Cosmological Co-evolution of Yang-Mills Fields and Perfect Fluids

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We study the co-evolution of Yang-Mills fields and perfect fluids in Bianchi type I universes. We investigate numerically the evolution of the universe and the Yang-Mills fields during the radiation and dust eras of a universe that is almost isotropic. The Yang-Mills field undergoes small amplitude chaotic oscillations, as do the three expansion scale factors which are also displayed by the expansion scale factors of the universe. The results of the numerical simulations are interpreted analytically and compared with past studies of the cosmological evolution of magnetic fields in radiation and dust universes. We find that, whereas magnetic universes are strongly constrained by the microwave background anisotropy, Yang-Mills universes are principally constrained by primordial nucleosynthesis and the bound is comparatively weak, and $\Omega_{YM} < 0.105 \Omega_{\text{rad}}$.

I. INTRODUCTION

There is considerable interest in the generation and evolution of magnetic fields in cosmological models\textsuperscript{1}. This interest has a double focus. There is the hope that an explanation might be found for the existence of significant magnetic fields in galaxies. This may require seed fields to originate very early in the history of the universe, or even form part of the initial conditions. We have also discovered that the cosmological evolution of anisotropic stresses, like electric and magnetic fields, has interesting mathematical features that create unexpected physical consequences. These features appear when a homogeneous and anisotropic stress with a trace-free energy-momentum tensor is present during the radiation era of the universe. The isotropic black-body radiation and the anisotropic stresses evolve to first order in the same way, but there is a greatly slowed (logarithmic) decay of the shear anisotropy caused by the pressure anisotropy of the anisotropic fluid\textsuperscript{2,3}. If the evolution were to be linearised around the isotropic model then a zero eigenvalue would exist. When the equation of state of the accompanying perfect fluid changes from radiation to dust the evolution of the shear is still dominated by the pressure anisotropy of the anisotropic stresses but the zero eigenvalue disappears and the shear falls as a reduced power of the cosmic time. In the study of these effects made in ref\textsuperscript{4} we considered the effects of anisotropic stresses whose trace-free anisotropic pressure tensor $\pi_{ab}$ was proportional to a density $\mu_A$, so

$$\pi_{ab} = C_{ab}\mu_A, \quad (1.1)$$

with $C_{ab}$ constant and $C^a_{\ a} = 0$. Barrow and Maartens considered the generalisation of these results to general inhomogeneous cosmological models close to isotropy using the covariant formalism\textsuperscript{5} and also some applications to Kaluza-Klein cosmologies\textsuperscript{6}. The key feature in the evolution of almost-isotropic cosmological models containing perfect fluids plus a magnetic field is that during the radiation era the ratio of the shear to Hubble expansion rate falls only logarithmically in time. The universe evolves towards an attractor state in which it is proportional to the ratio of the magnetic and perfect-fluid densities. This means that the ratio of these densities at the epoch of last-scattering determines the large-angle temperature anisotropy of the cosmic microwave background (CMB). This enables us to use observations of the CMB to place strong bounds on any homogeneous cosmological magnetic field\textsuperscript{7}, or other anisotropic stress defined by a constant $C_{ab}$,\textsuperscript{8}. An important extension of these studies is to consider anisotropic stresses with more general pressure tensors, for example those with time-dependent $C_{ab}$.

An important case with time-dependent $C_{ab}$, which generalises the electromagnetic field, is that of a Yang-Mills stress. This has been studied for pure Yang-Mills fields and no accompanying fluid by several authors in simple Bianchi type I anisotropic universes\textsuperscript{8,9,10}, in all class A Bianchi type universes\textsuperscript{11}, and in Kantowski-Sachs universes\textsuperscript{12}. These studies reveal that the Yang-Mills field creates a form of chaotic behaviour during the evolution of the Yang-Mills stress\textsuperscript{10}. This is not surprising because such behaviour is present in Minkowski space-time and is nothing to do with the cosmological evolution or the spacetime curvature. It should not be confused with the chaotic behaviour of general relativistic origin that is found in the ‘Mixmaster’ universes of Bianchi types VIII and
IX, even in vacuum\(^1\). Therefore the Yang-Mills chaos occurs in the simplest axisymmetric universes of Bianchi type I and is present even when the anisotropy level is arbitrarily small. This raises the interesting question of whether the chaotic evolution might leave an imprint in the temperature anisotropy of the CMB.

However, all of these earlier studies consider the evolution of Einstein-Yang-Mills equations only for the case of a pure Yang-Mills stress in an anisotropic cosmological model. Our experience with the behaviour of magnetic fields in anisotropic universes teaches us that it is important to include the presence of a perfect fluid in order to find the realistic evolution of the Yang-Mills stress during the radiation and dust-dominated eras of the cosmological expansion. In this paper we consider this generalisation by analysing the evolution of spatially homogeneous, anisotropic universes of Bianchi type I containing both a perfect fluid and an anisotropic Yang-Mills stress. By combining analytic and numerical studies we determine the evolution of the shear and the Yang-Mills stress during the early universe in order to discover if it is possible for it to leave an observable signature in the CMB.

In section 2 we set out the Einstein-Yang-Mills equations for the Bianchi class A universes. In section 3 we specialise to consider in detail the evolution for the Bianchi type I universe containing Yang-Mills fields and perfect fluids, focussing on the physically significant cases of pressureless dust and black-body radiation. In section 4 we describe the numerical and analytical results and in section 5 compare them in detail with the situation that results when a pure magnetic field replaces the Yang-Mills field. We consider the observational bounds that can be placed on the magnetic and Yang-Mills fields in the presence of perfect fluids using the microwave background temperature anisotropy and the primordial nucleosynthesis of helium-4 in section 6. We summarise our results in section 7.

