamically generated" resonances, i.e. states which do not survive in the large \( N_c \) limit \cite{18,19}, have also been performed. In this scheme the nature of the scalars below 1 GeV can also be related to four quark configurations. This can be viewed as a further indication that scalar quarkonia states are located at masses above 1 GeV.

In this work, starting from an effective chiral Lagrangian \cite{13} derived in Chiral Perturbation Theory (ChPT) \cite{20,21,22}, we perform a tree-level analysis of the strong and electromagnetic decays of scalar mesons settled in the energy range between 1 and 2 GeV. The scalar glueball is introduced as an extra-flavor singlet composite field with independent couplings to pseudoscalar mesons (and to photons, although suppressed). Although a chiral approach cannot be rigorously justified at this energy scale, since loop corrections could be large, we intend to use this framework as a phenomenological tool to extract possible glueball-quarkonia mixing scenarios from the observed decays.

The scalar glueball \( G \) mixes with the scalar quarkonia fields \( N \equiv \sqrt{1/2}(\bar{u}u + \bar{d}d) = \bar{m}m \) and \( S \equiv \bar{s}s \) in accord with flavor blindness. We also consider in this scheme a possible direct mixing of the quarkonia fields \( N \) and \( S \). The origin of such mixing can be driven by instantons \cite{23}. The presence of a (even small) flavor mixing in the scalar-isoscalar sector can sensibly affect the phenomenology.

In the presented approach the glueball decay into two pseudoscalar mesons is occurring by two mechanisms: a) through mixing, that is the glueball \( G \) acquires a quarkonium component, which subsequently decays into two pseudoscalars; b) direct decay of the glueball component \( G \) into two pseudoscalars without an intermediate scalar quarkonium \cite{6,8,10}. In \cite{6,10} this direct decay is argued to be suppressed as based on arguments of the strong coupling expansion, while in the phenomenological fit of \cite{8} it dominates. Here we also address the question, if the direct decay is needed to explain the decay phenomenology of the scalar-isoscalar resonances \( f_0(1370) \), \( f_0(1500) \) and \( f_0(1710) \).

For consistency we also consider the strong decays of the isovector and isodoublet scalar states as well, trying to highlight the difficulties and the open issues, also comparing with previous works on this subject. Following the idea of \cite{20} we analyze deviations from the large \( N_c \) limit in the framework of the proposed scalar-quarkonia assignment (which differs from \cite{20}), which turn out to be relatively small in the phenomenology.

The paper is organized as follows: in Section II we discuss the effective Lagrangian related to the scalar quarkonia-glueball mass spectrum and the strong and electromagnetic decays. In Section III we determine a phenomenological fit to the experimental data listed in \cite{1} by first neglecting the direct decay of the glueball component. In Section IV we also allow for a direct glueball decay studying its influence on the results. There we use the lattice data of \cite{24}, where an approximate calculation of the glueball decays into two pseudoscalar mesons has been performed, to constrain our analysis. Finally, in Section V we summarize our results and draw conclusions.

II. THE MODEL

A. The Lagrangian

The strong and electromagnetic decays of scalar mesons are based on an effective chiral Lagrangian \( \mathcal{L}_{\text{eff}} \) as derived in Chiral Perturbation Theory (ChPT) \cite{20,21,22}. The Lagrangian involves the nonets of pseudoscalar and of scalar
mesons,
\[ \mathcal{P} = \frac{1}{\sqrt{2}} \sum_{i=0}^{8} P_i \lambda_i, \quad \mathcal{S} = \frac{1}{\sqrt{2}} \sum_{i=0}^{8} S_i \lambda_i, \]  

the electromagnetic field and, in addition, a new degree of freedom, the bare glueball field \( G \), which is treated as a flavor-blind mesonic field. The lowest order effective Lagrangian \( \mathcal{L}_{\text{eff}} \) in the large \( N_c \) limit including \( 1/N_c \) corrections reads
\[
\mathcal{L}_{\text{eff}} = \frac{F^2}{4} \langle D_\mu U D^\mu U^\dagger + \chi_+ \rangle + \frac{1}{2} \langle D_\mu S D^\mu S - M^2 S S^2 \rangle + \frac{1}{2} \langle \partial_\mu G \partial_\mu G - M^2_G G^2 \rangle \\
+ c^e_3 \langle S u_\mu u^\mu \rangle + c^e_m \langle S \chi_+ \rangle + \frac{c^e_3}{\sqrt{3}} G \langle u_\mu u^\mu \rangle + \frac{c^e_m}{\sqrt{3}} G \langle \chi_+ \rangle \\
+ c^e_3 \langle S F^+_{\mu\nu} F^{+\mu\nu} \rangle + \frac{c^e_3}{\sqrt{3}} G \langle F^+_{\mu\nu} F^{+\mu\nu} \rangle + \mathcal{L}_{\text{mix}}^P + \mathcal{L}_{\text{mix}}^S.
\]

Here the symbol \( \langle ... \rangle \) denotes the trace over flavor matrices. The constants \( c^e_3, c^e_m, c^e_3, c^e_m, c^e_3 \) and \( c^e_3 \) define the coupling of scalar fields and of the bare glueball to pseudoscalar mesons and photons, respectively. We use the standard notation for the basic blocks of the ChPT Lagrangian \cite{21}: \( U = \hat{u}^2 = \exp(i\mathcal{P} \sqrt{2}/F) \) is the chiral field collecting pseudoscalar fields in the exponential parametrization, \( D_\mu \) denotes the chiral and gauge-invariant derivative, \( u_\mu = iu^\dagger D_\mu u^\dagger \) is the chiral field, \( \chi_+ = u^\dagger \chi u^\dagger \pm u \chi \dagger u^\dagger \), \( \chi = 2B(s + ip) \), \( s = M + \ldots \) and \( F^+_{\mu\nu} = u^\dagger F_{\mu\nu} Qu + u F_{\mu\nu} Q u^\dagger \), where \( F_{\mu\nu} \) is the stress tensor of the electromagnetic field. The charge and the mass matrix of current quarks are denoted by \( Q = e \text{ diag}(2/3, -1/3, -1/3) \) and \( M = \text{ diag}(\hat{m}, \tilde{m}, m_s) \) (we restrict to the isospin symmetry limit with \( m_u = m_d = \hat{m} \)); \( B \) is the quark vacuum condensate parameter and \( F \) the pion decay constant.

The masses of the octet pseudoscalar mesons in the leading order of the chiral expansion (first term of the Lagrangian) are given by
\[
M^2_\eta = 2\hat{m}B; \quad M^2_K = (\hat{m} + m_s)B; \quad M^2_{\pi^0} = \frac{2}{3}(\hat{m} + 2m_s)B.
\]

The contribution to the mass of \( \eta^0 \) in leading order is
\[
M^2_{\eta^0} = \frac{2}{3}(2\hat{m} + m_s)B,
\]
i.e. \( \eta^0 \) is a Goldstone boson in the large \( N_c \) and in the chiral limits.

Following \cite{22}, we encode in \( \mathcal{L}_{\text{mix}}^P \) an extra-contribution to the mass of \( \eta^0 \) (due to the axial anomaly) and the \( \eta^0-\eta^8 \) mixing term:
\[
\mathcal{L}_{\text{mix}}^P = -\frac{1}{2} \gamma_P (\eta^0)^2 - z_P \eta^0 \eta^8,
\]
(the parameters \( \gamma_P \) and \( z_P \) are in turn related to the parameters \( M_{\eta^1} \) and \( \tilde{d}_m \) of \cite{22}). The physical diagonal states \( \eta \) and \( \eta' \) are given by
\[
\eta^0 = \eta' \cos \theta_P - \eta \sin \theta_P, \quad \eta^8 = \eta' \sin \theta_P + \eta \cos \theta_P,
\]
where \( \theta_P \) is the pseudoscalar mixing angle. We follow the standard procedure \cite{20, 22, 22, 22, 26} and diagonalize the corresponding \( \eta^0-\eta^8 \) mass matrix to obtain the masses of \( \eta \) and \( \eta' \). By using \( M_\pi = 139.57 \) MeV, \( M_K = 493.677 \) MeV (the physical charged pion and kaon masses), \( M_\eta = 547.75 \) MeV and \( M_{\eta'} = 957.78 \) MeV the mixing angle is
Lagrangian \( L \) quarkonia states and in the glueball-quarkonia sector. With the use of Eqs. (7) and (8) the explicit expression of the pseudoscalar mesons (for a detailed discussion see Refs. [27]).

non-zero mixing, as described by a non-vanishing parameter \( \theta \) consistently use the corresponding tree-level result of the absolute value of the pseudoscalar mixing angle [25]; in our work we restrict to the tree-level evaluation, we therefore determine as \( \theta_p = -9.95^\circ \), which corresponds to the tree-level result (see details in Ref. [23]). Correspondingly one finds \( M_{\rho^0} = 948.10 \text{ MeV} \) and \( z_p = -0.105 \text{ GeV} \). Higher order corrections in ChPT cause a doubling of the absolute value of the pseudoscalar mixing angle [25]; in our work we restrict to the tree-level evaluation, we therefore use the corresponding tree-level result of \( \theta_p = -9.95^\circ \). In the present approach we do not include the neutral pion when considering mixing in the pseudoscalar sector, because we work in the isospin limit. This mixing is small, and can be safely neglected when studying the decay of scalar resonances into two pseudoscalars. Similarly, for all pseudoscalar mesons we use the unified leptonic decay constant \( F \), which is identified with the pion decay constant \( F = F_\pi = 92.4 \text{ MeV} \). A more accurate analysis including higher orders should use the individual couplings of the pseudoscalar mesons (for a detailed discussion see Refs. [27]).

### B. Scalar quarkonia - glueball mixing

In this subsection we discuss the glueball-quarkonia mixing. For this reason we restrict to the following part of the effective Lagrangian [2]:

\[
L^S = -\frac{1}{2} \langle D_\mu S D^\mu S - M^2_S S^2 \rangle + \frac{1}{2} (\partial_\mu G \partial^\mu G - M^2_G G^2) + L^S_{mix}
\]

where, besides the scalar quarkonia nonet, also the scalar glueball field \( G \) has been introduced. In the large \( N_c \) limit all the states of the quarkonia nonet have the same nonet mass \( M_S \) and the glueball is decoupled from the quarkonia sector. Deviations from this limit are encoded in \( L^S_{mix} \), where the glueball-quarkonia mixing is introduced and where the degeneracy of the nonet states is lifted.

For what concerns the explicit mass term and the next-to-leading order \( 1/N_c \) terms in the quarkonia sector we follow [20]. Including additionally a possible breaking of the Gell-Mann-Okubo (GMO) mass relation and the glueball-quarkonia mixing under the hypothesis of flavor blindness, we have:

\[
L^S_{mix} = e^S_m \langle S^2 \chi \rangle + k^S_m S_0 \langle \chi \rangle - \frac{M^2_{S_0}}{2} S^2 - \gamma S_0 - \frac{M^2_{S_0}}{2} S^2_0 - \sqrt{3} f GS_0.
\]

The parameter \( e^S_m \) describes the strength of flavor symmetry breaking derived from the non-zero values of the current quark masses, the parameters \( k^S_m \) and \( \gamma S_0 \) describe the order \( 1/N_c \) terms. The parameter \( \gamma S_a \) and the related term in the Lagrangian is not derived in the \( N_c \) expansion (it is in fact absent in [20]), but it describes violations of the GMO mass formula (indeed a result from higher orders in the chiral expansion). Finally, the parameter \( f \) is the glueball-quarkonia mixing strength. Note that \( G \), being a flavor singlet, couples only to the flavor singlet quarkonium state \( S_0 \) in the flavor-blind mixing limit. The glueball and the quarkonia sector decouple in the large \( N_c \) limit [30]: a non-zero mixing, as described by a non-vanishing parameter \( f \), takes into account a possible deviation from the large \( N_c \) limit, as the parameters \( k^S_m \) and \( \gamma S_0 \) in the quarkonia sector.

