The peremptory influence of a uniform background for trapping neutral fermions with an inversely linear potential

Antonio S. de Castro

Universidade de Coimbra
Centro de Física Computacional
P-3004-516 Coimbra Portugal
and
UNESP - Campus de Guaratinguetá
Departamento de Física e Química
12516-410 Guaratinguetá SP - Brasil

Electronic mail: castro@feg.unesp.br
Abstract

The problem of neutral fermions subject to an inversely linear potential is revisited. It is shown that an infinite set of bound-state solutions can be found on the condition that the fermion is embedded in an additional uniform background potential. An apparent paradox concerning the uncertainty principle is solved by introducing the concept of effective Compton wavelength.
1 Introduction

The four-dimensional Dirac equation with an anomalous magnetic-like (tensor) coupling describes the interaction of neutral fermions with electric fields and can be reduced to the two-dimensional Dirac equation with a pseudoscalar coupling when the fermion is limited to move in just one direction. Therefore, the investigation of the simpler Dirac equation in a 1+1 dimension with a pseudoscalar potential might be of relevance to a better understanding of the problem of neutral fermions subject to electric fields in the more realistic 3+1 world.

The bound states of fermions in one-plus-one dimensions by a pseudoscalar double-step potential [1] and their scattering by a pseudoscalar step potential [2] have already been analyzed in the literature providing the opportunity to find some quite interesting results. Indeed, the two-dimensional version of the anomalous magnetic-like interaction linear in the radial coordinate, christened by Moshinsky and Szczepaniak [3] as Dirac oscillator, has also received attention. Nogami and Toyama [4], Toyama et al. [5] and Toyama and Nogami [6] studied the behaviour of wave packets under the influence of that conserving-parity potential whereas Szmytkowski and Gruchowski [7] proved the completeness of the eigenfunctions. More recently Pacheco et al. [8] studied some thermodynamics properties of the 1+1 dimensional Dirac oscillator, and a generalization of the Dirac oscillator for a negative coupling constant was presented in Ref. [9]. The two-dimensional generalized Dirac oscillator plus an inversely linear potential has also been addressed [10].

In recent papers, Villalba [11] and McKeon and Van Leeuwen [12] considered a pseudoscalar Coulomb potential \( V = \lambda/r \) in 3+1 dimensions and concluded that there are no bounded solutions. The reason attributed in Ref. [12] for the absence of bounded solutions is that the different parity eigenstates mix. Furthermore, the authors of Ref. [12] assert that the absence of bound states in this system confuses the role of the \( \pi \)-meson in the binding of nucleons. Such an intriguing conclusion sets the stage for the analyses by other sorts of pseudoscalar potentials. A natural question to ask is if the absence of bounded solutions by a pseudoscalar Coulomb potential is a characteristic feature of the four-dimensional world. In Ref. [9] the Dirac equation in one-plus-one dimensions with the pseudoscalar power-law potential \( V = \mu |x|^{\delta} \) was approached and there it was concluded that only for \( \delta > 0 \) there can be a binding potential. That conclusion renders a sharp contrast to
the result found in [12] since Ref. [9] shows that it is possible to find bound states for fermions interacting by a pseudoscalar potential in 1+1 dimensions, to tell the truth there is confinement, notwithstanding the spinor is not an eigenfunction of the parity operator. One might ponder that the underlying reason is the way the spinors are affected by the behaviour of the potentials at the origin as well as at infinity because this is the difference between the Coulomb potentials in those two dissimilar worlds. Nevertheless, two more recent works show that it is not the case.

It was shown in Ref. [13] that the presence of a uniform background potential is a sine qua non condition for furnishing bounded solutions for the pseudoscalar screened Coulomb potential (∼ e−|x|/λ). This last interesting work encourages the inclusion of a uniform background for other sorts of potentials which otherwise are not able to hold bounded solutions. The parity-violating inversely linear potential (1/|x|) is not a binding potential [9] and the purpose of the present work to investigate the influence of a uniform background on its spectrum. We show that an infinite set of bounded solutions can come into existence because the fermion embedded in the uniform background acquires both effective mass and effective coupling constant. Beyond its importance as a new solution for a fundamental equation in physics, the problem analyzed in the present work adds a new contrast to the conclusions in Ref. [12]. Furthermore, it shows the decisive and masterful influence of a uniform background to furnish an infinite set of bound-state solutions.