II. THE EINSTEIN-YANG-MILLS EVOLUTION EQUATIONS

The Einstein equations to be solved for universes containing a perfect fluid and a Yang-Mills (YM) field are

\[
G_{\mu \nu} = 8\pi GT_{\mu \nu}, \tag{2.1}
\]

\[
T_{\mu \nu} = T_{YM\mu \nu} + T_{m\mu \nu}, \tag{2.2}
\]

\[
T_{m\mu \nu} = \text{diag}(\gamma \mu_m, p_m, p_m, p_m), \tag{2.3}
\]

\[
T_{YM\mu \nu} = \frac{1}{g_{YM}} \left[ F^{(A)}_{\mu \lambda} F^{(A)}_{\nu \lambda} - \frac{1}{4} g_{\mu \nu} F^{(A)}_{\lambda \sigma} F^{(A)}_{\lambda \sigma} \right], \tag{2.4}
\]

where \(F^{(A)}_{\mu \nu}\) is the field strength of YM field, “\(A\)” describes the components of the internal \(SU(2)\) space, and \(\mu_m\) and \(p_m\) have an equation of state

\[
p_m = (\gamma - 1) \mu_m.
\]

We shall assume that \(T_{YM\mu \nu}\) and \(T_{m\mu \nu}\) are separately conserved and there is no energy exchange between them. Hence, from the vanishing covariant divergence of \(T_{YM\mu \nu}\) we find that the YM equations are

\[
\bar{F}^{\mu \nu} - \bar{A}_\nu \times \bar{F}^{\mu \nu} = 0, \tag{2.5}
\]

where \(g_{YM}\) is the self-coupling constant of YM field, and \(\bar{F}^{\mu \nu} = F^{(A)\mu \nu} \bar{\tau}_A\) and \(\bar{A}_\nu = A^{(A)}_\nu \bar{\tau}_A\) with \(\bar{\tau}_A\) being the \(SU(2)\) basis. Setting our units as \(8\pi G/g_{YM}^2 = 1\) \(^2\) and \(c = \hbar = 1\), the basic equations become free of the value of \(g_{YM}\).

We adopt the orthonormal-frame formalism, which has been developed in ref. \(^{18}\). In Bianchi-type spacetimes, there exists a 3-dimensional homogeneous spacelike hypersurface \(\Sigma_t\) which is parameterized by a time coordinate \(t\). The timelike basis is given by \(e_0 = \partial_t\). The triad basis \(\{e_a\}\) on the hypersurface \(\Sigma_t\) is defined by the commutation function \(\gamma^c_{\ a\ b}\) as

\[
[e_a, \ e_b] = \gamma^c_{\ a\ b} e_c. \tag{2.6}
\]

\(^1\) The most general cosmological analysis of the Yang-Mills chaos is that carried out for all Ellis-MacCallum Class A Bianchi-type vacuum cosmologies by Jin and Maeda \(^{12}\) and they are able to consider the simultaneous presence of the Yang-Mills chaos and Mixmaster chaos \(^{13, 14, 15, 16, 17}\).

\(^2\) This choice of units has the following consequences. If \(g_{YM} = 1\), then \(8\pi G = 1\); if \(g_{YM} = 10^{-10}\), then \(8\pi G = 10^{-20}\) and in this case the unit of time is \(10^{10} t_{pl} \sim 10^{-34}\) sec and the present Hubble time is \(H_0^{-1} = 1.2 \times 10^{10} \text{yr} = 4 \times 10^{17} \text{sec} = 10^{51}\).
It is convenient to decompose $\gamma_{ab}^c$ into the geometric variables denoted conventionally by the Hubble expansion $H$, the acceleration $\dot{u}_a$, the shear $\sigma_{ab}$, and the vorticity $\omega_{ab}$, and the variables $\Omega_a$ (the rotation of $e_a$), and the variables $a_a$ and $n_{ab}$, which distinguish the type of Bianchi model, so

$$
\gamma_{ab}^c = -\sigma_b^a - H \delta_b^a - \epsilon^{c}{}_{bc}(\omega^c + \Omega^c),
$$

(2.7)

$$
\gamma_{0a} = \dot{u}_a,
$$

(2.8)

$$
\gamma_{0b} = -2\epsilon_{ab}^c \omega_c,
$$

(2.9)

$$
\gamma_{ab}^c = \epsilon_{abcd}n^d + a_a \delta_b^c - a_b \delta_a^c.
$$

(2.10)

If Bianchi spacetimes are expressed in this way, the vorticity $\omega_{ab}$ always vanishes because they are hypersurface orthogonal.

In order to analyze the EYM system, we shall study the simplest case. First, using the gauge freedom of the YM field, we set $A_0^\alpha(t) = 0$, which simplifies the vector potential to $A = A^\alpha_a(t) \epsilon^\alpha_a \omega_a$, where $\omega_a$ is the dual basis of $e_a$ [13]. Next we shall restrict attention to cosmologies of Bianchi type class A, in which the vector $a_a$ vanishes. We can also diagonalize $n_{ab}$, i.e. $n_{ab} = \text{diag}(n_1, n_2, n_3)$, using the remaining freedoms of a time-dependent rotation of the triad basis. Then, we can show that if $\sigma_{ab}$, $A_0^\alpha$ and $A_0^\beta$ do not have off-diagonal components initially, the equations of motion guarantee that those variables will remain diagonal during the subsequent evolution for the class A Bianchi spacetimes (see the Appendix in [11]). As a result, $\Omega_a$ vanishes and the number of basic variables defining the initial value problem reduces from 21 (12 for spacetime $[H, N_a, \sigma_{ab}, \Omega_a]$ and 9 for YM field $[A_0^a]$) to 10 (7 for spacetime $[H, N_a, \sigma_{aa}]$ and 3 for YM field $[Y_0^a]$).