The terms contained in \( L^S_{mix} \) lead to mass shifts of the nonet masses, while also introducing mixing both among quarkonia states and in the glueball-quarkonia sector. With the use of Eqs. (7) and (8) the explicit expression of the Lagrangian \( L^S \) reads:

\[
L^S = -\frac{1}{2} \bar{a}_0 (\Box + M^2_{a_0}) a_0 - K^*_{0} (\Box + M^2_{K^*_{0}}) K^*_0 - \frac{1}{2} G (\Box + M^2_G) G \\
- \frac{1}{2} M^2_{S_0} S_0^2 - \frac{1}{2} M^2_{S_0} S_0^2 - z S_0 S_0 - \sqrt{3} f GS_0
\]

(9)
where \( \tilde{a}_0 \) is the isovector collecting \( a_0^\pm \) and \( a_0^0 \) fields; \( K_0^* \) and \( K_0^\dagger \) are the doublets of \((K^0_+, \bar{K}^*0^0)\) and \((K^0_-, \bar{K}^*0^0)\) mesons, respectively (see A), and where the corresponding masses are given by (see Eqs. (7) and (8)):

\[
\begin{align*}
M^2_{a_0} &= M^2_S - 4\epsilon^2_m M^2_\pi, \\
M^2_{K_0^*} &= M^2_S - 4\epsilon^2_m M^2_K, \\
M^2_{\Sigma_0^n} &= M^2_S (1 + \gamma_{S_{\Sigma_0^n}}) - 4\epsilon^2_m (4M^2_K - M^2_\pi), \\
M^2_{\Sigma_0^0} &= M^2_S (1 + \gamma_{S_{\Sigma_0^0}}) - 4\epsilon^2_m (2M^2_K + M^2_\pi), \\
z_S &= \frac{8\sqrt{2}}{3} \left( \epsilon^2_m + \frac{\sqrt{3}}{2}k^S_m \right) (M^2_K - M^2_\pi). 
\end{align*}
\]

By inverting we get:

\[
\begin{align*}
M^2_S &= M^2_{a_0} + \frac{M^2_\pi(M^2_{a_0} - M^2_{K_0^*})}{(M^2_\pi - M^2_{a_0})}, \\
\epsilon^2_m &= \frac{M^2_{a_0} - M^2_{K_0^*}}{4(M^2_\pi - M^2_{a_0})}, \\
\gamma_{S_{\Sigma_0^n}} &= \frac{M^2_{\Sigma_0^n} - M^2_\Sigma + \frac{2}{3}\epsilon^2_m (4M^2_K - M^2_\pi)}{M^2_{\Sigma_0^n}}, \\
\gamma_{S_{\Sigma_0^0}} &= \frac{M^2_{\Sigma_0^0} - M^2_\Sigma + \frac{2}{3}\epsilon^2_m (2M^2_K + M^2_\pi)}{M^2_{\Sigma_0^0}}, \\
k^S_m &= \frac{2}{\sqrt{3}} \left( \frac{3z_S}{8\sqrt{2}(M^2_K - M^2_\pi)} - \epsilon^2_m \right).
\end{align*}
\]

We can immediately deduce \( M_S \) and \( \epsilon^2_m \) for the considered assignment (which differs from [20]) by using the experimental masses \( M_{a_0} = M_{a_0(1450)} = 1.474 \pm 0.019 \text{ GeV} \) and \( M_{K_0^*} = M_{K_0^*(1430)} = 1.416 \pm 0.006 \text{ GeV} \): \( M_S = 1.479 \text{ GeV}, \ \epsilon^2_m = 0.199 \).

Note that \( \epsilon^2_m \) is positive, contrary to the other nonets [20]. The numerical values for the other constants \( k^S_m, \gamma_{S_{\Sigma_0^0}} \) and \( \gamma_{S_{\Sigma_0^n}} \) (which should be small if the large \( N_c \) limit and chiral symmetry still applies approximately for the scalar nonet) are determined by a fit to data.

Finally we discuss the mass relation in the octet sector. From the first three equations in (11) we have:

\[
3M^2_{\Sigma_0} = 4M^2_{K_0^*} - M^2_{a_0} + 3\gamma_{S_{\Sigma_0}} M^2_S,
\]

where the term proportional to \( \gamma_{S_{\Sigma_0}} \) includes deviations from the GMO limit.

Glueball-quarkonia mixing is usually set up in the basis of fields \( N \equiv \sqrt{1/2}(u\bar{d} + d\bar{u}) \equiv \bar{u}n \) and \( S \equiv \bar{s}s \) instead of the octet and singlet fields \( S_0 \) and \( S_8 \). The connection is given by

\[
S_0 = \sqrt{2/3} N + \sqrt{1/3} S, \quad S_8 = \sqrt{1/3} N - \sqrt{2/3} S.
\]

Therefore, the scalar-isoscalar part of the Lagrangian [9] involving the fields \( N, S \) and \( G \) is rewritten as:

\[
\mathcal{L}^S = -\frac{1}{2} \Phi (\Box + M^2_S) \Phi
\]
with

\[ \Phi = \begin{pmatrix} N \\ G \\ S \end{pmatrix}, \quad M^2_\Phi = \begin{pmatrix} M^2_N \sqrt{2}f \varepsilon \\ \varepsilon f \\ M^2_S \end{pmatrix}. \] (16)

The bare masses \( M_N, M_S \) and the flavor mixing parameter \( \varepsilon \) are determined by inserting (14) into (9) with:

\[ M^2_N = \frac{2}{3} M^2_{S0} + \frac{1}{3} M^2_{S8} + \frac{2\sqrt{2}}{3} z_S, \]
\[ M^2_S = \frac{1}{3} M^2_{S0} + \frac{2}{3} M^2_{S8} - \frac{2\sqrt{2}}{3} z_S, \]
\[ \varepsilon = \frac{\sqrt{2}}{3} (M^2_{S0} - M^2_{S8}) - \frac{1}{3} z_S. \] (17)

The inverted relations are:

\[ M^2_{S0} = \frac{2}{3} M^2_N + \frac{1}{3} M^2_S + \frac{2\sqrt{2}}{3} \varepsilon, \]
\[ M^2_{S8} = \frac{1}{3} M^2_N + \frac{2}{3} M^2_S - \frac{2\sqrt{2}}{3} \varepsilon, \]
\[ z_S = \sqrt{\frac{2}{3}} (M^2_N - M^2_S) - \frac{1}{3} \varepsilon. \] (18)

As we will show in the next section, as a result of the fit we will determine the values \( M_N, M_S, \) and \( \varepsilon \), from which we can deduce \( M_{S0}, M_{S8} \) and \( z_S \) or, equivalently by (11), the parameters \( k^S_{m_n}, \gamma_{S0} \) and \( \gamma_{S8} \) of the Lagrangian (7).

The parameter \( f \) is the quarkonia-glueball mixing strength, analogous to the parameter \( z \) of Refs. [6, 7, 8, 9, 10].

The mixing strength \( z \) refers to the quantum mechanical case, where the mass matrix is linear in the bare mass terms. The connection between \( f \) and \( z \), discussed in Refs. [9, 12], leads to the approximate relation \( f \simeq 2z M_G \).

For the glueball-quarkonia mixing we work in the limit of flavor blindness. The issue of flavor blindness breaking at the mixing level has been considered in [6, 8, 9, 10], where flavor mixing is considered to be of higher-order. However, a substantial \( N-S \) mixing in the scalar sector is the starting point of the analysis of Refs. [28, 29]. The origin of quarkonia flavor mixing is, according to [28, 29], connected to instantons as in the pseudoscalar channel, but with opposite sign (see also [23]). Such a phase structure is also found in the NJL model including the six-point 't Hooft interaction term [31, 32]). The mixed physical fields are predicted to be a higher lying state of flavor structure \( [N \sqrt{2} - S]/\sqrt{3} \) and a lower one with \( [N + S \sqrt{2}]/\sqrt{3} \).

Here we study the case \( \varepsilon \neq 0 \), more precisely \( \varepsilon > 0 \), which leads to the same phase structure as in Ref. [28, 29], but the quantitative results and interpretation will differ.

For the glueball-quarkonia mixing we work in the limit of flavor blindness. The issue of flavor blindness breaking at the mixing level has been considered in [7, 8, 12]. This effect can be taken into account when introducing in the mass mixing matrix \( M^2_\Phi \), defined in (15), an additional parameter \( r \) as:

\[ M^2_\Phi = \begin{pmatrix} M^2_N \sqrt{2}f \cdot r \varepsilon \\ \varepsilon f \cdot r M^2_S \end{pmatrix}. \] (19)

For \( r = 1 \) we regain the original expression, \( r \neq 1 \) takes into account a possible deviation from this limit. Determination of this parameter on the Lattice [7] results in \( r = 1.20 \pm 0.07 \), in the fit of [8] a value of \( r = 1 \pm 0.3 \) is obtained. In the microscopic quark/gluon model of [12] the value \( r \sim 1.1-1.2 \) is deduced. All these findings point to possible small deviations from the flavor-blind mixing configuration. Hence, in the following we will restrict to the limit \( r = 1 \).
The orthogonal physical states assigned as \( f_1 \equiv f_0(1370) \), \( f_2 \equiv f_0(1500) \) and \( f_3 \equiv f_0(1710) \) resulting from Eq. (15) are obtained by diagonalization of \( M^s_3 \) (Eq. (10)) with the transformation matrix \( B \) as

\[
BM^s_3B^T = M^f_3 = \begin{pmatrix}
M^s_{f_1} & 0 & 0 \\
0 & M^s_{f_2} & 0 \\
0 & 0 & M^s_{f_3}
\end{pmatrix},
\]

where the eigenvalues of \( M^f_3 \) represent the masses of the physical states \( f_1 \equiv f_0(1370) \), \( f_2 \equiv f_0(1500) \) and \( f_3 \equiv f_0(1710) \). The physical states \(|i\rangle\), with \( i = f_1, f_2, f_3 \), are then given in terms of the bare states as

\[
|i\rangle = \sum_{j=N,G,S} B_{ij} \ |j\rangle.
\]

In a covariant framework, where mixing of bound states is studied on a elementary level, it is not possible to define an orthogonal mixing matrix \( B \) \((B \cdot B^T = B^T \cdot B = I_3)\) connecting the physical to the bare fields \([12, 31, 32]\). However, as shown in the model of \([12]\) for the glueball-quarkonia system, deviations from orthogonality of the mixing matrix \( B \) are small, therefore justifying a Klein-Gordon mixing scenario as in the present approach.

### C. Strong and electromagnetic decays of scalar states

The generic expression for the strong decay width of a scalar state \( s \) (both quarkonia and gluonium) into two pseudoscalar mesons \( p_1 \) and \( p_2 \) is given by:

\[
\Gamma_{s \rightarrow p_1p_2} = \frac{\lambda^{1/2}(M_s^2, M_{p_1}^2, M_{p_2}^2)}{16 \sqrt{\pi} M_s^3} \gamma_{sp_1p_2} \ |M_{s \rightarrow p_1p_2}|^2
\]

where \( \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz \) is the Källen triangle function. The factor \( \gamma_{sp_1p_2} = 1/2 \) or 1 stands for identical or different particles in the final state. In the case of a degenerate isomultiplet an average over the isospin configurations is understood.

The matrix elements \( M_{s \rightarrow p_1p_2} \) are expressed in terms of parameters \( c_d^s, c_m^s, c_d^0, c_m^0 \) (see Eq. (2)). The parameters \( c_d^s \) and \( c_m^s \) refer to the scalar quarkonia decays, \( c_d^0 \) and \( c_m^0 \) to the direct glueball decays. The complete expressions for the two-pseudoscalar decay widths (matrix elements included) of the scalar resonances are given in the Appendix [13].

The decay of the bare glueball states embedded in the physical \( f_0 \) states can proceed in two ways (see Introduction). Mixing expressed by the parameter \( f \) corresponds to the conversion of a bare glueball to a quarkonia state, which in turn decays into a pseudoscalar meson pair. For \( f \neq 0 \) no physical \( f_0 \) state is a pure glueball, and the decays of the quarkonia components are driven by the amplitudes \( B_{iN} \) and \( B_{iS} \), which depend on \( f \).

The direct decay of the glueball component, without proceeding via an intermediate scalar quarkonium state, is contained in the parameters \( c_d^0 \) and \( c_m^0 \). For a microscopic description of this mechanism we refer to Refs. [4, 10], where the possible transition of the glueball to two pseudoscalar mesons is described by processes containing four internal quark/antiquark lines. The parameters \( c_d^0 \) and \( c_m^0 \) are attached to the gluonic amount \( B_{iG} \) of the state \( i \), where \( i = f_1, f_2, f_3 \). With the normalization adopted in Eq. (2) the limit \( c_d^0 = c_d^s \), \( c_m^0 = c_m^s \) refers to a direct glueball decay strength equivalent to the decay of a flavor-singlet quarkonia state. In the large \( N_c \) limit the quarkonia decay constants \( c_d^0 \) and \( c_m^0 \) scale as \( N_c^{1/2} \), while the glueball decay constants \( c_d^0 \) and \( c_m^0 \) scale as \( N_c \) [30]. Large \( N_c \) arguments therefore suggest that the direct glueball decay is suppressed with respect to quarkonia decays. When performing in
the following a fit to data, we first study the large \( N_c \) limit by setting the direct glueball decay parameters \( c^d_g \) and \( c^m_g \) to zero.

The matrix element for the two-photon decay of the scalar \((0^{++})\) state has the manifestly gauge-invariant form

\[
M_{s \rightarrow \gamma \gamma} = e^2 g_{s \gamma \gamma} (g^\mu \nu q_1 q_2 - q_1^\mu q_2^\nu) \epsilon_\mu(q_1) \epsilon_\nu(q_2)
\]

where \( q_1 \) and \( q_2 \) are the photon four-momenta, \( p = q_1 + q_2 \) is the scalar state momentum and \( g_{s \gamma \gamma} \) is the \( s \gamma \gamma \) coupling constant. The decay width of the transition \( s \rightarrow \gamma \gamma \) is given by

\[
\Gamma_{s \rightarrow \gamma \gamma} = \frac{1}{32 \pi M_s} \sum_{pol} |M_{s \rightarrow \gamma \gamma}|^2 = \frac{\pi}{4} \alpha^2 g_{s \gamma \gamma}^2 M_s^3,
\]

where \( \alpha = e^2/(4\pi) = 1/137 \) is the fine structure constant.