2 The Dirac equation with a pseudoscalar potential in a 1+1 dimension

The 1+1 dimensional time-independent Dirac equation for a fermion of rest mass m coupled to a pseudoscalar potential reads

\[ H\psi = E\psi, \quad H = \alpha p + \beta mc^2 + \beta\gamma^5V \]  

where \( E \) is the energy of the fermion, \( c \) is the velocity of light and \( p \) is the momentum operator. We use \( \alpha = \sigma_1 \) and \( \beta = \sigma_3 \), where \( \sigma_1 \) and \( \sigma_3 \) are Pauli matrices, and \( \beta\gamma^5 = \sigma_2 \). Provided that the spinor is written in terms of the upper and the lower components, \( \psi_+ \) and \( \psi_- \) respectively, the Dirac equation decomposes into:
\(-E \pm mc^2\) \(\psi_\pm = i\hbar c \psi'_\pm \pm iV \psi_\mp \) \hspace{1cm} (2)

where the prime denotes differentiation with respect to \(x\). In terms of \(\psi_+\) and \(\psi_-\) the spinor is normalized as \(\int_{-\infty}^{+\infty} dx \left( |\psi_+|^2 + |\psi_-|^2 \right) = 1\) so that \(\psi_+\) and \(\psi_-\) are square integrable functions.

In the nonrelativistic approximation (potential energies small compared to \(mc^2\) and \(E \approx mc^2\)) Eq. (1) becomes

\[\psi_- = \left( \frac{p}{2mc} + i \frac{V}{2mc^2} \right) \psi_+\] \hspace{1cm} (3)

\[\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{V^2}{2mc^2} + \frac{\hbar V'}{2mc} \right) \psi_+ = \left( E - mc^2 \right) \psi_+\] \hspace{1cm} (4)

Eq. (3) shows that \(\psi_-\) is of order \(v/c \ll 1\) relative to \(\psi_+\) and Eq. (4) shows that \(\psi_+\) obeys the Schrödinger equation (at this point the author digresses to make his apologies for mentioning in former papers \((1)-(2)\) that the pseudoscalar potential does not present any contributions in the nonrelativistic limit). Note that the Dirac equation is not invariant under \(V \rightarrow V + \text{const.}\). Therefore, the absolute values of the energy have physical significance and the freedom to choose a zero-energy is lost. This last statement remains truthfully in the nonrelativistic limit. It is also noticeable that the pseudoscalar coupling results in the Schrödinger equation with an effective potential in the nonrelativistic limit, and not with the original potential itself. Indeed, this is the side effect which in a 3+1 dimensional space-time makes the linear potential to manifest itself as a harmonic oscillator plus a strong spin-orbit coupling in the nonrelativistic limit \((3)\). The form in which the original potential appears in the effective potential, the \(V^2\) term, allows us to infer that even a potential unbounded from below could be a confining potential. This phenomenon is inconceivable if one starts with the original potential in the nonrelativistic equation. It has already been verified that a constant added to the screened Coulomb potential is undoubtedly physically relevant \((13)\). As a matter of fact, it plays a crucial role to ensure the existence of bounded solutions. Nevertheless, the resulting potential does not present any nonrelativistic limit.