In order to discuss the dynamics most efficiently, it is convenient to introduce the Hubble-normalized variables, which are defined as follows:

$$
\Sigma_{ab} \equiv \frac{\sigma_{ab}}{H}, \quad N_a \equiv \frac{n_a}{H}.
$$

(2.11)

We also re-express the shear variables of the Bianchi spacetime as

$$
\Sigma_+ = \frac{1}{2} (\Sigma_{22} + \Sigma_{33}), \quad \Sigma_- = -\frac{1}{2\sqrt{3}} (\Sigma_{22} - \Sigma_{33}), \quad \Sigma^2 = \Sigma_+^2 + \Sigma_-^2.
$$

The diagonal components of the YM field potential are described by new variables $a(t)$, $b(t)$, and $c(t)$ as

$$
a \equiv A_1^1, \quad b \equiv A_2^2, \quad c \equiv A_3^3.
$$

and a new time variable, $\tau$, defined by $d\tau = H dt$ is introduced so that $\tau$ denotes the e-folding number of the scale length.

Using these variables we can write down the evolution and constraint equations explicitly. They consist of the generalized Friedmann equation, the dynamical Einstein equations and the YM evolution equations. The generalized Friedmann equation, which is the constraint equation, is

$$
\Sigma^2 + \Omega_{YM} + \Omega_m + K = 1
$$

(2.12)

where $\Omega_{YM}$ is the density parameter of the YM field, i.e. the Hubble-normalized energy density of the YM field, which is defined by

$$
\Omega_{YM} \equiv \frac{\mu_{YM}}{3H^2}
$$

(2.13)

$$
= \frac{1}{6} \left[ (a' + (2\Sigma_+ + 1)a)^2 + \left\{ b' + (\Sigma_+ + \sqrt{3}\Sigma_- + 1)b \right\}^2 + \left\{ c' + (\Sigma_+ - \sqrt{3}\Sigma_- + 1)c \right\}^2
\right.
\left. + \left( Na + \frac{bc}{H} \right)^2 + \left( Nb + \frac{ca}{H} \right)^2 + \left( Nc + \frac{ab}{H} \right)^2 \right];
$$

(2.13)

$\Omega_m$ is the density parameter of perfect fluid, defined by $\Omega_m \equiv \mu_m/(3H^2)$, and $K$ is the Hubble normalized curvature, which is defined by

$$
K \equiv \frac{(-3)R}{6H^2} = \frac{1}{12} \left\{ N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1) \right\}.
$$
Note that the positive 3-curvature corresponds to $K < 0$. Hence, except for the Bianchi type IX models we have $K \geq 0$. The energy density of YM field and perfect fluid are always positive definite. Thus we find that $\Sigma^2$, $\Omega_{YM}$, and $K$ are restricted to the domains $0 \leq \Sigma^2 \leq 1$, $0 \leq \Omega_{YM} \leq 1$, and $0 \leq K \leq 1$, except for the Type IX universes.

The dynamical Einstein equations are
\begin{align}
H' &= -(1 + q)H, \\
\Sigma'_+ &= (q - 2)\Sigma_+ - S_+ + \Pi_+, \\
\Sigma'_- &= (q - 2)\Sigma_- - S_- + \Pi_-, \\
N'_1 &= (q - 4\Sigma_+)N_1, \\
N'_2 &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, \\
N'_3 &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3,
\end{align}
where $q$ is deceleration parameter:
\begin{equation}
q \equiv 2\Sigma^2 + \frac{1}{2}\Omega + \frac{p}{2H^2} = 2\Sigma^2 + \Omega_{YM} + \frac{1}{2}(3\gamma - 2)\Omega_m.
\end{equation}

The quantities $S_\pm$ are given only by the $N_a$:
\begin{align}
S_+ &= \frac{1}{6} \{(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)\}, \\
S_- &= \frac{1}{2\sqrt{3}}(N_3 - N_2)(N_1 - N_2 - N_3).
\end{align}

Last, we define the Hubble-normalized anisotropic pressures of YM field, $\Pi_+$ and $\Pi_-$, so that
\begin{align}
\Pi_+ &= -\frac{1}{6} \left[ -2 \left\{ a' + (-2\Sigma_+ + 1)a \right\}^2 + \left\{ b' + (\Sigma_+ + \sqrt{3}\Sigma_- + 1)b \right\}^2 + \left\{ c' + (\Sigma_+ - \sqrt{3}\Sigma_- + 1)c \right\}^2 \\
&\quad - 2 \left( N_1a + \frac{bc}{H} \right) + \left( N_2b + \frac{ca}{H} \right) + \left( N_3c + \frac{ab}{H} \right) \right], \\
\Pi_- &= \frac{\sqrt{3}}{6} \left[ -\left\{ b' + (\Sigma_+ + \sqrt{3}\Sigma_- + 1)b \right\}^2 + \left\{ c' + (\Sigma_+ - \sqrt{3}\Sigma_- + 1)c \right\}^2 \\
&\quad - \left( N_2b + \frac{ca}{H} \right) + \left( N_3c + \frac{ab}{H} \right) \right].
\end{align}

Here we can see the complexity introduced by the YM dynamics. We can see how the anisotropic pressures are driven by the directional scale factors and their time derivatives, and not solely by the fractions of the density as was the case with the simple model of anisotropic stresses defined by eq. (1.1).