The coupling constants \( g_{s \gamma \gamma} \) for the bare states \( S_0, S_8, G \) and for the isovector state \( a_0^0 \) are directly calculated using the effective Lagrangian (2):

\[
g_{S0 \gamma \gamma} = \frac{32}{3\sqrt{3}} e^s, \quad g_{S8 \gamma \gamma} = \frac{8}{3} \sqrt{\frac{2}{3}} e^s,
\]

\[
g_{G \gamma \gamma} = \frac{32}{3\sqrt{3}} e^e, \quad g_{a_0 \gamma \gamma} = \frac{8}{3} \sqrt{2} e^e.
\]

The parameter \( e^s \) refers to the quarkonia components, while \( e^e \) contains the direct coupling of the glueball component to the electromagnetic fields. Latter coupling constant \( e^e \) is supposed to be suppressed, since gluons do not couple directly to the photon field. However, an intermediate state of two vector mesons for example can in the framework of vector meson dominance generate a coupling of the glueball to the two-photon final state; this coupling is supposed to be suppressed and will not be considered in the numerical analysis. Using the identities for the field transformations (14) and (21) we can derive the couplings for the bare \( N \equiv \sqrt{1/2(\bar{u}u + \bar{d}d)} \) and \( S \equiv \bar{s} s \) states and finally for the three physical scalar-isoscalar states \( i = f_1, f_2, \) and \( f_3 \):

\[
g_{N \gamma \gamma} = \frac{5}{\sqrt{2}} g_{S \gamma \gamma} = \frac{40}{9} \sqrt{2} e^e, \quad g_{i \gamma \gamma} = \sum_{j=N,G,S} B_{ij} g_{j \gamma \gamma}.
\]

### III. PHENOMENOLOGICAL FIT WITHOUT DIRECT GLUEBALL DECAY

#### A. General considerations

In the following we determine a best fit of the parameters entering in Eqs. (2) to the experimental averages of masses and decay modes listed in Ref. [1]. We first analyze the case of a non-decaying glueball, i.e. \( c^d_g = c^m_g = 0 \), where the decays are dominated by the quarkonia components (as in the original work of [8]) in line with large \( N_c \) arguments. The phenomenological analysis of Ref. [10] confirmed this trend, but, as already mentioned, the fit of [8] shows a strong contribution from the direct decays of the glueball configuration.

The parameters of the model entering in the fit are the three bare masses \( M_N, M_G, M_S \), the two mixing parameters \( f \) and \( \varepsilon \) and the two quarkonia decay parameters \( c^s_d \) and \( c^s_m \):

\[
M_N, M_G, M_S, f, \varepsilon, c^s_d, c^s_m.
\]

As an experimental input we use the following accepted values from [1]:
(a) The scalar-isoscalar $f_0$ masses with the corresponding values of

\[ M_{f_0(1370)} = 1.35 \pm 0.15 \text{ GeV}, \]
\[ M_{f_0(1500)} = 1.507 \pm 0.005 \text{ GeV}, \]
\[ M_{f_0(1710)} = 1.714 \pm 0.005 \text{ GeV}. \]

(b) The partial decay widths of $f_2 \equiv f_0(1500)$ with:

\[ \Gamma_{f_2 \rightarrow \pi\pi} = 0.0380 \pm 0.0050 \text{ GeV}, \]
\[ \Gamma_{f_2 \rightarrow \pi K} = 0.0094 \pm 0.0017 \text{ GeV}, \]
\[ \Gamma_{f_2 \rightarrow \eta\eta} = 0.0056 \pm 0.0014 \text{ GeV}. \]

(c) The two accepted ratios for $f_3 \equiv f_0(1710)$:

\[ \frac{\Gamma_{f_3 \rightarrow \pi\pi}}{\Gamma_{f_3 \rightarrow \pi K}} = 0.20 \pm 0.06, \]
\[ \frac{\Gamma_{f_3 \rightarrow \eta\eta}}{\Gamma_{f_3 \rightarrow \pi K}} = 0.48 \pm 0.15. \]

(d) The state $f_0(1710)$ has only been observed in the decays into two pseudoscalar mesons \[^{[1]}\]. The decay into the final state $4\pi$, which can be fed by higher meson resonances, is suppressed \[^{[33]}\]. We therefore impose the additional condition that the sum of partial decay widths into two pseudoscalar mesons ($\Gamma_{f_3}^{2P}$) saturates the total width ($\Gamma_{f_3}^{\text{tot}}$):

\[ (\Gamma_{f_3}^{2P}) = (\Gamma_{f_3}^{\text{tot}}) = 140 \pm 10 \text{ MeV}. \]

Such a constraint is necessary to obtain meaningful total decay widths: without this condition on the full width a minimum for $\chi^2$ is obtained where $(\Gamma_{f_3}^{\text{tot}})$ is larger than 1 GeV, a clearly unacceptable solution.

(e) The two decay ratios of the $I = 1$ state $a_0(1450)$:

\[ \frac{\Gamma_{a_0 \rightarrow \pi\eta'}}{\Gamma_{a_0 \rightarrow \pi\eta}} = 0.35 \pm 0.16, \]
\[ \frac{\Gamma_{a_0 \rightarrow KK}}{\Gamma_{a_0 \rightarrow \pi\eta}} = 0.88 \pm 0.23. \]

The total width of $a_0(1450)$ into two pseudoscalars is not known, because of the uncertainty of other decay widths as for $\omega\pi\pi$.

(f) The $I = 1/2$ state $K_0^*(1450)$ decays dominantly into $K\pi$ with the corresponding width

\[ \Gamma_{K_0^* \rightarrow K\pi} = 273 \pm 51 \text{ MeV}. \]

The only accepted average not included in the fit is $\Gamma_{f_2 \rightarrow \eta\eta'}$ for the reason that the decay channel $\eta\eta'$ is produced at threshold. Therefore, a significant distortion due to the finite width of the state is expected.

For the $N = 12$ experimental values listed in Eqs. \[^{[23]}\]–\[^{[39]}\] we perform a $\chi^2$ fit with

\[ \chi^2 = \chi^2[M_N, M_G, M_S, f, \varepsilon, c_2, c_4, c_6, c_8] = \sum_{i=1}^{N=12} \frac{(A_i^{\text{theory}} - A_i^{\text{exp}})^2}{\Delta A_i^2}, \]

(40)
where $A_i^{\exp}$ represents the i-th experimental result, $\Delta A_i$ is its error and $A_i^{\text{theory}}$ is the corresponding theoretical expression, depending on the parameters of $\theta$.

For the case, where the direct glueball is suppressed (i.e. we set $c_d^b = c_m^b = 0$), two local minima for $\chi^2$ are obtained, the consequences of which we will describe in the following. The first solution was already analyzed in [13], while the second one is a novel solution with some peculiar characteristics.

### B. First solution and implications

**Fit results:** From the first solution (I) we extract following fit parameters:

$$
M_N = 1.455 \text{ GeV}, \quad M_{G} = 1.490 \text{ GeV}, \quad M_{S} = 1.697 \text{ GeV},
$$

$$
f = 0.065 \text{ GeV}^2, \quad \varepsilon = 0.211 \text{ GeV}^2,
$$

$$
c_d^b = 8.48 \text{ MeV}, \quad c_m^b = 2.59 \text{ MeV}; \quad \chi^2_{\text{tot}} = 29.01. \tag{41}
$$

The corresponding fit results are reported in Table 1.

**Table 1.** Fitted mass and decay properties of scalar mesons.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp</th>
<th>Theory</th>
<th>$\chi^2_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{f_1}$ (MeV)</td>
<td>1350 ±150</td>
<td>1417</td>
<td>0.202</td>
</tr>
<tr>
<td>$M_{f_2}$ (MeV)</td>
<td>1507 ± 5</td>
<td>1507</td>
<td>~ 0</td>
</tr>
<tr>
<td>$M_{f_3}$ (MeV)</td>
<td>1714 ± 5</td>
<td>1714</td>
<td>0.003</td>
</tr>
<tr>
<td>$\Gamma_{f_2 \to \pi\pi}$ (MeV)</td>
<td>38.0 ± 4.6</td>
<td>38.52</td>
<td>0.011</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \bar{K}K}$ (MeV)</td>
<td>9.4 ± 1.7</td>
<td>10.36</td>
<td>0.322</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \eta\eta}$ (MeV)</td>
<td>5.6 ± 1.3</td>
<td>1.90</td>
<td>8.109</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \pi\pi}/\Gamma_{f_3 \to \bar{K}K}$</td>
<td>0.20 ± 0.06</td>
<td>0.212</td>
<td>0.036</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \eta\eta}/\Gamma_{f_3 \to \pi\pi}$</td>
<td>0.48 ± 0.15</td>
<td>0.249</td>
<td>2.446</td>
</tr>
<tr>
<td>$\Gamma_{a_0 \to \pi\pi}/\Gamma_{a_0 \to \pi 0\pi 0}$</td>
<td>0.88 ± 0.23</td>
<td>0.838</td>
<td>0.032</td>
</tr>
<tr>
<td>$\Gamma_{a_0 \to \eta\eta}/\Gamma_{a_0 \to \pi\pi}$</td>
<td>0.35 ± 0.16</td>
<td>0.288</td>
<td>0.150</td>
</tr>
<tr>
<td>$\Gamma_{K_0^{*+} \to \pi\pi}$ (MeV)</td>
<td>273 ± 51</td>
<td>59.10</td>
<td>17.590</td>
</tr>
<tr>
<td>$(\Gamma_{f_3})_{2P}$ (MeV)</td>
<td>140 ± 10</td>
<td>143.27</td>
<td>0.110</td>
</tr>
<tr>
<td>$\chi^2_{\text{tot}}$</td>
<td>-</td>
<td>-</td>
<td>29.01</td>
</tr>
</tbody>
</table>

**Bare masses:** The bare non-strange quarkonia field $N$ has a mass of $M_N = 1.455$ GeV, which is, as desired, similar to the scale set by the isotriplet combination $a_0(1450)$ with a mass of $M_{a_0} = 1.474 \pm 0.019$ GeV [1]. The mass of the bare glueball $M_{G} = 1.490$ GeV is in agreement with the lattice results [2] and with the phenomenological analyses of [8], [10]. The bare state $S$ has a mass of $M_{S} = 1.697$ GeV, which is about $\sim 200$ MeV heavier than the $N$ state, an acceptable mass difference like in the tensor meson nonet.

**Mixing parameters** For the glueball-quarkonia mixing parameter we get $f = 0.065 \text{ GeV}^2$, which by the approximate relation $f \simeq 2z M_G$ [8]–[12] corresponds to $z \simeq 21.8$ MeV.

The results of Refs. [8], [10], [12] are $z = 85 \pm 10$ MeV, $z = 80$ MeV and $z \simeq 62$ MeV, respectively, i.e. of the same order, but larger. The introduction of additional flavor mixing between the quarkonia configurations in the fit, as done here, leads to a reduction of the strength parameter $f$. The lattice result of Ref. [4] with $43 \pm 31$ MeV is in agreement with the present evaluation, but has a large uncertainty. A mixing strength of the same order is found in the lattice evaluation of [15].
The flavor mixing parameter resulting from the fit is $\varepsilon = 0.211$ GeV$^2$. In the limit $f = 0$ the mixed physical states are $|f_1\rangle = 0.97 |N\rangle + 0.26 |S\rangle$ and $|f_3\rangle = -0.26 |N\rangle + 0.97 |S\rangle$ (and, of course, $|f_2\rangle = |G\rangle$). The phase structure of the mixed states is, as discussed previously, as in $23, 28, 29$. But here the strength of flavor mixing is smaller, resulting in mixed states, which are dominantly $N$ or $S$. The influence however of (an even small) flavor mixing in strong and electromagnetic decays may be non-negligible.

**Mixing matrix:** The mixing matrix $B$ relating the physical to the bare states in the present fit is expressed as:

$$
\begin{pmatrix}
|f_1\rangle & = & |f_0(1370)\rangle \\
|f_2\rangle & = & |f_0(1500)\rangle \\
|f_3\rangle & = & |f_0(1710)\rangle
\end{pmatrix} =
\begin{pmatrix}
0.86 & 0.45 & 0.24 \\
-0.45 & 0.89 & -0.06 \\
-0.24 & -0.06 & 0.97
\end{pmatrix}
\begin{pmatrix}
|N\rangle & = & |\bar{n}n\rangle \\
|G\rangle & = & |gg\rangle \\
|S\rangle & = & |ss\rangle
\end{pmatrix} .
$$

(42)

The physical resonances are dominated by the diagonal bare components, qualitatively in line with Refs. $3, 6, 8, 10$. Since the glueball does not contribute to the decay, the relative phase with respect to the quarkonia components is at this stage irrelevant. By inverting $f \to -f$ we would find the same results for the decays, but opposite glueball-quarkonia phases. In turn, the relative phases of the $N$ and $S$ components are not symmetric under $\varepsilon \to -\varepsilon$. As discussed above, in $f_0(1370)$ they are in phase, while in $f_0(1710)$ they are out of phase. The state $|f_0(1500)\rangle$ behaves like a $N$ state with a decreased width, while the $S$ component is small $10, 34$. Thus the decay into $\bar{K}K$ is smaller than for the $\pi\pi$ channel. In the present solution (I) the $N$ and $S$ state components are in phase contrary to the results of $6, 10$. However, the other solution (II), presented later on, shows again an opposite phase in $f_2 = f_0(1500)$, but with a large $\bar{s}s$ amount.