For \(E \neq \pm mc^2\), the coupling between the upper and the lower components of the Dirac spinor can be formally eliminated when Eq. (2) is written as second-order differential equations:
\[- \frac{\hbar^2}{2} \psi''_{\pm} + \left( \frac{V^2}{2c^2} \pm \frac{\hbar}{2c} V' \right) \psi_{\pm} = \frac{E^2 - m^2 c^4}{2c^2} \psi_{\pm} \]  

These last results show that the solution for this class of problem consists in searching for bounded solutions for two Schrödinger equations. It should not be forgotten, though, that the equations for \( \psi_{+} \) or \( \psi_{-} \) are not indeed independent because \( E \) appears in both equations. Therefore, one has to search for bound-state solutions for both signals in (5) with a common eigenvalue. At this stage on can realize that the Dirac energy levels are symmetrical about \( E = 0 \). It means that the potential couples to the positive-energy component of the spinor in the same way it couples to the negative-energy component. In other words, this sort of potential couples to the mass of the fermion instead of its charge so that there is no atmosphere for the spontaneous production of particle-antiparticle pairs. No matter the intensity and sign of the potential, the positive- and the negative-energy solutions never meet. Thus there is no room for transitions from positive- to negative-energy solutions. This all means that Klein’s paradox never comes to the scenario. The solutions for \( E = \pm mc^2 \), excluded from the Sturm-Liouville problem, can be obtained directly from the Dirac equation (2).

The solutions for \( E = \pm mc^2 \), excluded from the Sturm-Liouville problem, can be obtained directly from the Dirac equation (2). One can observe that such sort of isolated solutions can be written for \( E = +mc^2 \) as

\[
\psi_{-} = N_{-} \exp[-v(x)]
\]

\[
\psi'_{+} - v' \psi_{+} = +i \frac{2mc}{\hbar} N_{-} \exp[-v(x)]
\]

and for \( E = -mc^2 \) as

\[
\psi_{+} = N_{+} \exp[+v(x)]
\]

\[
\psi'_{-} + v' \psi_{-} = -i \frac{2mc}{\hbar} N_{+} \exp[+v(x)]
\]

where \( N_{+} \) and \( N_{-} \) are normalization constants and \( v(x) = \int x dy V(y) / (hc) \). Of course well-behaved eigenstates are possible only if \( v(x) \) has a distinctive leading asymptotic behaviour.
3 The inversely linear potential plus a uniform background

Now let us concentrate our attention on the potential in the form

$$V = -\frac{\hbar c q}{|x|} + V_0$$

where $V_0$ and the dimensionless coupling constant, $q$, are real numbers. In this case the possible isolated solutions corresponding to $E = \pm mc^2$ are excluded from our consideration because they do not fulfill the conditions of continuity and normalizability simultaneously. On the other side, the Sturm-Liouville problem corresponding to Eq. (5) becomes

$$H_{\text{eff}} \psi_{\pm} = -\frac{\hbar^2}{2m_{\text{eff}}} \psi_{\pm}'' + V_{\text{eff}}^{[\pm]} \psi_{\pm} = E_{\text{eff}} \psi_{\pm}$$

where

$$V_{\text{eff}}^{[\pm]}(x) = -\frac{\hbar c q_{\text{eff}}}{|x|} + A_{\pm}(x), \quad A_{\pm}(x) = \frac{\hbar^2}{2m_{\text{eff}}} q [q \mp \text{sgn}(x)]$$

and

$$E_{\text{eff}} = \frac{E^2 - m_{\text{eff}}^2 c^4}{2m_{\text{eff}} c^2}, \quad m_{\text{eff}} = m \sqrt{1 + \frac{V_0^2}{m^2 c^4}}, \quad q_{\text{eff}} = q \frac{V_0}{m_{\text{eff}} c^2}$$

Therefore, one has to search for bounded solutions of a particle in an effective Kratzer-like potential [14].