The isotropic matter conservation equation is
\begin{equation}
\Omega_m' = \{2q - (3\gamma - 2)\} \Omega_m
\end{equation}
and the YM evolution equations are
\begin{align}
a'' &= (q - 2)a' + (q - 1 - 4K + 4\Sigma^2 - 4\Sigma_+ - N_1(N_2 + N_3) + 2\Pi_+)a - \frac{1}{H}(N_1 + N_2 + N_3)bc - \frac{1}{H^2}a(b^2 + c^2), \\
b'' &= (q - 2)b' + (q - 1 - 4K + (\Sigma_+ + \sqrt{3}\Sigma_-)(\Sigma_+ + \sqrt{3}\Sigma_- + 2) - N_2(N_3 + N_1) - \Pi_+ - \sqrt{3}\Pi_-)b - \frac{1}{H}(N_1 + N_2 + N_3)ca - \frac{1}{H^2}b(c^2 + a^2), \\
c'' &= (q - 2)c' + (q - 1 - 4K + (\Sigma_+ - \sqrt{3}\Sigma_-)(\Sigma_+ - \sqrt{3}\Sigma_- + 2) - N_3(N_1 + N_2) - \Pi_+ + \sqrt{3}\Pi_-)c - \frac{1}{H}(N_1 + N_2 + N_3)ab - \frac{1}{H^2}c(a^2 + b^2).
\end{align}

The basic variables are therefore $[H, \Sigma_+, \Sigma_-, \Omega_{YM}, N_1, N_2, N_3, a, b, c]$, and they are determined uniquely and completely by Eqs. (2.14) - (2.19), (2.26) - (2.28), along with the constraint equation (2.12).
III. BIANCHI I CO-EVOLUTION OF YANG-MILLS FIELDS AND PERFECT FLUIDS

We compute the solutions of the dynamical equations (2.14)-(2.19) and (2.26)-(2.28) numerically and illustrate the typical evolutionary behaviours. In this section, we consider for simplicity only the Bianchi type I spacetime (so \( K = 0 \)) with black-body radiation (\( \gamma = 4/3 \)) or dust (\( \gamma = 1 \)) and a YM field. We set initial value of \( H \) as \( H_{ini} = 10^{-10} \) (our unit), which is not a restriction because our time unit depends on \( g_{YM} \) which determines the physical identity of the YM field and can be scaled.

First, we display the behavior of the three leading quantities, \( \Sigma^2 \), \( \Omega_{YM} \), and \( \Omega_m \). Figure 1 shows \( \Sigma^2 \to 0 \), i.e., spacetime approaches to the flat isotropic FRW universe. We also find that \( \Omega_{YM} \) does not damp, but approaches a constant. It is interesting to compare this behavior with that which obtains close to isotropy in universes containing radiation and a magnetic field. There we found that \( \Sigma \) and \( \Omega_{mag} \) both fell slowly as \( (\ln t)^{-1} \) as \( t \) increased, with asymptotic approach to an attractor where \( \Omega_{mag}/\Omega_{rad} = \lambda \Sigma \), where \( \lambda \approx O(1) \) is a calculable constant depending on the orientation of the field and the number of effectively massless spin states in the radiation background.

Second, we can display the evolution of the components of the YM field. Figure 2 shows that \( a \), \( b \), and \( c \) follow a sequence of damped chaotic oscillations, and decay as \( e^{-\tau} \). Figure 2 also shows that the frequencies of the oscillations of \( a \), \( b \), and \( c \) are growing, although their amplitudes are decaying. This is further supported by Figure 3 which shows that the mean amplitudes of \( a' \), \( b' \), and \( c' \) remain nearly constant.
FIG. 3: Typical behaviors of the time derivative of the principal components of the Yang-Mills field. The mean amplitudes of this shows amplitudes of $a'$, $b'$, and $c'$ remain nearly constant.

Third, we show the behavior of $\Pi_{\pm}$, the Hubble-normalized anisotropic pressures. Figure 4 shows that the $\Pi_{\pm}$ oscillate in a complicated fashion with approximately constant mean amplitudes. This is consistent with the observation that $\Omega_{\text{YM}} \propto \Omega_{\text{m}}$ but it is surprising that the pressure anisotropy remains constant despite the fall in the shear. These chaotic oscillations are a familiar feature of the YM evolution which resembles the chaotic Hamiltonian dynamics of a point bouncing inside a potential whose four steep walls in the $x-y$ plane are formed by the branches of the rectangular hyperbolae $x^2 y^2 = \text{constant}$ [10, 11].

In the next section we try to understand these features emerging from the numerical studies of Bianchi I spacetime with the YM field and radiation or dust.

A. The approach to isotropy

In this section we consider Bianchi I spacetime again, i.e., $N_a = 0$ and $K = 0$. We assume that the FRW limit holds, that is $\Sigma \to 0$, so the dynamics can be close to isotropy. This results in

$$q = \Omega_{\text{YM}} + \frac{1}{2}(3\gamma - 2)\Omega_{\text{m}}, \quad (3.1)$$