**Large $N_c$ constants:** From the fit parameters of $41$, we determine $\gamma_{S_0}$, $\gamma_{S_s}$, and $k^S_m$ by using $18$ and $41$:

$$
M_S = 1.479 \text{ GeV}, \quad e^S_m = 0.199, \\
\gamma_{S_0} = 0.225, \quad \gamma_{S_s} = 0.236, \quad k^S_m = -0.818 .
$$

(43)

The values of $\gamma_{S_0}$ and $k^S_m$ are smaller than in $20$. Also, the violation of the GMO relation, encoded in $\gamma_{S_s}$, is small, indicating that higher order corrections in the chiral expansion are possibly not too large to invalidate the present study. In the present scenario the smallness of the glueball mixing parameter $f$ can also be interpreted as a small violation of the large $N_c$ limit. The present results show that large $N_c$ and chiral symmetry can, although violated at some level, be a useful guideline to scalar meson physics.

**Resonance $f_0(1370)$:** The experimental uncertainties of the $f_0(1370)$ resonance are large, no average or fit is presented in $1$. The main problem connected with this resonance is its large width (200-500 MeV) and its partial overlap with the broad low-lying $\sigma \equiv f_0(400 - 1200)$. However, the results from WA102 $33$ indicate a large $N = \pi \eta$ component in its wave function. Results from Crystal Barrel (summarized in $35$, and subsequently analyzed in $36$) confirm such a trend (see also $37$ for a recent review).

**Table 2.** Decays of $f_1 = f_0(1370)$.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (WA102)</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{f_1 \to \bar{K}K} / \Gamma_{f_1 \to \pi\pi}$</td>
<td>$0.46 \pm 0.19$</td>
<td>0.34</td>
</tr>
<tr>
<td>$\Gamma_{f_1 \to \eta\eta} / \Gamma_{f_1 \to \pi\pi}$</td>
<td>$0.16 \pm 0.07$</td>
<td>0.06</td>
</tr>
<tr>
<td>($\Gamma_{f_1})_{2P}$ (MeV)</td>
<td>&quot;small&quot;</td>
<td>166</td>
</tr>
</tbody>
</table>
Predictions for the two-pseudoscalar decay modes are in acceptable agreement with the results of WA102 as shown in Table 2. The measured ratio $\Gamma_{f_1 \rightarrow 4\pi}/\Gamma_{f_1 \rightarrow 3\pi} = 34.0^{+22}_{-29}$, although the errors are large, points to a dominant $4\pi$ contribution to the total width. Our prediction gives however a sizable contribution of the two-pseudoscalar decay mode and is therefore not in agreement with such a large $4\pi$ decay mode.

In the original work of [1], a quarkonium $\eta n$ state has a very large two-pseudoscalar width with $\Gamma_{\eta n \rightarrow \pi\pi} = 270 \pm 25$ MeV, $\Gamma_{\eta n \rightarrow \pi\pi K} = 195 \pm 20$ MeV and $\Gamma_{\eta n \rightarrow \eta\eta} = 95 \pm 10$ MeV, i.e. $(\Gamma_{\eta n})_{2P} \sim 500$ MeV. The $f_1 \equiv f_0(1370)$ is in [6] dominantly $\eta n$, therefore one expects a large value for $(\Gamma_{f_1})_{2P}$ in contrast to the presented experimental analyses listed above. A large value for $(\Gamma_{\eta n})_{2P}$ is also predicted in [10]: for the mixed state $f_0(1370)$ one has $(\Gamma_{f_1})_{2P} = 115.7$ MeV comparable to the present study.

On the contrary, in the study of [8], small two-pseudoscalar partial widths are obtained by the following mechanism: the glueball decay amplitude in $f_0(1370)$ to two pseudoscalar is large and interferes destructively with the $\eta n$ component (the phases quarkonia-glueball in [8] are inverted with respect to [12]). As a result in the two-pseudoscalar decay width $(\Gamma_{f_1})_{2P}$ is smaller than $(\Gamma_{f_2})_{2P}$ with $(\Gamma_{f_2})_{2PS}/(\Gamma_{f_1})_{2PS} = 10.0 \pm 3.0$. At the same time $(\Gamma_{f_1})_{2P}/(\Gamma_{f_2})_{2P} = 0.7 \pm 0.2$ is obtained.

In the present fit an inert glueball (as in the original work of [1] and as in [10], where the glueball is allowed to decay, but the quarkonia components still dominate) we find the following decay widths into two pseudoscalars pairs:

$$(\Gamma_{f_1})_{2P} = 166$ MeV > $(\Gamma_{f_2})_{2P} = 143$ MeV > $(\Gamma_{f_2})_{2P} = 51$ MeV. \hspace{1cm} (44)$$

An analysis by Crystal Barrel [30] also indicates sizable partial decay widths of the $4\pi$ decay channels: $\Gamma_{f_1 \rightarrow \sigma\sigma} = 120.5 \pm 65$ MeV and $\Gamma_{f_1 \rightarrow \rho\rho} = 62.2 \pm 28.8$ MeV. The same analysis gives the following two-pseudoscalar partial widths [30,37]: $\Gamma_{f_1 \rightarrow \pi\pi} = 21.7 \pm 9.9$ MeV, $\Gamma_{f_1 \rightarrow \pi\pi K} = (7.9\pm2.7$ MeV) to $(21.2\pm7.2$ MeV), $\Gamma_{f_1 \rightarrow \eta\eta} = 0.41\pm0.27$ MeV. On the contrary, the analysis of [38] reports the ratio $\Gamma_{f_1 \rightarrow \pi\pi}/(\Gamma_{f_1})_{tot} = 0.26 \pm 0.09$, pointing to a large $\pi\pi$ (ergo to a large two-pseudoscalar) partial decay width for $f_0(1370)$. This experimental result is therefore in disagreement with the analysis of [38]. New results on $f_0(1370)$ would be crucial to disentangle the scalar puzzle and to understand if a destructive glueball/quarkonia interference as in [6] is necessary.

**Resonance $f_0(1500)$**: The theoretical partial widths of the $f_0(1500)$ are in good agreement with the data (see Table 1) apart from a slight underestimate of the $2\eta$ channel. We also obtain $\Gamma_{f_2 \rightarrow \eta\eta'} = 0.036$ MeV as compared to the experimental value of $\Gamma_{f_2 \rightarrow \eta\eta'} = 2 \pm 1$ MeV. Taking into account the finite width of the resonance will lead to an increase of the theoretical value.

**Resonance $f_0(1710)$**: For the decays of $f_0(1710)$ we summarize our results compared to the data of WA102 [33] in Table 3.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (WA102)</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{f_3 \rightarrow K\bar{K}}/\Gamma_{f_3 \rightarrow \pi\pi}$</td>
<td>$5.0 \pm 0.7$</td>
<td>4.70</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \rightarrow \eta\eta}/\Gamma_{f_3 \rightarrow \pi\pi}$</td>
<td>$2.4 \pm 0.6$</td>
<td>1.17</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \rightarrow \eta\eta'}/\Gamma_{f_3 \rightarrow \pi\pi}$</td>
<td>&lt; 0.18</td>
<td>1.59</td>
</tr>
<tr>
<td>$(\Gamma_{f_3})_{2P}$ (MeV)</td>
<td>&quot;dominant&quot;</td>
<td>143.27</td>
</tr>
</tbody>
</table>

The first two ratios, already included in the fit of Table 2, can be reproduced. The theoretical ratio $\Gamma_{f_3 \rightarrow \eta\eta'}/\Gamma_{f_3 \rightarrow \pi\pi}$, which is not included in [6], is in complete disagreement with the WA102 result. The dominance of the $\eta\eta'$ mode over
ππ is a solid prediction in the framework of solution I, which does not depend very much on the choice of parameters. A confirmation of the experimental result could possibly hint at a different mixing scenario or at a sizable role of direct glueball decay.

**Resonance a_0(1450):** The ratios of two-pseudoscalar decay modes of a_0(1450), included in the fit of Table 2, are well reproduced. The prediction for the two-pseudoscalar width of (Γ_{a_0})_{2P} = 84.26 MeV is smaller than the total width of 265 ± 13 MeV. However, the experimental ratio (Γ_{a_0→ωππ}/Γ_{a_0→πη}) is not known: no average or fit is listed in PDG2000 [42]. The experimental value from [39], which is 10.7 ± 2.3, would imply a dominant ωππ mode and in turn a rather small two-pseudoscalar partial decay widths. This finding is in disagreement with the results of [8, 12]. The ratio (Γ_{a_0→ωππ}/Γ_{a_0→πη} = [(Γ_{a_0})_{tot} - (Γ_{a_0})_{2P}]/ Γ_{a_0→πη}) ~ 4.5.

**Resonance K_0^*(1430):** Our result for Γ_{K_0^*→Kπ} underestimates the experimental value by a factor of about 5 (see Table 1). Furthermore, for the additional Kπ decay channel we get Γ_{K_0^*→Kπ}/Γ_{K_0^*→πK} is 0.026.

In [40], a value of Γ_{K_0^*→Kπ} = 340 MeV is predicted, but, as discussed above, (Γ_{a_0})_{2P} is of the order of 1 GeV, much larger than the full width. Similarly, in [41] with Γ_{K_0^*→Kπ} = 200 ± 20 and (Γ_{a_0})_{2P} = 390 ± 110 MeV the first result underestimates while the second overshoots the experimental value. A full analysis in the 3P_0 model [41] results in Γ_{K_0^*→πK} = 166 MeV and Γ_{N→ππ} = 271 MeV; unfortunately the resonance a_0(1450) is not discussed in [41]. The authors of [41] also tried to adjust Γ_{K_0^*→πK} to its experimental value, and then calculate the 2π partial width of a N state, obtaining Γ_{N→ππ} ~ 450 MeV. The last result implies a very large two-pseudoscalar and full width for a N state. A full experimental determination of all relevant decay modes involving a_0(1450) and K_0^*(1430) would certainly help to clarify this issue. We refer to section III D for a further discussion of this problem.

If f_0(1370) is dominantly ¯nμ, as in [10, 12], there is, as discussed above, an incompatibility of the present experimental small two-pseudoscalar partial decay widths [39] and various model calculations. At the same time, a consistent understanding of the isodoublet states K_0^*(1450) and the isovectors a_0(1450) is still incomplete.

**Two-photon decays:** As a further consequence we discuss the two-photon decay rates of the scalar resonances. We assume that the coupling c_γ is suppressed with respect to c_γ, i.e. we set the glueball-photon coupling c_γ to zero. The ratios of radiative decay widths as a prediction of the fit are:

\[
\Gamma_{f_1→2γ} : \Gamma_{f_2→2γ} : \Gamma_{f_3→2γ} : \Gamma_{a_0^*→2γ} = 1 : 0.305 : 0.002 : 0.471,
\]

which are independent of the coupling c_γ. The result for Γ_{f_2→2γ}/Γ_{f_1→2γ} is in qualitative agreement with the results of [8, 12]. The ratio Γ_{f_3→2γ}/Γ_{f_1→2γ}, however, is considerably smaller than in the previous works. The suppression of Γ_{f_3→2γ} originates from the destructive interference between the N and S components, which in turn is traced to the flavor mixing with ε > 0 in accord with the phases of [28, 29]. Another interesting prediction is the ratio Γ_{a_0^*→2γ}/Γ_{f_1→2γ}, which is relatively large.

The experimental status of the two-photon decays is still incomplete. For the f_0(1370) two values are indicated in PDG2000 [42]: 3.8 ± 1.5 keV and 5.4 ± 2.3 keV. However, it is not clear if the two-photon signal comes from the f_0(1370) or from the high mass end of the broad f_0(400 − 1200). The PDG currently [1] seems to favor this last possibility, but the data could also be valid for the f_0(1370). We therefore interpret the two experimental values as an upper limit for the two-photon decay width of the f_0(1370). Signals for two-photon decays of f_0(1500) and f_0(1710)
have not yet been seen, the following upper limits are reported \[^1\]:

\[
\begin{align*}
\Gamma_{f_0(1500)\to 2\gamma}(\Gamma_{f_0(1500)\to \pi\pi}/\Gamma_{f_0(1500)\text{tot}}) &< 0.46 \text{ keV}, \\
\Gamma_{f_0(1710)\to 2\gamma}(\Gamma_{f_0(1710)\to K\bar{K}/\Gamma_{f_0(1710)\text{tot}}}) &< 0.11 \text{ keV}.
\end{align*}
\] (46)

Using the known branching ratio \(\Gamma_{f_0(1500)\to 2\gamma}/\Gamma_{f_0(1500)\text{tot}}\) one gets \(\Gamma_{f_0(1500)\to 2\gamma} < 1.4 \text{ keV}\) \[^34\]. An accepted fit for \(\Gamma_{f_0(1710)\to K\bar{K}/\Gamma_{f_0(1710)\text{tot}}}\) is not reported in \[^1\]. Using the value from \[^43\] with \(\Gamma_{f_0(1710)\to K\bar{K}/\Gamma_{f_0(1710)\text{tot}}} = 0.38^{+0.03}_{-0.13}\) we find an upper limit of the order of \(\Gamma_{f_0(1710)\to 2\gamma} \sim 0.3 \text{ keV}\).