3.1 Qualitative analysis

Before proceeding, it is useful to make some qualitative arguments regarding the Kratzer-like potential and its possible solutions. Although the potential given by (3) with $q < 0$ gives rise to an ubiquitous repulsive potential in a non-relativistic theory, the possibility of such a sort of potential to bind fermions, if $V_0 < 0$, is already noticeable in the nonrelativistic limit of the Dirac equation (see Eq. (4)). Furthermore, $V_0$ must be different from zero, otherwise
there would be a repulsive effective potential as long as the condition \( A_+ < 0 \) and \( A_- < 0 \) is never satisfied simultaneously. The parameters of the effective potential with \( V_0 \neq 0 \) and \( q_{\text{eff}} > 0 \) fulfill the key conditions to furnish spectra discrete with \( E_{\text{eff}} < 0 \), corresponding to \( |E| < mc^2 \). The Dirac eigenenergies belonging to \( |E| > mc^2 \) correspond to the continuum. Note that the parameters of the effective potential are related in such a manner that the change \( q \rightarrow -q \) induces the change \( A_{\pm} \rightarrow A_{\mp} \). The combined transformation \( q \rightarrow -q \) and \( V_0 \rightarrow -V_0 \) has as effect \( V_{\text{eff}}^{[\pm]} \rightarrow V_{\text{eff}}^{[\mp]} \), meaning that the effective potential for \( \psi_+ \) transforms into that one for \( \psi_- \) and vice versa. On the other hand, the change \( x \rightarrow -x \) induces the change \( V_{\text{eff}}^{[\pm]}(-x) \rightarrow V_{\text{eff}}^{[\mp]}(x) \) \((A_{\pm}(-x) \rightarrow A_{\mp}(x))\), implying that \( |\psi_{\pm}(-x)| \) behaves like \( |\psi_{\mp}(x)| \). One can see that \( \psi_{\pm} \) is subject to a potential-well structure for \( V_{\text{eff}}^{[\pm]} \) when \(|q| > 1\). For \(|q| \leq 1\) the effective potential has a potential-well structure on one side of the \( x \)-axis and a singular at the origin on the other side. The singularity is given by \(-1/|x|\) when \( |q| = 1\), and \(-1/|x|^2\) when \( |q| < 1\). It is worthwhile to mention at this point that the singularity at \( x = 0 \) never menaces the fermion to collapse to the center [15] because in any condition \( A_{\pm} \) is never less than the critical value \( A_c = -\hbar^2/(8m_{\text{eff}}) \). The Schrödinger equation with the Kratzer-like potential is an exactly solvable problem and its solution, for a repulsive inverse-square term in the potential \((A_{\pm} > 0)\), can be found on textbooks [15]-[16]. Since we need solutions involving a repulsive as well as an attractive inverse-square term in the potential, the calculation including this generalization is presented.

Since \( |\psi_{\pm}(-x)| \) behaves like \( |\psi_{\mp}(x)| \) we can concentrate our attention on the half-line and impose boundary conditions on \( \psi_{\pm} \) at \( x = 0 \) and \( x = \infty \). Square-integrability requires that \( \psi_{\pm}(\infty) = 0 \) and the boundary condition at the origin comes into existence by demanding that the effective Hamiltonian given [19] is Hermitian, viz.

\[
\int_0^\infty dx \psi_{k}^* (H_{\text{eff}} \psi_{k}) = \int_0^\infty dx \ (H_{\text{eff}} \psi_{k})^* \psi_{k}
\]

(12)

where \( \psi_k \) is an eigenfunction corresponding to an effective eigenvalue \((E_{\text{eff}})_k\). In passing, note that a necessary consequence of Eq. (12) is that the eigenfunctions corresponding to distinct effective eigenvalues are orthogonal. It can be shown that (12) is equivalent to

\[
\lim_{x \rightarrow 0} \left( \psi_k^* \frac{d\psi_{k'}}{dx} - \frac{d\psi_k^*}{dx} \psi_{k'} \right) = 0
\]

(13)
3.2 Quantitative analysis

According to the previous qualitative analysis, it is convenient to define the dimensionless quantities $z$ and $B$,

$$
z = \frac{2}{h} \sqrt{-2m_{\text{eff}}E_{\text{eff}}} |x|, \quad B = q_{\text{eff}} \sqrt{-\frac{m_{\text{eff}}}{2E_{\text{eff}}}} \tag{14}$$

and reduce (9) to the form

$$
\psi''_{\pm} + \left( -\frac{1}{4} + \frac{B}{z} - \frac{2m_{\text{eff}}A_{\pm}}{h^2 z^2} \right) \psi_{\pm} = 0 \tag{15}
$$