$$\Omega_{\text{YM}} + \Omega_{\text{m}} = 1. \quad (3.2)$$
These conditions imply
\[ q = 1 + \frac{1}{2}(3\gamma - 4)\Omega_m, \] (3.3)
Motivated by the results of the numerical analysis, we also assume that \( \Omega_{YM} \rightarrow \text{const.} \) and \( \Omega_m \rightarrow \text{const.} \) These assumptions reduce (2.26) to the condition that
\[ \Omega'_m = \{2q - (3\gamma - 2)\} \Omega_m = 0, \] (3.4)
and if \( \Omega_m \neq 0 \), we have
\[ q = \frac{1}{2}(3\gamma - 2). \] (3.5)
Hence the evolution of \( H \) is given by \( H' = -(1 + q)H = -\frac{3}{2} \gamma H \). Solving this equation, we have
\[ H \propto \exp\left(-\frac{3}{2} \gamma \tau\right). \] (3.6)
This means that to leading order the YM field does not determine the expansion rate in FRW limit.
From eqs. (3.3) and (3.5), we get
\[ 3\gamma - 4 = (3\gamma - 4)\Omega_m. \] (3.7)
Therefore, in the radiation case (\( \gamma = 4/3 \)), \( \Omega_m \) can converge to any value. Otherwise (\( \gamma \neq 4/3 \)), \( \Omega_m \) becomes dominant, and we must have \( \Omega_m \rightarrow 1 \). In the radiation case (\( \gamma = 4/3 \)), the results become
\[ q = 1, \quad \Omega_{YM} + \Omega_m = 1, \quad H \propto e^{-2\tau}. \]
Hereafter, we will consider only the radiation case.
From our numerical results, we know that the YM field oscillates with growing frequency while its amplitude is damped. Therefore, we assume that \( a \rightarrow 0, a' \neq 0 \). Because the \( \Pi \) are both finite\(^3\), eq. (2.26) becomes
\[ a'' = -a' - \frac{1}{H^2}a(b^2 + c^2) = -a' - e^{-2\tau}a(b^2 + c^2). \] (3.8)
If we divide \( a \) into an amplitude and a phase, with \( a = e^{-p\tau}e^{i\theta_1} \), where \( p \) is the damping rate and \( \theta_1(\tau) \) is the time-dependent oscillation frequency, then
\[ a' = -pa + i\theta_1'a = (-p + i\theta_1')a \] (3.9)
In order to keep \( a' \neq 0, \theta_1' \) has to grow like
\[ \theta_1' \sim O(e^{p\tau}). \]
Differentiating eq. (3.9), this gives
\[ a'' = -pa' + i\theta_1''a + i\theta_1'a' \]
\[ = \{p^2 - \theta_1'^2 + i(\theta_1'' - 2p\theta_1')\}a, \] (3.10)
where \( \theta_1'^2 \) grows \( O(e^{2p\tau}) \) and \( \theta_1'' \) depends on the phase of \( \theta_1' \). Substituting in eq. (3.8) yields
\[ p^2 - \theta_1'^2 + i(\theta_1'' - p\theta_1') = \quad -(-p + i\theta_1') - e^{4\tau}(b^2 + c^2) = -p - \theta_1' + e^{4\tau}e^{-2p\tau}(e^{2i\theta_2} + e^{2i\theta_3}), \] (3.11)
where, for the simplicity, we assume \( b \) and \( c \) damp at the same rate as \( a \), so \( b = e^{-p\tau}e^{i\theta_2} \) and \( c = e^{-p\tau}e^{i\theta_3} \). Picking out the dominant terms in eq. (3.10), we have
\[ -\theta_1'^2 + i\theta_1'' = -e^{(4-2p)\tau}(e^{2i\theta_2} + e^{2i\theta_3}) = -e^{(4-2p)\tau}2\cos(\theta_2 - \theta_3)e^{i(\theta_2 + \theta_3)}. \] (3.12)

\(^3\) In Bianchi I, II, and VI\(_0\) spacetimes, \(-1 \leq \Pi_+ \leq 2\) and \(-\sqrt{3} \leq \Pi_- \leq \sqrt{3}\) (see [11]).
It is generic to assume that there is no relation between \( \theta_2 \) and \( \theta_3 \), so \( \theta_2 \neq \theta_3 \). As a result, the characteristic frequencies of the above equation will be neither purely real nor purely imaginary. Therefore it is likely that \( O(\theta'_{\tau}^2) \sim O(\theta''_{\tau}) \), which means that the leading-order solution of the above equation grows as \( e^{2p\tau} \), i.e., \( p = 1 \).

From this, we can estimate \( \Pi_{\pm} \) as follows,

\[
\Pi_+ = \frac{1}{6} \left( -2a^2 + b'^2 + c'^2 - 2e^{2i(\theta_2 + \theta_1)\tau} + e^{2i(\theta_3 + \theta_1)\tau} + e^{2i(\theta_1 + \theta_2)\tau} \right) \sim O(\text{const.})
\]

\[
\Pi_- = \frac{\sqrt{3}}{6} \left( -b'^2 + c' - 2e^{2i(\theta_3 + \theta_1)\tau} + e^{2i(\theta_1 + \theta_2)\tau} \right) \sim O(\text{const.}).
\]

(3.13)

It follows from our discussion that \( \Pi_+ \neq 0 \) despite the fact that \( \Sigma \to 0 \). This result is quite unexpected.

It is worth checking explicitly that these non-zero anisotropic pressures, \( \Pi_{\pm} \), do not break isotropy. That is, we need to confirm whether these non-zero \( \Pi_{\pm} \) make \( \Sigma \not\to 0 \) or not. Using eqs. (2.24) and (2.25), and the property that \( \Sigma_\pm \ll \Pi_{\pm} \), and assuming \( A^2 + A' \sim B^2 + B' \sim C^2 + C' \), where \( A \equiv a' + (-2\Sigma_+ + 1)a, B \equiv b' + (\Sigma_+ + \sqrt{3}\Sigma_- + 1)b, C \equiv c' + (\Sigma_+ - \sqrt{3}\Sigma_- + 1)c, \), \( A \equiv N_1 a + \sqrt{8} \), \( B \equiv N_2 b + \sqrt{8}, \) and \( C \equiv N_3 c + \sqrt{8}, \) we have

\[
\Sigma' = \frac{(\Sigma^2)'}{2\Sigma} = \frac{1}{\Sigma} (\Sigma_+ \Sigma'_+ + \Sigma_- \Sigma'_-) \sim \frac{1}{\Sigma} (\Sigma_+ \Pi_+ + \Sigma_- \Pi_-)
\]

\[
= \frac{1}{3} \left[ (A^2 + A'^2) \cos \Psi + (B^2 + B') \cos \left( \Psi + \frac{2}{3} \pi \right) + (C^2 + C') \cos \left( \Psi - \frac{2}{3} \pi \right) \right]
\]

\[
\sim \frac{1}{3} \left[ (A^2 + A'^2) \cos \left( \Psi + \frac{2}{3} \pi \right) + \cos \left( \Psi - \frac{2}{3} \pi \right) \right] = 0,
\]

(3.14)

where \( \Psi \equiv \tan^{-1}(\Sigma_-/\Sigma_+) \). This shows that the non-zero \( \Pi_{\pm} \) modes do not break isotropy. The assumption, \( A^2 + A'^2 \sim B^2 + B' \sim C^2 + C' \), is justified by numerical solutions. Figure 5 shows the time averages of \( A^2 + A'^2 \sim B^2 + B' \sim C^2 + C' \) over a period of 0.1\( \tau \), which confirms our assumption.