For an absolute prediction of the two-photon decay widths we use \(c_s^e = 0.0138 \text{ GeV}^{-1}\) as determined in the model approach of Ref. \[^12\]. For the non-strange quarkonium state we get \(\Gamma_{N\to 2\gamma} = 0.969 \text{ keV}\), while for the isovector and mixed scalars we have:

\[
\begin{align*}
\Gamma_{f_1\to 2\gamma} &= 0.703 \text{ keV} , \Gamma_{f_2\to 2\gamma} = 0.235 \text{ keV} , \\
\Gamma_{f_3\to 2\gamma} &= 0.002 \text{ keV} , \Gamma_{a_0^\pi\to 2\gamma} = 0.362 \text{ keV}.
\end{align*}
\] (47)

The results for the mixed states are below the current upper limits (the original results presented in \[^13\] contain a slight missprint, the correct numbers are reported here. The physical considerations are not affected from this slight change).

The estimate for the \(2\gamma\) decay of the bare quarkonium state \(N \equiv \bar{n}n\) of 0.969 keV is smaller than the one of \[^34\], where the following expression has been used:

\[
\Gamma_{\pi n\to 2\gamma}(0^{++}) = k \left(\frac{M_N(0^{++})}{M_N(2^{++})}\right)^3 \Gamma_{\pi n\to 2\gamma}(2^{++})
\] (48)

The coefficient \(k\) is 15/4 in a non-relativistic calculation, but becomes smaller when considering relativistic corrections \[^44\]. In \[^34\] a range of values for \(k\) from 2 to 15/4 is considered. Our chiral Lagrangian approach combined with \[^12\] points to a smaller value of \(k\). Using our result for \(\Gamma_{N\to 2\gamma}(0^{++})\) and taking the value \(\Gamma_{N\to 2\gamma}(2^{++}) = 2.60 \pm 0.24 \text{ keV}\) \[^1\] at \(M_N(2^{++}) = 1.27 \text{ GeV}\) we get \(k \sim 0.25\). This result is model dependent, since it relies on the parameters for the covariant description of the scalar mesons used in \[^12\]. A fully covariant treatment may imply strong deviations from the non-relativistic limit.

**Discussion:** The largest contribution to \(\chi^2\) is due to the underestimate of the \(K_0^* \to K\pi\) width: from Table 1 we have \(\chi^2_{\text{tot}}/N = 2.42\). When excluding the data point for \(\Gamma_{K_0^*\to K\pi}\) in the fit, a very similar minimum compared to \((41)\) is found with:

\[
\begin{align*}
M_N &= 1.442 \text{ GeV} , \ M_G = 1.485 \text{ GeV} , \ M_S = 1.695 \text{ GeV} , \ f = 0.080 \text{ GeV}^2 , \\
\varepsilon &= 0.225 \text{ GeV}^2 , \ c_d^s = 8.12 \text{ MeV} , \ c_m^s = 3.57 \text{ MeV} ; \ \chi^2_{\text{tot}} = 11.19.
\end{align*}
\] (49)

In this case we have \(\chi^2_{\text{tot}}/N = 1.02\), corresponding to a good description of data. The discussion about the isoscalar states and \(a_0(1450)\) remains unchanged.

The underestimate of \(K_0^* \to K\pi\) constitutes an open problem of the scalar analysis. As already discussed, a consistent understanding of the complete scalar nonet is lacking in other approaches as well. We will further discuss this issue in section \[^11\].

Aside from this difficulty, the rest of the accepted data in \[^1\] is well described. The quality of the current fit seemingly excludes a sizable direct decay of the scalar glueball component. Concerning results for data, which are not
reported as average or fit in, the situation is less clear: the predicted full two-pseudoscalar width of $f_1 = f_0(1370)$ is large, when confronted with the WA102 result (see Table 2 and Refs. 35, 36, 37, but also the different result of 38). The ratio $\Gamma_{f_1 \to \eta'}/\Gamma_{f_1 \to \pi\pi}$ (Table 3) is also problematic in the present solution. An accepted average, in particular for these values, would help in clarifying these issues.

C. Second solution and implications

Fit results: Solution II is obtained for following fit parameters:

$$M_N = 1.298 \text{ GeV}, \ M_G = 1.513 \text{ GeV}, \ M_S = 1.593 \text{ GeV}, \ f = 0.400 \text{ GeV}^2, \ e = 0.015 \text{ GeV}^2, \ c_s = 7.48 \text{ MeV}, \ c_m = 6.42 \text{ MeV}; \ \chi^2_{\text{tot}} = 24.61.$$  \hspace{1cm} (50)

The corresponding results are reported in Table 4.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (MeV)</th>
<th>Theory $\chi^2_{\text{tot}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{f_1}$</td>
<td>$1350 \pm 150$</td>
<td>$1142$</td>
</tr>
<tr>
<td>$M_{f_2}$</td>
<td>$1507 \pm 5$</td>
<td>$1508$</td>
</tr>
<tr>
<td>$M_{f_3}$</td>
<td>$1714 \pm 5$</td>
<td>$1713$</td>
</tr>
<tr>
<td>$\Gamma_{f_2 \to \pi\pi}$</td>
<td>$38.0 \pm 4.6$</td>
<td>$37.31$</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \pi\pi}$</td>
<td>$9.4 \pm 1.7$</td>
<td>$10.08$</td>
</tr>
<tr>
<td>$\Gamma_{f_2 \to \eta\eta}$</td>
<td>$5.6 \pm 1.3$</td>
<td>$4.70$</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \eta\eta}$</td>
<td>$0.20 \pm 0.06$</td>
<td>$0.216$</td>
</tr>
<tr>
<td>$\Gamma_{f_2 \to \eta\eta}/\Gamma_{f_3 \to \eta\eta}$</td>
<td>$0.48 \pm 0.15$</td>
<td>$0.248$</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \eta\eta}/\Gamma_{f_3 \to \pi\pi}$</td>
<td>$0.88 \pm 0.23$</td>
<td>$1.078$</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \eta\eta}/\Gamma_{f_3 \to \eta\eta}$</td>
<td>$0.35 \pm 0.16$</td>
<td>$0.291$</td>
</tr>
<tr>
<td>$\Gamma_{K_S^0 \to \eta\eta}$</td>
<td>$273 \pm 51$</td>
<td>$53.75$</td>
</tr>
<tr>
<td>$\chi^2_{\text{tot}}$</td>
<td>$140 \pm 10$</td>
<td>$143.94$</td>
</tr>
</tbody>
</table>

A first look to (50) shows some peculiar differences when compared to the set of (41). In the following we discuss the implications and the differences of this second solution.

Bare masses: The bare masses $M_N$ and $M_S$ are smaller than in solution I, their mass difference is still around 200 GeV (Eq. 50). The bare glueball mass is about $\sim 1.5$ GeV, as before, but now is much closer to $M_S$. This small mass difference leads to a strong mixing between the glueball and the bare $S = \bar{7}s$ state.

Mixing parameters: The quarkonia flavor mixing $e = 0.015 \text{ GeV}^2$ is very small in this solution and has practically no influence on the phenomenology. On the contrary $f = 0.400 \text{ GeV}^2$ is much larger, leading to a strong glueball-quarkonia mixing. In this respect there is a clear difference between the two solutions. Using the approximate relation $f \simeq 2z_{MG}$, we find in this case $z \simeq 130 \text{ MeV}$, which is larger than the results from other works listed in Sec. III.B.3, but still in qualitative agreement.

Mixing matrix: The mixing matrix $B$, relating the physical to the bare states, for the second solution is expressed as:

$$
\begin{pmatrix}
|f_1\rangle &=& |f_0(1370)\rangle \\
|f_2\rangle &=& |f_0(1500)\rangle \\
|f_3\rangle &=& |f_0(1710)\rangle \\
\end{pmatrix}
= 
\begin{pmatrix}
0.81 & 0.54 & 0.19 \\
-0.49 & 0.49 & 0.72 \\
-0.30 & -0.68 & 0.67 \\
\end{pmatrix}
\begin{pmatrix}
|N\rangle &=& |\bar{u}\bar{n}\rangle \\
|G\rangle &=& |gg\rangle \\
|S\rangle &=& |\bar{s}s\rangle \\
\end{pmatrix}. \hspace{1cm} (51)
$$
The large mixing parameter $f$ causes the glueball configuration to be spread out among the $f_0$ states: $f_1 \equiv f_0(1370)$ is still dominantly $\bar{u}u$, $f_2 \equiv f_0(1500)$ is mostly $\bar{s}s$, but with a sizable out-of-phase $\bar{u}u$ amplitude (the opposite phase of $\bar{u}u$ and $\bar{s}s$ was first considered in [6] as a mechanism to explain the large $\pi\pi/KK$ ratio). In $f_3 \equiv f_0(1710)$ both the gluonium and the $\bar{s}s$ components are large, where the gluonic component is slightly larger. Remarkably, although the bare level ordering is still $M_N < M_G < M_S$, the largest $\bar{s}s$ amount is contained in $f_2$, while the largest gluonic component is present in $f_3$. The mixing matrix resembles some features of the results of [7] although the bare level ordering is different.

Large $N_c$ constants: From the parameters in (50) we determine $\gamma_{S_0}$, $\gamma_{S_8}$ and $k_m^S$ by using (11) and (18):

$$
M_S = 1.479 \text{ GeV}, \ e_m^S = 0.199,
\gamma_{S_8} = 0.139, \  \gamma_{S_0} = -0.032, \ k_m^S = -0.786. \quad (52)
$$

The same considerations as for solution I also hold here.

Resonance $f_0(1370)$: The results for $f_1 = f_0(1370)$ are summarized in Table 5.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (WA102)</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{f_1 \to \pi \pi}$</td>
<td>$0.46 \pm 0.19$</td>
<td>0.27</td>
</tr>
<tr>
<td>$\Gamma_{f_1 \to \eta \eta}$</td>
<td>$0.16 \pm 0.07$</td>
<td>0.02</td>
</tr>
<tr>
<td>$(\Gamma_{f_1})_{2P}$ (MeV)</td>
<td>&quot;small&quot;</td>
<td>56.79</td>
</tr>
</tbody>
</table>

The main difference with respect to solution I concerning $f_1 = f_0(1370)$ is the decreased theoretical two-pseudoscalar width (mostly caused by the smaller physical mass, see Table 4). In the present fit we find the following decay widths into two pseudoscalar pairs:

$$(\Gamma_{f_3})_{2P} = 144 \text{ MeV} > (\Gamma_{f_1})_{2P} = 57 \text{ MeV} > (\Gamma_{f_2})_{2P} = 52 \text{ MeV}.$$  

Here the decay pattern is in better agreement with the analysis of [36, 37] than the one of solution I.

Resonance $f_0(1500)$: The theoretical partial widths of $f_0(1500)$ are in rather good agreement with the data (see Table 2). We also obtain $\Gamma_{f_2 \to \eta \eta'} = 1.5 \text{ MeV}$ as compared to the experimental value of $\Gamma_{f_2 \to \eta \eta'} = 2 \pm 1 \text{ MeV}$ without invoking further threshold corrections.

Resonance $f_0(1710)$: For the decays of $f_0(1710)$ we summarize our results compared to the data of WA102 [33] in Table 6.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (WA102)</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{f_3 \to \pi K}$</td>
<td>$5.0 \pm 0.7$</td>
<td>4.63</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \eta \eta}$</td>
<td>$2.4 \pm 0.6$</td>
<td>1.15</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \to \eta \eta'}$</td>
<td>$&lt; 0.18$</td>
<td>0.36</td>
</tr>
<tr>
<td>$(\Gamma_{f_3})_{2P}$ (MeV)</td>
<td>&quot;dominant&quot;</td>
<td>143.94</td>
</tr>
</tbody>
</table>

The theoretical $\eta \eta'/\pi \pi$ ratio is now smaller than in solution I. Although the prediction is still larger than the upper limit set by WA102, it does not represent such an evident mismatch.
The two experimental ratios are satisfactorily described. Otherwise, the discussion of solution I is still valid here.

Resonance $K_0^*$ (1430): The underestimate of $\Gamma_{K_0^* \to K\pi}$ is also present in the second solution (see Table 4). Furthermore, for the additional $K\eta$ decay channel we get $\Gamma_{K_0^* \to K\eta}/\Gamma_{K_0^* \to \pi K} = 0.05065$.

Two-photon decays: The two-photon decay ratios resulting from solution II read:

$$\Gamma_{f_1 \to 2\gamma} : \Gamma_{f_2 \to 2\gamma} : \Gamma_{f_3 \to 2\gamma} : \Gamma_{a_0^0 \to 2\gamma} = 1 : 0.253 : 0.055 : 0.493.$$  (54)

The two $\gamma$ results are in qualitative agreement with [12].

Following the arguments of the previous section by using the result of [12] we find:

$$\Gamma_{f_1 \to 2\gamma} = 0.350 \text{ keV}, \quad \Gamma_{f_2 \to 2\gamma} = 0.088 \text{ keV},$$

$$\Gamma_{f_3 \to 2\gamma} = 0.019 \text{ keV}, \quad \Gamma_{a_0^0 \to 2\gamma} = 0.172 \text{ keV}.$$  (55)

As before, the rates are below the upper limits set by current data, but they depend on the model-dependent evaluation of [12].