Now the prime denotes differentiation with respect to $z$. The normalizable asymptotic form of the solution as $z \to \infty$ is $e^{-z/2}$. As $z \to 0$, when the term $1/z^2$ dominates, the solution behaves as $z^{s_{\pm}}$, where $s_{\pm}$ is a solution of the algebraic equation

$$
s_{\pm}(s_{\pm} - 1) - \frac{2m_{\text{eff}}A_{\pm}}{h^2} = 0 \tag{16}
$$

The boundary condition at $x = 0$ demands $s_{\pm} \geq 1/2$ so that the solution of (16) is given by

$$
s_{\pm} = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{8m_{\text{eff}}A_{\pm}}{h^2}} \right) = \frac{1}{2} |q + \frac{1}{2}| \tag{17}
$$

The solution for all $z$ can be expressed as $\psi_{\pm} = z^{s_{\pm}} e^{-z/2} w_{\pm}$, where $w_{\pm}$ is solution of Kummer’s equation [17]

$$
z w''_{\pm} + (b_{\pm} - z) w'_{\pm} - a_{\pm} w_{\pm} = 0 \tag{18}
$$

with

$$
a_{\pm} = s_{\pm} - B, \quad b_{\pm} = 2s_{\pm} \tag{19}
$$

Then $w_{\pm}$ is expressed as $M(a_{\pm}, b_{\pm}, z)$ and in order to furnish normalizable $\psi_{\pm}$, the confluent hypergeometric function must be a polynomial. This demands that $a_{\pm} = -n_{\pm}$, where $n_{\pm}$ is a nonnegative integer in such a way that $B > 0$ (corresponding to $q_{\text{eff}} > 0$) and $M(a_{\pm}, b_{\pm}, z)$ is proportional to the associated Laguerre polynomial $L^{b_{\pm} - 1}_{n_{\pm}}(z)$, a polynomial of degree $n_{\pm}$. This requirement,
combined with the first equation of (19), also implies into quantized effective eigenvalues:

\[ E_{\text{eff}} = -m_{\text{eff}} c^2 \frac{q_{\text{eff}}^2}{2 (s_\pm + n_\pm)^2}, \quad n_\pm = 0, 1, 2, \ldots \]  

(20)

with eigenfunctions given by

\[ \psi_\pm = N_\pm z^{s_\pm} e^{-z/2} L_{n_\pm}^{2s_\pm-1}(z) \]  

(21)

where \( N_\pm \) is a normalization constant. Note that the behaviour of \( \psi_\pm \) at very small \( z \) implies into the Dirichlet boundary condition \( \psi_\pm(0) = 0 \). This boundary condition is essential whenever \( A_\pm \neq 0 \), nevertheless it also develops for \( A_\pm = 0 \).

The necessary conditions for binding fermions in the Dirac equation with the effective Kratzer-like potential have been put forward. The formal analytical solutions have also been obtained. Now we move on to consider a survey for distinct cases in order to match the common effective eigenvalue. As we will see this survey leads to additional restrictions on the solutions, including constraints involving the nodal structure of the Dirac spinor.

From (17) one sees that for \( q \geq 1/2 \) one has \( s_+ = q \) and \( s_- = s_+ + 1 \), for \( q \leq -1/2 \) one has \( s_+ = -q + 1 \) and \( s_- = s_+ - 1 \), and for \(-1/2 < q < +1/2 \) one has \( s_+ = -q + 1 \) and \( s_- = s_+ + 2 \). Therefore, demanding a common eigenvalue implies that for \( q \geq 1/2 \) (\( q \leq -1/2 \)) one has \( n_- = n - 1 \) (\( n_- = n + 1 \)), where \( n = n_+ \). On the other hand, for \(-1/2 < q < +1/2 \) one has \( n_- = n - 2q \), showing to be an unacceptable possibility because it does not provide an integer value for \( n - n_- \). As an immediate consequence of this analysis one can see that the solutions split into two distinct classes. In order to write \( \psi_\pm \) on the whole line we recur again to the observation that \(|\psi_\pm(-x)|\) behaves like \(|\psi_\pm(x)|\). Nevertheless, the matter is a little more complicated because the potential presents a singularity at the origin so that \( \psi_\pm \) can present a discontinuity there. In fact, the first-order differential equation given by (2) implies that \( \psi_\pm \) can be discontinuous wherever the potential undergoes an infinite jump. In the specific case under consideration, the effect of the singularity of the potential can be evaluated by integrating (2) from \(-\delta \) to \(+\delta \) and taking the limit \( \delta \to 0 \). The connection condition relating \( \psi_\pm(+\delta) \) and \( \psi_\pm(-\delta) \) can be summarized as
\[ \psi_{\pm}(+\delta) - \psi_{\pm}(-\delta) = \mp q \int_{-\delta}^{+\delta} dx \frac{\psi_{\pm}}{|x|} \quad (22) \]