![FIG. 5: The typical behaviors of the \( \langle A^2 + A'^2 \rangle, \langle B^2 + B' \rangle, \) and \( \langle C^2 + C' \rangle, \) where \( \langle \cdots \rangle \) means time average per 0.1\( \tau \). This shows that \( A^2 + A'^2 \sim B^2 + B' \sim C^2 + C' \), where \( \langle \cdots \rangle \) means time average per 0.1\( \tau \). This shows that \( A^2 + A'^2 \sim B^2 + B' \sim C^2 + C' \), and it is found that \( H \propto e^{-2\tau}, a \propto O(e^{-\tau}), \Sigma' \to 0, \) and the normalised pressures \( \Pi_{\pm} \) oscillate with nearly constant amplitudes. In the next section we provide some further details of this evolution which can be obtained from the numerical studies.

IV. A SURVEY OF TYPICAL EVOLUTIONARY BEHAVIORS

Our numerical studies of the Bianchi I expansion dynamics were performed for various combinations of radiation or dust and YM field. We give here a summary of the principal conclusions from these investigations. These results
are stable against small changes in the initial data ($H_{\text{ini}}$ and $\Sigma_{\text{ini}}$) and appear to be robust. This robustness is a consequence of the generic nature of the chaotic oscillations of the YM field and the fact that these chaotic oscillations have a small effect on the expansion dynamics of the universe.

A. Radiation and YM field

1. YM field dominates: $\Omega_{\text{YM}} \gg \Omega_{\text{rad}}$

In the case where the YM field density dominates the density of the isotropic black-body radiation field, we find that $\Omega_{\text{YM}} \sim \text{const.}$ and $\Omega_{\text{rad}} \sim \text{const.}$ during the evolution. If we define an averaged expansion scale factor for the expanding universe, $\ell(t)$, by

$$ H \equiv \frac{d\ell}{dt} \ell $$

then the normalised shear is found to fall as $\Sigma \propto \ell^{-1.01}$ in the numerical integrations, while the Hubble expansion rate falls at the rate expected in a FRW model, with $H \propto \ell^{-2.00}$, so the mean scale factor evolves as $\ell \propto t^{0.500} \sim t^{1/2}$, as in a FRW universe.

2. Radiation dominates: $\Omega_{\text{YM}} \ll \Omega_{\text{rad}}$

If the black-body radiation density dominates the YM field then we still find $\Omega_{\text{YM}} \sim \text{const.}$ and $\Omega_{\text{rad}} \sim \text{const.}$ and the normalised shear falls almost linearly with the scale factor, as $\Sigma \propto \ell^{-1.07}$, with $H \propto \ell^{-2.00}$ and $\ell \propto t^{0.500} \sim t^{1/2}$ as in the FRW model. Thus we see that the evolution in both of the radiation plus YM field situations is similar, with $\Sigma \propto \ell^{-1}$ and $\ell \propto t^{1/2}$ holding to an excellent approximation. Although the anisotropic pressure can be significant, it is not driving significant shear expansion anisotropy.

B. Dust and YM field

1. YM field dominates: $\Omega_{\text{YM}} \gg \Omega_{\text{dust}}$

When the YM field dominates over the dust we are in the very early stages of the overall evolution because on average the dust density redshifts away more slowly than the YM field. When the YM field still dominates, so $\Omega_{\text{YM}} \sim 1$, the numerical evolution gives

$$ \Omega_{\text{dust}} \propto \ell^{0.983} \quad \Sigma \propto \ell^{-1.01} $$

while $H \propto \ell^{-2.00}$ so $\ell \propto t^{0.500} \sim t^{1/2}$. This reflects the assumption of the YM field domination and is not inconsistent with eq. (3.6) because the assumption used there, $\Omega_{\text{YM}}$ and $\Omega_{\text{m}} \rightarrow \text{const.}$, is violated. We see that the damping rate of shear and the expansion law are the same as that in the radiation-dominated case. It is understandable that $\Omega_{\text{dust}} \propto \ell^{0.983}$, because $\mu_{\text{dust}} \propto \ell^{-3}$ and $H \propto \ell^{-1} \propto \ell^{-2}$. This results in the dependence $\Omega_{\text{dust}} \equiv \mu_{\text{dust}}/3H^2 \propto \ell$, as expected. Thus, $\Omega_{\text{dust}}$ grows and will eventually become dominant and the assumption that $\Omega_{\text{YM}} \gg \Omega_{\text{dust}}$ will eventually fail.