Discussion: If in this case we exclude the $K_0^* \to K\pi$ mode from the fit (which generates the by far largest contribution to $\chi^2$), the following set of parameters is found:

\begin{align*}
M_N &= 1.313 \text{ GeV}, \quad M_G = 1.511 \text{ GeV}, \quad M_S = 1.594 \text{ GeV}, \quad f = 0.395 \text{ GeV}^2, \\
\varepsilon &= 0.002 \text{ GeV}^2, \quad c^*_d = 7.30 \text{ MeV}, \quad c^*_m = 6.61 \text{ MeV}; \quad \chi^2_{tot} = 5.97.  \quad (56)
\end{align*}

We then obtain $\chi^2_{tot}/N = 0.54$, corresponding also in this case to a good description of the remaining data. Again, there is no phenomenological evidence for a direct decay of the glueball component.

Apart from the decay $K_0^* \to K\pi$, both solutions (I and II) describe the data [28]-[38] well. There are, however, differences when comparing to the WA102 results. The second solution comes in this respect closer to these data, but the experimental results are not yet conclusive.

D. $K_0^*(1430)$

The most striking mismatch with the data is in both analyzed scenarios I and II the theoretical underestimate of the $K_0^* \to K\pi$ mode (Tables 1 and 4). The corresponding partial $\chi^2_{K_0^* \to K\pi}$ is by far the dominant contribution to $\chi^2_{tot}$. This mismatch also holds when including direct glueball decays in the analysis as shown in the next section.

As a further attempt, following [41], we can also pursue another strategy: in a first fit we fix the quarkonium decay parameters $c^*_d$ and $c^*_m$ in order to reproduce $K_0^* \to K\pi$ (Eq. (54)) and the two ratios for $a_0$ given in [38] and [39]. Then we obtain $c^*_d = 17.94 \text{ MeV}$ (larger by a factor 2 when compared to the previous fit in Tables 1 and 4), and $c^*_m = 7.35 \text{ MeV}$. With these values one has $\Gamma_{K_0^* \to \pi K} = 281 \text{ MeV}$, $\Gamma_{a_0 \to \pi K}/\Gamma_{a_0 \to \pi\eta} = 0.88$ and $\Gamma_{a_0 \to \pi\eta}/\Gamma_{a_0 \to \pi\eta} = 0.30$ and $\Gamma_{a_0 \to 2P} = 381.17 \text{ MeV}$ (already larger than the experimental result).

The corresponding decay width of the $N$ state into two pseudoscalars is $\Gamma_{N \to 2P} \sim 900 \text{ MeV}$ (for a mass of $M_N \sim 1.4 \text{ GeV}$). We then find similar results as in [41], implying that the discussed trend is rather model independent. If at this stage we fix $c^*_d = 17.94 \text{ MeV}$ and $c^*_m = 7.35 \text{ MeV}$ and we make a second fit of the remaining free parameters ($M_N, M_G, M_S, f, \varepsilon, c^*_d, c^*_m$) to the values reported in [28]-[38], we find minima with $\chi^2_{tot} \sim 500,$
which are clearly unacceptable. This is a further confirmation of the incompatibility of results in the scalar mesonic sector. If we attempt to reproduce the width of \( K_0^*(1430) \), the other results are off. As noted above, this problem arises in other models as well.

In [1] only the \( K_0^* \rightarrow K \pi \) mode is quoted in the list of the decay modes. Our suggestion is that a strong coupling to the scalar mesons below 1 GeV, \( K_0^*(800) \) and \( \sigma \) takes place. Both states are very broad and could eventually influence the decay strengths of \( K_0^*(1430) \). The theoretical description of such a phenomenon is however beyond the goals of the present tree-level study.

IV. PHENOMENOLOGY INVOLVING THE DIRECT GLUEBALL DECAY MECHANISM

A. Flavor symmetry limit

The analysis of the previous section did not reveal a phenomenological need to include the direct two-pseudoscalar decay of the glueball component. In this section we analyze this additional mechanism when including it in the fit. We start first by including the interaction term proportional to \( c_d g_m \), while neglecting the flavor symmetry breaking contribution \( c_s d_m \).

The ratio \( c_d^g/c_d^s \) is a measure of the direct glueball decay strength. The analogous quantity used in [8] is, because of the different normalization, related as \( c_d^g/c_d^s \leftrightarrow \sqrt{3/2} r_2 \). For the different solutions in the fit of [8] the parameter \( r_2 \) varies between 1 and 5. Note that the case \( r_2 = 1 \) corresponds to a direct glueball decay strength into two pseudoscalars \( \sim 1.22 \) larger than the quarkonium strength. In [13] the bare masses have been kept fixed when the glueball decay parameters have been introduced. Here we release this constraint by leaving the bare masses free. In the fit we again find two different solutions, which correspond to the ones (I and II) analyzed in the previous section.

The \( \chi^2 \) minimum corresponding to solution I but now with inclusion of \( c_d^g \) in the fit is:

\[
M_N = 1.416 \text{ GeV}, \quad M_G = 1.493 \text{ GeV}, \quad M_S = 1.694 \text{ GeV}, \\
f = 0.075 \text{ GeV}^2, \quad \varepsilon = 0.241 \text{ GeV}^2, \quad c_d^g = 8.73 \text{ MeV}, \\
c_d^s m = 1.48 \text{ MeV}, \quad c_d^g = -0.94 \text{ MeV}; \quad \chi^2_{tot} = 28.70.
\]  

(57)

A comparison with the previous fit [41] shows that the parameters practically did not change: with \( c_d^g = -0.94 \text{ MeV} \) the direct glueball decay is suppressed, resulting in \( |c_d^g/c_d^s| = 0.11 \ll 1 \). Also, the total \( \chi^2 \) is only slightly smaller than in [41].

The minimum corresponding to solution II reads:

\[
M_N = 1.401 \text{ GeV}, \quad M_G = 1.461 \text{ GeV}, \quad M_S = 1.609 \text{ GeV}, \\
f = 0.376 \text{ GeV}^2, \quad \varepsilon = -0.082 \text{ GeV}^2, \quad c_d^g = 7.63 \text{ MeV}, \\
c_d^s m = 7.09 \text{ MeV}, \quad c_d^g = 1.82 \text{ MeV}; \quad \chi^2_{tot} = 22.87.
\]  

(58)

A comparison with [60] shows a clear similarity, although the bare masses are somewhat shifted. Again, direct glueball decay is suppressed with \( |c_d^g/c_d^s| = 0.24 \).

The inclusion of \( c_d^g \) in the fit does not lead to a drastic change of the previous fit parameters. The bare glueball decay strength is strongly suppressed with respect to the quarkonium one, thus in agreement with the analysis of the previous section and with large \( N_c \) considerations.
B. Flavor symmetry breaking and lattice results

Inclusion of the flavor symmetry breaking term in direct glueball decay, that is the term with \( c_m^g \), results in nine free parameters (the set of (27) and \( c_d^g, c_m^g \)). A direct fit of these parameters to the data generates various minima, which are not well pronounced. For example, solutions are found where the flavor symmetry breaking term in the glueball sector \( c_m^g \) is exceedingly large, dominating the decay mechanism. Instead, to study the effect of flavor symmetry breaking in glueball decay we resort to a first lattice study \( [24] \) to fix the decay parameters \( c_d^g \) and \( c_m^g \).

In \( [24] \) a full two-pseudoscalar decay width of the scalar glueball with about 100 MeV is deduced. The corresponding mass of 1.7 GeV led the authors of \( [24] \) to interpret the resonance \( f_0(1710) \) as mainly gluonic. However, in the cited lattice analysis it is not clear if the glueball decay mechanism occurs partially by mixing with scalar quarkonia (here parametrized by \( f \)), or by direct decay (parametrized by \( c_d^g \) and \( c_m^g \)). In \( [4] \) a scenario is studied, where the lattice results of \( [24] \) are explained by mixing only (hence \( c_d^g \) and \( c_m^g \) are set to zero). The physical state \( f_0(1710) \) is mainly gluonic, but because of mixing, it acquires a large \( \pi \bar{\pi} \) amount, which it turn explains the decay pattern. The corresponding mixing matrix is then similar to the results of \( [6] \) (and to some aspects of our solution II).

In \( [4] \) it is stated that the amplitudes deduced in \( [24] \) probably include significant contributions from mixing of the scalar glueball with quarkonium, although not proven. If this is the case, only the mixing mechanism contributes to glueball decay and then we are back to the previous solutions, where the constants \( c_d^g \) and \( c_m^g \) are negligible. Here we also intend to investigate the opposite case, where the decay couplings calculated in \( [24] \) arise from direct glueball decay.

The decay widths of the glueball can be expressed as \( [24] \) (see also \( [2] \)):

\[
\begin{align*}
\Gamma_{G \to \pi\pi} &= \frac{3 \lambda^{1/2}(M_G^2, M_\pi^2, M_\rho^2)}{32 \pi M_G^3} \left(y_{G\pi\pi} M_\rho\right)^2, \\
\Gamma_{G \to \pi K} &= \frac{\lambda^{1/2}(M_G^2, M_K^2, M_\rho^2)}{8 \pi M_G} \left(y_{G\pi K} M_\rho\right)^2, \\
\Gamma_{G \to \eta\eta} &= \frac{\lambda^{1/2}(M_G^2, M_\pi^2, M_\eta^2)}{32 \pi M_G^2} \left(y_{G\eta\eta} M_\rho\right)^2,
\end{align*}
\]  

(59)

where \( M_\rho = 775 \) MeV is the \( \rho \) mass. The lattice results for the decay constants \( y_{G\pi\pi}, y_{G\pi K}, \) and \( y_{G\eta\eta} \) are:

\[
y_{G\pi\pi} = 0.834^{+0.603}_{-0.579}, \quad y_{G\pi K} = 2.654^{+0.372}_{-0.402}, \quad y_{G\eta\eta} = 3.099^{+0.364}_{-0.423}.
\]  

(60)

Note that in the flavor-symmetry limit we would expect \( y_{G\pi\pi}, y_{G\pi K}, y_{G\eta\eta} = 1 : 1 : 1 \). Although the errors are large, the lattice results show a sizable deviation from this limit. As already noted, it is however not clear, if and to what extent mixing with quarkonia is included in these amplitudes. Interpreting the lattice results in the context of the direct glueball decay mechanism implies a large symmetry violation parameter \( c_m^g \).

The corresponding decays can be derived from the expressions of Appendix B by setting the glueball-quarkonia mixing to zero, that is by considering the scalar-isoscalar decay for \( M_i = M_G \) and \( B_{iN} = B_{iS} = 0 \) and, of course, \( B_{iG} = 1 \). The explicit expressions for the \( \pi\pi \) and \( \pi K \) modes of the direct glueball decay are:

\[
\begin{align*}
\Gamma_{G \to \pi\pi} &= \frac{3 \lambda^{1/2}(M_G^2, M_\pi^2, M_\rho^2)}{32 \pi M_G^3} \left(\frac{2}{\sqrt{3} F^2} \left( [M_G^2 - 2M_\pi^2] c_d^g + 2M_\pi^2 c_m^g \right) \right)^2, \\
\Gamma_{G \to \pi K} &= \frac{\lambda^{1/2}(M_G^2, M_K^2, M_\rho^2)}{8 \pi M_G} \left(\frac{2}{\sqrt{3} F^2} \left( [M_G^2 - 2M_K^2] c_d^g + 2M_K^2 c_m^g \right) \right)^2.
\end{align*}
\]  

(61)
Matching the expressions for the $\pi\pi$ and $K\bar{K}$ decay modes of (61) to (59) using $M_G = 1.7$ GeV (as in [24]) we obtain for the decay constants:

\[ c_d^0 = 1.34 \text{ MeV} \quad \text{and} \quad c_m^0 = 24.6 \text{ MeV}. \] (62)

For a bare glueball mass of $M_G = 1.5$ GeV we find rather similar values of $c_d^0 = 1.72$ MeV and $c_m^0 = 25$ MeV; hence we have a rather slight dependence on $M_G$ within a reasonable range of values. In the following we take the values evaluated at $M_G = 1.7$ GeV.

Using the values of (62) we can also determine the $\eta\eta$ decay amplitude. Compared to the lattice result of $\frac{\Gamma_{G \eta\eta}}{M_{\rho}} = 2.40^{+0.28}_{-0.33}$ MeV we get the value of $2.025$ GeV. The corresponding decay widths for the bare glueball are (for $M_G = 1.7$ GeV):

\[ \Gamma_{G \rightarrow \pi\pi} = 7.23 \text{ MeV}, \quad \Gamma_{G \rightarrow K\bar{K}} = 80.61 \text{ MeV}, \] (63)
\[ \Gamma_{G \rightarrow \eta\eta} = 18.35 \text{ MeV}, \quad \Gamma_{G \rightarrow \eta\eta'} = 11.73 \text{ MeV}. \] (64)

For a lower value of mass $M_G$ the only significantly affected mode is $\eta\eta'$, since threshold effects become important (note that the $\eta\eta'$ mode is entirely generated by the flavor symmetry breaking term proportional to $c_m^0$).

With the direct glueball decay including flavor symmetry violation fixed by (62), we now perform a fit with the remaining free parameters $M_N$, $M_G$, $M_S$, $f$, $\varepsilon$, $c_d^*$, $c_m^*$. Two solutions (III and IV) are obtained, which correspond to the bare level orderings $M_N < M_G < M_S$ and $M_N < M_G < M_S$, which we analyze in the following. Although the direct glueball decay dominates in these cases, the two solutions have similarities to the ones discussed in detail in the previous section.