Substitution of (21) into (22) allows us to conclude that \( \psi_{\pm}(+\delta) = \psi_{\pm}(-\delta) \) in all the circumstances. The continuity of the spinor at the origin does \( \psi_{\pm}(-x) \) to differ from \( \psi_{\pm}(+x) \) just by a factor related to the relative normalization. Therefore,

A) For \( q \geq 1/2 \) (\( V_0 > 0 \)):

\[ E = \pm mc^2 \sqrt{1 + \frac{V_0^2}{m^2 c^4} \left[ 1 - \left( \frac{q}{n + q} \right)^2 \right]}, \quad n = 1, 2, 3, \ldots \]

\[ \psi_+ = e^{-z/2} \left[ \theta(-x) z^{q+1} L_{n+1}^{2q+1}(z) + \theta(+x) z^{q} L_{n}^{2q-1}(z) \right] \quad (23) \]

\[ \psi_- = N e^{-z/2} \left[ \theta(-x) z^{q} L_{n}^{2q-1}(z) + \theta(+x) z^{q+1} L_{n+1}^{2q+1}(z) \right] \]

B) For \( q \leq -1/2 \) (\( V_0 < 0 \)):

\[ E = \pm mc^2 \sqrt{1 + \frac{V_0^2}{m^2 c^4} \left[ 1 - \left( \frac{q}{n - q + 1} \right)^2 \right]}, \quad n = 0, 1, 2, \ldots \]

\[ \psi_+ = e^{-z/2} \left[ \theta(-x) z^{-q} L_{n+1}^{-2q-1}(z) + \theta(+x) z^{-q+1} L_{n}^{-2q+1}(z) \right] \quad (24) \]

\[ \psi_- = N e^{-z/2} \left[ \theta(-x) z^{-q+1} L_{n}^{-2q+1}(z) + \theta(+x) z^{-q} L_{n+1}^{-2q-1}(z) \right] \]

where \( N \) is a normalization constant and \( \theta(x) \) is the Heaviside step function. The preceding results show explicitly that for any \( V_0 \neq 0 \) (recall that \( \text{sgn} (qV_0) > 0 \)) there is an infinite set of bound-state solutions with eigenergies into the spectral gap between \( -mc^2 \) and \( +mc^2 \) arranged symmetrically about \( E = 0 \). It is also evident that the ultrarelativistic zero-eigenmodes are allowed as \( |q| \to \infty \). If \( V_0 = 0 \), though, there are no bounded solutions at all, as predicted by the qualitative arguments.
4 Conclusions

We have succeed in searching for exact bounded solutions for massive neutral fermions by considering a violating-parity inversely linear potential plus a uniform background, despite the mixing of parities. A remarkable feature of this problem is the possibility of trapping a neutral fermion with an uncertainty in the position that can shrink without limit as $|V_0|$ increases. At first glance it seems that the uncertainty principle dies away provided such a principle implies that it is impossible to localize a particle into a region of space less than half of its Compton wavelength (see, e.g., Ref. [18]). However, a result consistent with the uncertainty principle can be obtained by using the effective Compton wavelength $\lambda_c = \hbar/(m_{\text{eff}}c)$. It means that the localization of a fermion under the influence of this sort of pseudoscalar potential does not require any minimum value in order to ensure the single-particle interpretation of the Dirac equation.

Acknowledgments

This work was supported in part by means of funds provided by CNPq and FAPESP.
References