2. Dust dominates: $\Omega_{\text{YM}} \ll \Omega_{\text{dust}}$

This is the natural situation for the universe to evolve into after the radiation-dominated era. Our numerical studies find that in the dust-dominated phase the YM density falls as

$$ \Omega_{\text{YM}} \propto \ell^{-0.954} $$
while the normalised shear falls as
\[ \Sigma \propto \ell^{-1.48} \sim \ell^{-3/2} \]
and the Hubble rate falls as \( H \propto \ell^{-1.52} \). Hence, we obtain a close approximation to the evolution expected in a dust-dominated FRW universe, with \( \ell \propto t^{0.658} \sim t^{2/3} \). As expected, the shear falls off more rapidly than in the radiation-dominated case because of the growing influence of the isotropising dust density. It is also interesting that the shear falls off more rapidly than in the simpler dust plus magnetic universes \([4]\). We note that the fall of the shear in the YM case is close to the \( \Sigma \propto \ell^{-3/2} \) fall-off that would occur if the YM field were absent in a dust-dominated Bianchi I universe. We see how this arises by noting that
\[ \Sigma' = -\frac{3}{2} \Sigma + \Pi_+ \sim -\frac{3}{2} \Sigma + \]
because \( |\Pi_+| < \Omega_{YM} \), and therefore,
\[ \Sigma_+ \sim \exp\left(-\frac{3}{2} t\right) \sim \ell^{-3/2}. \]

V. COMPARISON OF YANG-MILLS AND MAGNETIC FIELDS

One of the original motivations for our study of the evolution of YM fields in the presence of perfect fluids was the unusual behavior found in the case of a pure magnetic field and a perfect fluid, notably in the situation where the perfect fluid is black-body radiation. Since the YM field is a generalisation of a magnetic field it is instructive to compare and contrast the results for these two cases.

In the case of an almost isotropic universe (\( \ell \propto t^{1/2} \)) containing a pure magnetic field (or other anisotropic stresses with pressure anisotropy of the form \([\mathbf{1}])\) and black-body radiation with \( \Omega_{\text{mag}} \ll \Omega_{\text{rad}} \), the expansion-normalised shear falls logarithmically in time, and
\[ \Sigma \propto \mu_{\text{mag}}/\mu_{\text{rad}} \propto 1/\ln t \propto 1/\ln \ell \]
This unusual ‘critical’ evolution arises because there is a zero eigenvalue when we linearise the Bianchi type I magnetic radiation universe around the isotropic Friedmann radiation universe \([\mathbf{3}])\). Note also that the ratio of the magnetic to the black-body density is not constant as is often assumed, but falls slowly due to the coupling to the shear \([2]\).

There is a late-time attractor with \( \mu_{\text{mag}}/\mu_{\text{rad}} \propto \Sigma \) and
\[ \Omega_{\text{mag}} \equiv \mu_{\text{mag}}/3H^2 \propto \mu_{\text{mag}}/\mu_{\text{rad}} \propto \Sigma. \]
It is interesting that \( \Omega_{\text{mag}} \propto \Sigma \). This means that \( \Omega_{\text{mag}} \) can be constrained directly by the CMB temperature anisotropy, \( \Delta T/T \), since the presently observed \( \Delta T/T \propto \Sigma_{\text{rec}} \) where \( \Sigma_{\text{rec}} \sim 1100 \) is the recombination redshift. After accounting for the short period of dust dominated evolution from the equal-density redshift, \( z_{\text{eq}} \sim 10^4 \), to \( z_{\text{rec}} \) this leads to strong limits on any spatially homogeneous cosmological magnetic field today of \( 3.4 \times 10^{-9} (\Omega_{0}b_{0}^2)^{1/2}G \), \([7]\), or on any anisotropic stress with pressures of the form \([\mathbf{4}]\), see ref. \([8]\) for details and examples.

In a dust-dominated era (\( \Omega_{\text{mag}} \ll \Omega_{\text{dust}} \)) the magnetic stresses still slow the fall off of the shear to \( \Sigma \propto \ell^{-1} \propto t^{-2/3} \), whereas we would have had \( \Sigma \propto \ell^{-3/2} \) if the magnetic field was absent. The magnetic density evolves as
\[ \Omega_{\text{mag}} \propto \mu_{\text{mag}}/\mu_{\text{dust}} \propto \ell^{-1}. \]

The results for the different magnetic and YM evolutions are summarised in Table IV:

<table>
<thead>
<tr>
<th>Material content</th>
<th>Shear Evolution, ( \Sigma )</th>
<th>Anisotropic stress density</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dust + magnetic field</td>
<td>( \Sigma \propto \ell^{-1} \propto t^{-2/3} )</td>
<td>( \Omega_{\text{mag}} \propto \Sigma \propto \ell^{-2/3} )</td>
</tr>
<tr>
<td>Dust + YM field</td>
<td>( \Sigma \propto \ell^{-3/2} \propto t^{-1} )</td>
<td>( \Omega_{YM} \propto \ell^{-2/3} )</td>
</tr>
<tr>
<td>Radiation + magnetic field</td>
<td>( \Sigma \propto (\ln \ell)^{-1} \propto (\ln t)^{-1} )</td>
<td>( \Omega_{\text{mag}} \propto \Sigma \propto (\ln \ell)^{-1} )</td>
</tr>
<tr>
<td>Radiation + YM field</td>
<td>( \Sigma \propto \ell^{-1} \propto t^{-1/2} )</td>
<td>( \Omega_{YM} = \text{const.} )</td>
</tr>
</tbody>
</table>

TABLE I: Evolution of the normalised shear, \( \Sigma \equiv \sigma/H \).
VI. OBSERVATIONAL BOUNDS

A. The microwave background

The key difference that these results display between the magnetic and YM fields is the rapid fall-off in the shear that accompanies the YM evolution. As a result the YM field density does not determine a shear attractor at the end of the radiation era and does not have a significant effect on the CMB anisotropy in the way that the magnetic field does. As a corollary, observations of the CMB temperature anisotropy do not provide a direct and powerful upper bound on the present YM field density in the way that they do for a magnetic field density. A scenario with which $\mu_{YM} \sim \mu_{rad}$ is not constrained by shear anisotropy of CMB, although it may be constrained by Big Bang nucleosynthesis (BBN), as we discuss below.