C. Third solution and implications

Solution III is obtained for the set of parameters:

\[ M_N = 1.359 \text{ GeV}, \quad M_G = 1.435 \text{ GeV}, \quad M_S = 1.686 \text{ GeV}, \]
\[ f = 0.212 \text{ GeV}^2, \quad \varepsilon = 0.277 \text{ GeV}^2, \]
\[ c_d^* = 8.28 \text{ MeV}, \quad c_m^* = 7.21 \text{ MeV}; \quad \chi_{tot}^2 = 21.56, \] (65)

where the fit results are listed in Table 7. The mixing matrix is similar to the one of solution I and explicitly reads:

\[
\begin{pmatrix}
|f_1\rangle & \equiv & |f_0(1370)\rangle \\
|f_2\rangle & \equiv & |f_0(1500)\rangle \\
|f_3\rangle & \equiv & |f_0(1710)\rangle
\end{pmatrix} = \begin{pmatrix}
0.79 & 0.56 & 0.26 \\
-0.58 & 0.81 & 0.02 \\
-0.20 & -0.16 & 0.97
\end{pmatrix} \begin{pmatrix}
|N\rangle & \equiv & |\bar{n}n\rangle \\
|G\rangle & \equiv & |gg\rangle \\
|S\rangle & \equiv & |\bar{s}s\rangle
\end{pmatrix}.
\] (66)

The decay rates for $f_1 = f_0(1370)$ and $f_3 = f_0(1710)$ as compared to the WA102 data are summarized in Tables 8 and 9.

The two-photon decay ratios resulting from solution III read:

\[ \Gamma_{f_1 \rightarrow 2\gamma} : \Gamma_{f_2 \rightarrow 2\gamma} : \Gamma_{f_3 \rightarrow 2\gamma} : \Gamma_{a_0^0 \rightarrow 2\gamma} = 1 : 1.018 : 0.025 : 0.494. \] (67)
The results of the fit summarized in Table 7 are acceptable, apart from the already discussed underestimate of the $K^*$ decay width. The corresponding prediction for the WA120 data on $f_0(1370)$ has problems: the predicted ratio $\Gamma_{f_3 \rightarrow \eta \eta}/\Gamma_{f_3 \rightarrow \pi \pi}$ is larger than unity and the full two-pseudoscalar decay width is very large (we refer to the discussion of solution I on the issue of the latter point). For the state $f_0(1710)$ we obtain a large ratio $\Gamma_{f_2 \rightarrow \eta \eta}/\Gamma_{f_3 \rightarrow \pi \pi}$, again as in solution I, in contrast to the WA120 result. For the two-photon decays we have a large ratio $\Gamma_{f_3 \rightarrow \gamma \gamma}/\Gamma_{f_1 \rightarrow \gamma \gamma}$.

Table 7. Fitted mass and decay properties of scalar mesons.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (MeV)</th>
<th>Theory</th>
<th>$\chi^2_{\text{tot}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{f_1}$</td>
<td>1350 ± 150</td>
<td>1242</td>
<td>0.519</td>
</tr>
<tr>
<td>$M_{f_2}$</td>
<td>1507 ± 5</td>
<td>1507</td>
<td>0.003</td>
</tr>
<tr>
<td>$M_{f_3}$</td>
<td>1714 ± 5</td>
<td>1714</td>
<td>0.011</td>
</tr>
<tr>
<td>$\Gamma_{f_2 \rightarrow \pi \pi}$</td>
<td>38.0 ± 4.6</td>
<td>38.50</td>
<td>0.010</td>
</tr>
<tr>
<td>$\Gamma_{f_2 \rightarrow \eta \eta}$</td>
<td>9.4 ± 1.7</td>
<td>10.38</td>
<td>0.332</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \rightarrow \pi \pi}$</td>
<td>5.6 ± 1.3</td>
<td>3.65</td>
<td>2.292</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \rightarrow \eta \eta}$</td>
<td>0.20 ± 0.06</td>
<td>0.197</td>
<td>0.002</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \rightarrow \eta \eta}/\Gamma_{f_3 \rightarrow \pi \pi}$</td>
<td>0.48 ± 0.15</td>
<td>0.314</td>
<td>1.121</td>
</tr>
<tr>
<td>$\Gamma_{f_0 \rightarrow K K}/\Gamma_{f_0 \rightarrow \pi \pi}$</td>
<td>0.88 ± 0.23</td>
<td>1.079</td>
<td>0.745</td>
</tr>
<tr>
<td>$\Gamma_{f_0 \rightarrow \eta \eta}/\Gamma_{f_0 \rightarrow \pi \pi}$</td>
<td>0.35 ± 0.16</td>
<td>0.291</td>
<td>0.134</td>
</tr>
<tr>
<td>$\Gamma_{K_0^* \rightarrow K \pi}$</td>
<td>273 ± 51</td>
<td>71.02</td>
<td>16.221</td>
</tr>
<tr>
<td>$(\Gamma_{f_1})_{2P}$</td>
<td>140 ± 10</td>
<td>143.3</td>
<td>0.109</td>
</tr>
<tr>
<td>$\chi^2_{\text{tot}}$</td>
<td>-</td>
<td>-</td>
<td>21.560</td>
</tr>
</tbody>
</table>

Table 8. Decays of $f_1 = f_0(1370)$.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (WA120)</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{f_1 \rightarrow K K}/\Gamma_{f_1 \rightarrow \pi \pi}$</td>
<td>0.46 ± 0.19</td>
<td>1.10</td>
</tr>
<tr>
<td>$\Gamma_{f_1 \rightarrow \eta \eta}/\Gamma_{f_1 \rightarrow \pi \pi}$</td>
<td>0.16 ± 0.07</td>
<td>0.17</td>
</tr>
<tr>
<td>$(\Gamma_{f_1})_{2P}$ (MeV)</td>
<td>&quot;small&quot;</td>
<td>193.5</td>
</tr>
</tbody>
</table>

Table 9. Decays of $f_3 = f_0(1710)$.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (WA120)</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{f_3 \rightarrow K K}/\Gamma_{f_3 \rightarrow \pi \pi}$</td>
<td>5.0 ± 0.7</td>
<td>5.08</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \rightarrow \eta \eta}/\Gamma_{f_3 \rightarrow \pi \pi}$</td>
<td>2.4 ± 0.6</td>
<td>1.59</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \rightarrow \eta \eta}/\Gamma_{f_3 \rightarrow \pi \pi}$</td>
<td>&lt; 0.18</td>
<td>2.01</td>
</tr>
<tr>
<td>$(\Gamma_{f_3})_{2P}$ (MeV)</td>
<td>&quot;dominant&quot;</td>
<td>143.3</td>
</tr>
</tbody>
</table>

D. Fourth solution and implications

The solution IV corresponds to an inverted bare level ordering and a small glueball-quarkonia mixing with

$$M_N = 1.392 \text{ GeV}, \quad M_G = 1.712 \text{ GeV}, \quad M_S = 1.452 \text{ GeV},$$

$$f = -0.050 \text{ GeV}^2, \quad \varepsilon = 0.232 \text{ GeV}^2,$$

$$c_d^* = 6.66 \text{ MeV}, \quad c_m^* = 5.84 \text{ MeV}; \quad \chi^2_{\text{tot}} = 26.330.$$
Although the fit results given in Table 10 are acceptable, the smallness of the glueball-quarkonia mixing is clear contrast to other phenomenological studies \cite{8-13} and lattice result \cite{7, 24}.

The mixing matrix reads:

$$
\begin{pmatrix}
|f_1\rangle & |f_0(1370)\rangle \\
|f_2\rangle & |f_0(1500)\rangle \\
|f_3\rangle & |f_0(1710)\rangle
\end{pmatrix} =
\begin{pmatrix}
0.82 & -0.07 & 0.57 \\
-0.57 & \sim 0 & 0.82 \\
-0.06 & 0.99 & 0.04
\end{pmatrix}
\begin{pmatrix}
|N\rangle & |\bar{NN}\rangle \\
|G\rangle & |\bar{GG}\rangle \\
|S\rangle & |\bar{SS}\rangle
\end{pmatrix}.
$$

In this solution the state $f_0(1710)$ is very close to a pure gluonic configuration, which is traced to the small mixing parameter $f$. The states $f_0(1370)$ and $f_0(1500)$ are in turn dominated by the quarkonia components, but with strong mixing between $\bar{m}n$ and $\bar{s}s$. The decay rates for $f_1 = f_0(1370)$ and $f_3 = f_0(1710)$ as compared to the WA102 data are listed in Tables 11 and 12.

**Table 11.** Decays of $f_1 = f_0(1370)$.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (WA102)</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{f_1 \rightarrow K\bar{K}} / \Gamma_{f_1 \rightarrow \pi\pi}$</td>
<td>0.46 ± 0.19</td>
<td>0.73</td>
</tr>
<tr>
<td>$\Gamma_{f_1 \rightarrow \eta\eta} / \Gamma_{f_1 \rightarrow \pi\pi}$</td>
<td>0.16 ± 0.07</td>
<td>0.13</td>
</tr>
<tr>
<td>$(\Gamma_{f_1})^{2p}$ (MeV)</td>
<td>&quot;small&quot;</td>
<td>99.47</td>
</tr>
</tbody>
</table>

**Table 12.** Decays of $f_3 = f_0(1710)$.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Exp (WA102)</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{f_3 \rightarrow K\bar{K}} / \Gamma_{f_3 \rightarrow \pi\pi}$</td>
<td>5.0 ± 0.7</td>
<td>7.78</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \rightarrow \eta\eta} / \Gamma_{f_3 \rightarrow \pi\pi}$</td>
<td>2.4 ± 0.6</td>
<td>1.76</td>
</tr>
<tr>
<td>$\Gamma_{f_3 \rightarrow \eta\eta} / \Gamma_{f_3 \rightarrow \pi\pi}$</td>
<td>&lt; 0.18</td>
<td>1.00</td>
</tr>
<tr>
<td>$(\Gamma_{f_3})^{2p}$ (MeV)</td>
<td>&quot;dominant&quot;</td>
<td>142.95</td>
</tr>
</tbody>
</table>

Finally, the two-photon decay ratios resulting from solution IV read:

$$
\Gamma_{f_1 \rightarrow 2\gamma} : \Gamma_{f_2 \rightarrow 2\gamma} : \Gamma_{f_3 \rightarrow 2\gamma} : \Gamma_{a_0^0 \rightarrow 2\gamma} = 1 : 0.273 : 0.008 : 0.382.
$$

(70)
E. Discussion of solutions III and IV

The study of the previous subsections shows that a direct glueball decay with large violation of flavor symmetry is feasible. This feature is based on an interpretation of first lattice results, which corresponds to the limiting case of a direct glueball decay as contained in the parameters \( c_g^d \) and \( c_g^m \). This interpretation is in contrast to the phenomenological study of [8] and to the comments given in [24]. Furthermore, such a large value for the flavor symmetry breaking parameter \( c_g^m \) is not in agreement with large \( N_c \) arguments. In any case, considering that the interpretation of lattice results is not unique, from a phenomenological point of view it is interesting to analyze the situation, where a strong, flavor symmetry violating glueball decay is present.

A sizable direct glueball decay is, as already discussed, the result of the phenomenology given in [8]. However, in [8] the direct glueball decay pattern is flavor blind, which would correspond to a large \( c_g^d \) but to a suppressed value for \( c_g^m \). In this sense the solutions III and IV differ from the analysis of [8].

Care should also be taken when considering the two-photon decay in this scheme: if the two-pseudoscalar amplitudes are sizable, the same can also be expected for the transitions into two vector mesons (although not studied here). Invoking vector meson dominance sizable corrections to the two-photon final state are expected. The use of the limit \( c_g^e = 0 \) in the present case is therefore questionable.

V. CONCLUSIONS

In this paper we analyzed the two-pseudoscalar and the two-photon decays of the scalar states between 1-2 GeV in the framework of a chiral Lagrangian, where the glueball has been included as a flavor-blind composite mesonic field with independent couplings to pseudoscalar fields.

In a first step we have set the glueball-pseudoscalar couplings to zero and performed a fit to the accepted averages of PDG2004 [1]. We find two possible solutions (I and II), which, apart from the underestimate of the \( K_0^* \rightarrow K\pi \) mode, show good agreement with the data. The solutions I and II differ in the bare isoscalar masses, in the mixing matrix (in scheme I) the state \( f_0(1500) \) has the largest gluonic amount, while in scheme II the state \( f_0(1710) \) has the main gluonic component), in some predictions concerning other decay modes, which have been compared to the experimental results of [33]. From a phenomenological point of view, there is no striking hint for a direct glueball-pseudoscalar coupling.

We then enlarged our analysis by including the direct glueball decay parameter \( (c_g^d) \) in the flavor symmetry limit in the fit. A small value for this parameter is obtained, which in turn confirms the suppression of the direct glueball decay in agreement with large \( N_c \) arguments and with [8, 10], but contrary to the study of [8]. In a last step we also included the second glueball decay parameter \( (c_g^m) \) in the fit, which involves flavor symmetry breaking in the direct glueball decay. The minima in the fit are less pronounced, therefore we utilized the lattice results of [24] to determine the direct glueball decay parameters \( c_g^d \) and \( c_g^m \). The lattice data are interpreted such as they are matched by the direct glueball decay mechanism, although this procedure might be a limiting case. The resulting fits also generate a good description of the data, where either the \( f_0(1500) \) or the \( f_0(1710) \) contain a dominant glueball component. These solutions however should at present taken with some care and require further input either from an enlarged, reliable database or lattice constraints. For this reason our preferred solutions are I and II presented in section III.