The slow decay of the shear in the magnetic universe means that the CMB anisotropy places a stronger limit on the magnetic field density than can be obtained by considering its effects on the expansion rate of the universe at the time of BBN, $z_{BBN} \sim 10^{9} - 10^{10}$. However, in the YM case the rapid fall in the shear as we go forward in time means that a negligible shear at the time of recombination will be much larger at earlier epochs, with an enhancement factor of

$$\frac{\Sigma_{BBN}}{\Sigma_{rec}} \sim \frac{1 + 2 BBN}{1 + z_{eq}} \sim 10^{5}.$$  

Since $\Sigma_{BBN} < 0.35$ from the helium-4 abundance [18, 20, 21, 22, 23], we see that the BBN anisotropy bound is

$$\Sigma_{rec} \lesssim 3.5 \times 10^{-6}. \quad (6.1)$$

This is about an order of magnitude stronger than the limit imposed by the CMB since $\Delta T / T = \Sigma_{rec} \times f(\theta, \phi, \Omega_0)$,

$$\Sigma_{rec} < 10^{-5}. \quad (6.2)$$

As discussed in Barrow [24], this simple anisotropy evolution leads to an unphysical super-Planck anisotropy energy density in the shear modes at very early times if extrapolated backwards to $z \gg z_{eq}$ from modest values of shear at $z_{eq}$. A physically reasonable requirement (the Planck Equipartition Principle (PEP)) would be that at $t_{pl} \sim G^{1/2} \sim 10^{-43}$s all energy densities contributing to the cosmological dynamics should be bounded above by the Planck density $\mu_{pl} \sim t_{pl}^{-4} \sim 10^{52}gm.cm^{-3}$. This would require $\Sigma_{pl} \leq 1$ at $z_{pl} \sim 10^{32}$ and hence

$$\frac{\Sigma_{rec}}{\Sigma_{pl}} \sim \frac{1 + z_{rec}}{1 + z_{pl}} \sim 10^{-29}$$

and the residual anisotropy in the CMB on large angular scales in the YM Bianchi I universe would be completely negligible. By contrast, in the magnetic universe the slow logarithmic fall in $\Sigma$ allows an interesting anisotropy level $\Sigma_{rec} \sim O(10^{-5})$ to persist at recombination even if $\Sigma_{pl} \leq 1$. The predicted value of $\Delta T / T \sim \Sigma_{rec}$ depends upon the number of relativistic spin states contributing to the equilibrium radiation sea and can also differ if other forms of anisotropy are assumed [24]. In general, we would expect that YM fields in the more complex Bianchi type VII universes would display slow logarithmic decay of their shear regardless of the presence or absence of the YM fields. This is due to the anisotropic 3-curvature which mimics the presence of an anisotropic ‘fluid’ of long-wavelength gravitational waves satisfying (1.1) to a good approximation and leads to $\Sigma \propto (\ln t)^{-1}$, see refs. [25, 26, 27, 28, 29]. Note however, that these modes are excluded in anisotropic open universes if their spatial topology is compact [30, 31]: hyperbolic Bianchi spaces with compact topology must be isotropic.

B. Constraints from BBN

In the scenario containing YM fields studied above, the YM density $\Omega_{YM}$ does not evolve in proportional to the shear. The YM pressure anisotropy does not dominate the simple ‘adiabatic’ decay of the shear ($\sigma \propto t^{-3}$) that occurs in the absence of anisotropic sources or anisotropic 3-curvature. Therefore it is impossible to constrain $\Omega_{YM}$ by the
shear anisotropy alone, in the way that $\Omega_{\text{mag}}$ was constrained by shear. Primordial nucleosynthesis gives the strongest constraint on $\Omega_{\text{YM}}$. In our YM scenario, the expansion rate $H$ is larger than in the isotropic FRW case, because

$$H^2 = \left(\frac{8\pi G}{3}\right) (\mu_{\text{rad}} + \mu_{\text{YM}})$$

where $\mu_{\text{rad}} \propto T^4$ is fixed by observation. The increase in the expansion rate raises the temperature at which neutron-proton abundance ratio freezes out of equilibrium, and this leads to an enhancement in the final helium-4 abundance. The YM field evolves on average like a radiation field and we can use constraints on the density of dark radiation (DR) to limit its allowed effect in this process. This gave a bound of $\Omega_{\text{DR}} < 0.105 \Omega_{\text{rad}}$ at $2\sigma$ confidence level. Therefore, we expect the YM field density to be similarly constrained with the bound

$$\Omega_{\text{YM}} < 0.105 \Omega_{\text{rad}}.$$

A similar bound would be obtained from BBN considerations for the magnetic energy density. However, the bound previously obtained from the CMB anisotropy is far stronger in this case: $\Omega_{\text{mag}} < 10^{-5} \Omega_{\text{rad}}$.

**VII. CONCLUSIONS**

We have analysed the evolution of YM fields in anisotropic radiation and dust cosmologies which evolve close to isotropy. The YM field undergoes chaotic oscillations which produce small chaotic vibrations about the Friedmann expansion when the anisotropy level is small. We investigated the evolution of the shear anisotropy and the YM density during the radiation and dust eras. This revealed significant differences to the unusual situation that is known to exist in magnetic cosmologies containing perfect fluids. In particular, unlike in magnetic universes, there is no attractor in the radiation era which couple the YM energy density to the shear anisotropy. Consequently, there is no direct bound on the YM field density from the CMB temperature anisotropy. We have carried out a comparative analysis of the magnetic and YM universes which shows how magnetic universes of Bianchi type I are principally constrained by the CMB anisotropy whilst the YM universes of this type are constrained by the effects of the YM energy density on the synthesis of helium-4. The YM evolution reveals unusual features. Despite the fall in shear to Hubble expansion ratio, the Hubble-normalised pressure anisotropies induced by the YM field do not decay during the radiation era, but oscillate chaotically with constant amplitudes.

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