Although the presence of a strong direct glueball decay cannot be verified directly, the presence of a sizable glueball-
quarkonia mixing is essential to be in accord with the data. The magnitude is different in the two proposed solutions (smaller in I, largest in II), but is in line with other models and in magnitude qualitatively consistent with lattice results [15].

The starting point of the Lagrangian has been outlined in [20]. In the present work we added the glueball degree of freedom, both for mixing and decays. As a result of the fit, we also considered deviations from the large $N_c$ limit. We find that large $N_c$ arguments are still useful as a guideline in the scalar sector as well. Also a small deviation from the GMO octet mass relation is found.

A direct isoscalar $\bar{m}_n-\bar{s}_s$ quarkonia mixing has been introduced in the theoretical analysis. Instanton solutions of the QCD vacuum are believed to generate strong flavor-mixing in both the isoscalar and the scalar sectors. The presence of such mixing is established in the pseudoscalar nonet, while it is still an open question in the scalar-isoscalar mesonic sector. Our two proposed solutions differ in this point: while in the first one this flavor mixing sensibly affects the results, in the second solution it turns out to be negligible.

The problem of the $K_0^* \to K\pi$ mode has been discussed; although no final statement to this puzzle can be said, we compared predictions of various approach and attempted to highlight the difficulty in the scalar sector.

Although many experimental results can be reproduced, and the presence of a scalar glueball and its mixing with scalar quarkonia explains many features of the scalar meson spectroscopy, further work, both theoretically and experimentally, is needed to rule out some mixing scenarios in favour of others.

Acknowledgments

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We use the following phase conventions for the pseudoscalar $P = \{ \pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta^0, \eta^8 \}$ and scalar $S = \{ a^0_0, a^0_0, K^*_0, K^*_0, \bar{K}^0_0, S_0, S^8 \}$ meson fields (neglecting the mixing of the third and the eighth component):

\[
\begin{align*}
\pi^\pm &= \frac{1}{\sqrt{2}} (p_1 \mp ip_2), \quad \pi^0 = P_3, \quad K^\pm = \frac{1}{\sqrt{2}} (P_4 \mp ip_5), \\
K^0 &= \frac{1}{\sqrt{2}} (P_6 - ip_7), \quad \bar{K}^0 = \frac{1}{\sqrt{2}} (P_6 + ip_7), \quad \eta^0 = P_0, \quad \eta^8 = P_8,
\end{align*}
\]

(A1)

\[
\begin{align*}
a^0_0 &= \frac{1}{\sqrt{2}} (S_1 \mp i S_2), \quad a^0_0 = S_3, \quad K^*_0 &= \frac{1}{\sqrt{2}} (S_4 \mp i S_5), \\
K^*_0 &= \frac{1}{\sqrt{2}} (S_6 - i S_7), \quad \bar{K}^*_0 &= \frac{1}{\sqrt{2}} (S_6 + i S_7).
\end{align*}
\]

APPENDIX B: TWO-BODY $s \to p_1 p_2$ AND $s \to \gamma \gamma$ TRANSITIONS (MATRIX ELEMENTS AND DECAY WIDTHS)

1. Scalar-isoscalar strong decays

The strong decay widths of the scalar states are derived at tree-level from the following term of the Lagrangian (2):

\[
\mathcal{L}_{\text{decay}}^{\text{strong}} = c_d^s \langle S u_\mu u^\mu \rangle + c_m^s \langle S \chi_+ \rangle + \frac{c_g^9}{\sqrt{3}} \langle u_\mu u^\mu \rangle + \frac{c_g^8}{\sqrt{3}} \langle \chi_+ \rangle.
\]  

(B1)

The decay expression for the scalar-isoscalar states $|i\rangle$ with $i = f_1, f_2, f_3$ into $\pi\pi, K\bar{K}, \eta\eta$ and $\eta\eta'$ are given by the following expressions:

\[
\Gamma_{i \to \pi\pi} = \Gamma_{i \to \pi^+\pi^-} + \Gamma_{i \to \pi^0\pi^0} = \frac{3}{2} \Gamma_{i \to \pi\pi} = \frac{3 \lambda^{1/2}(M_i^2, M_1^2, M_2^2)}{32 \pi M_i^3} |M_{i\pi\pi}|^2,
\]  

(B2)

\[
\Gamma_{i \to K\bar{K}} = \Gamma_{i \to K^+\bar{K}^-} + \Gamma_{i \to K^0\bar{K}^0} = 2 \Gamma_{i \to K\bar{K}} = \frac{\lambda^{1/2}(M_i^2, M_1^2, M_2^2)}{8 \pi M_i^3} |M_{iK\bar{K}}|^2,
\]  

(B3)

\[
\Gamma_{i \to \eta\eta} = \frac{\lambda^{1/2}(M_i^2, M_1^2, M_2^2)}{32 \pi M_i^3} |M_{i\eta\eta}|^2,
\]  

(B4)

\[
\Gamma_{i \to \eta\eta'} = \frac{\lambda^{1/2}(M_i^2, M_1^2, M_2^2)}{16 \pi M_i^3} |M_{i\eta\eta'}|^2
\]  

(B5)

where $\lambda(x, y, z)$ is the Källen triangle function:

\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz.
\]  

(B6)

The matrix elements $M_{i\pi\pi}, M_{iK\bar{K}}, M_{i\eta\eta}$ and $M_{i\eta\eta'}$ are given by

\[
M_{i \to \pi\pi} = -\frac{2 B_{iN}}{F^2 \sqrt{2}} \left\{ [M_i^2 - 2 M_1^2] c_d^s + 2 M_2^2 c_m^s \right\} - \frac{2 B_{iG}}{F^2 \sqrt{3}} \left\{ [M_i^2 - 2 M_2^2] c_d^s + 2 M_2^2 c_m^s \right\}.
\]  

(B7)
\[ M_{i\rightarrow K^+K^-} = -\frac{B_{iN} + \sqrt{2}B_{iS}}{F^2\sqrt{2}} \left\{ [M_i^2 - 2M_K^2] c_d^a + 2M_K^2 c_m^a \right\} \]
\[ - \frac{2B_{iG}}{F^2\sqrt{3}} \left\{ [M_i^2 - 2M_K^2] c_d^a + 2M_K^2 c_m^a \right\}, \] (B8)

\[ M_{i\rightarrow \eta\eta} = -\frac{2c_m^2}{F^2\sqrt{2}} [M_i^2 - 2M_{\eta}^2] \left\{ B_{iN} \sin^2 \delta_P + B_{iS} \cos^2 \delta_P \sqrt{2} \right\} \]
\[ - \frac{4c_m}{F^2\sqrt{2}} \left\{ M_{\pi}^2 B_{iN} \sin^2 \delta_P + [2M_K^2 - M_{\eta}^2] B_{iS} \cos^2 \delta_P \right\} \]
\[ - \frac{2B_{iG}}{F^2\sqrt{3}} \left\{ c_d^a [M_i^2 - 2M_{\eta}^2] + 2c_m^a \left( M_{\pi}^2 + 2[M_K^2 - M_{\eta}^2] \cos^2 \delta_P \right) \right\}, \] (B9)

\[ M_{i\rightarrow \eta\eta'} = \sin \frac{2\delta_P}{F^2\sqrt{2}} \left\{ c_d^a [M_i^2 - M_{\eta}^2 - M_{\eta'}^2] [B_{iN} - B_{iS} \sqrt{2}] \right\} \]
\[ + 2c_m^a [M_i^2 B_{iN} + (M_{\pi}^2 - 2M_{\eta}^2) B_{iS} \sqrt{2}] \]
\[ - 4\sqrt{\frac{2}{3}} c_m^a B_{iG} [M_i^2 - M_{\pi}^2] \right\}, \] (B10)

where \( \delta_P = \theta_P - \theta_P' \) and \( \theta_P' \) is the ideal mixing angle with \( \sin \theta_P' = 1/\sqrt{3} \); the quantities \( B_{ij} \) are the elements of mixing matrix relating physical states \( i = f_1, f_2, f_3 \) and bare states \( j = N, G, S \) (see definitions in Eqs. (44) and (21)).

2. Isovector and isodoublet strong decays

The decay rates for \( a_0(1450) \) into \( K\bar{K}, \pi\eta \) and \( \pi\eta' \) are:

\[ \Gamma_{a_0 \rightarrow K\bar{K}} = \frac{1}{3} \left[ \Gamma_{a_0 \rightarrow K\bar{K}^0} + \Gamma_{a_0 \rightarrow K-K^0} + \Gamma_{a_0 \rightarrow K^+K^-} + \Gamma_{a_0 \rightarrow K^0\bar{K}^0} \right] \]
\[ = \Gamma_{a_0 \rightarrow K^+K^-} = \frac{\lambda^{1/2}(M_{a_0}^2, M_K^2, M_K^2)}{16\pi M_{a_0}^3} |M_{a_0^+K^+K^-}|^2, \] (B11)

\[ \Gamma_{a_0 \rightarrow \pi\eta} = \frac{1}{3} \left[ \Gamma_{a_0 \rightarrow \pi\eta^0} + \Gamma_{a_0 \rightarrow \pi^-\eta^+} + \Gamma_{a_0 \rightarrow \pi^0\eta^-} \right] \]
\[ = \Gamma_{a_0 \rightarrow \pi^+\eta} = \frac{\lambda^{1/2}(M_{a_0}^2, M_{\pi}^2, M_{\eta}^2)}{16\pi M_{a_0}^3} |M_{a_0^+\pi^+\eta}|^2, \] (B12)

\[ \Gamma_{a_0 \rightarrow \pi\eta'} = \frac{1}{3} \left[ \Gamma_{a_0 \rightarrow \pi\eta'^0} + \Gamma_{a_0 \rightarrow \pi^-\eta'^+} + \Gamma_{a_0 \rightarrow \pi^0\eta'^-} \right] \]
\[ = \Gamma_{a_0 \rightarrow \pi^+\eta'} = \frac{\lambda^{1/2}(M_{a_0}^2, M_{\pi}^2, M_{\eta'}^2)}{16\pi M_{a_0}^3} |M_{a_0^+\pi^+\eta'}|^2. \] (B13)

The matrix elements \( M_{a_0^+K^+K^0}, M_{a_0^+\pi^+\eta} \) and \( M_{a_0^+\pi^+\eta'} \) are given by:

\[ M_{a_0^+K^+K^0} = -\frac{1}{F^2} \left( [M_{a_0}^2 - 2M_K^2] c_d^a + 2M_K^2 c_m^a \right) \] (B14)

\[ M_{a_0^+\pi^+\eta} = \frac{\sin \delta_P \sqrt{2}}{F^2} \left( [M_{a_0}^2 - M_\pi^2 - M_\eta^2] c_d^a + 2M_\pi^2 c_m^a \right), \] (B15)

\[ M_{a_0^+\pi^+\eta'} = -\frac{\cos \delta_P \sqrt{2}}{F^2} \left( [M_{a_0}^2 - M_\pi^2 - M_\eta'^2] c_d^a + 2M_\pi^2 c_m^a \right). \] (B16)
The decay rates for $K_0^*$ (1430) into $K\pi$ and $K\eta$ are (considering the isodoublet $\{K_0^{*0}, K_0^{*+}\}$):

$$\Gamma_{K_0^* \to K\pi} = \frac{1}{2} \left[ \Gamma_{K_0^{*+} \to K\pi^0} + \Gamma_{K_0^{*0} \to K\pi^0} + \Gamma_{K_0^{*0} \to K\pi^+} + \Gamma_{K_0^{*+} \to K\pi^0} \right]$$

$$= \frac{3}{2} \Gamma_{K_0^{*+} \to K\pi^+} = \frac{3 \lambda^{1/2}(M_{K_0^*}^2, M_K^2, M_{\pi}^2)}{32\pi M_{K_0^*}^2} |M_{K_0^{*+} + K\pi^+}|^2 ,$$ (B17)

$$\Gamma_{K_0^* \to K\eta} = \frac{1}{2} \left[ \Gamma_{K_0^{*+} \to K\eta^0} + \Gamma_{K_0^{*0} \to K\eta^0} \right]$$

$$= \Gamma_{K_0^{*+} \to K\eta^0} = \frac{\lambda^{1/2}(M_{K_0^*}^2, M_K^2, M_{\eta}^2)}{16\pi M_{K_0^*}^2} |M_{K_0^{*+} + K\eta^+}|^2 ,$$ (B18)

The matrix elements $M_{K_0^{*+} + K\pi^+}$ and $M_{K_0^{*+} + K\eta^+}$ are given by

$$M_{K_0^{*+} + K\pi^+} = -\frac{1}{F^2} \left( |M_{K_0^{*+}} - M_{\pi}| c_d + |M_{K} + M_{\pi}| c_m \right) ,$$ (B19)

$$M_{K_0^{*+} + K\eta^+} = \frac{\cos \delta_p}{F^2} \left( |M_{K_0^{*+}} - M_{\eta} - M_{\pi}| c_d + |3M_{K} - M_{\pi}| c_m \right)$$

$$+ \frac{\sin \delta_p}{F^2 \sqrt{2}} \left( |M_{K_0^{*+}} - M_{\pi} - M_{\eta}| c_d + |M_{K} + M_{\pi}| c_m \right) .$$ (B20)