Scalar Field Contribution to Rotating Black Hole Entropy

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Abstract
Scalar field contribution to entropy is studied in arbitrary $D$ dimensional one parameter rotating spacetime by semiclassical method. By introducing the zenithal angle dependent cutoff parameter, the generalized area law is derived. The non-rotating limit can be taken smoothly and it yields known results. The derived area law is then applied to the Bañados-Teitelboim-Zanelli (BTZ) black hole in (2+1) dimension and the Kerr-Newman black hole in (3+1) dimension. The generalized area law is reconfirmed by the Euclidean path integral method for the quantized scalar field. The scalar field mass contribution is discussed briefly.

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1 Introduction

One of the important offshoots of general relativity is black hole physics. Extensive investigations in this field have thrown up riches of information that are significant from the theoretical and/or observational point of view [1, 2]. On the theoretical side, many exact black hole solutions in general relativity are well known [3, 4]. Among them, stationary, axi-symmetric exact solution with asymptotic flatness is known as the Kerr-Newman solution that represents charged, massive, rotating black hole in (3+1) dimensional spacetime [5, 6]. Stationary rotating black hole solution with a negative cosmological term is known as the BTZ solution which asymptotically approaches the anti-de Sitter spacetime in (2+1) dimension [7]. Recently, Kerr-AdS black hole solutions in higher dimensions is studied with the interest of AdS/CFT correspondence [8, 9, 10]. Black holes are assumed to attract many matter fields due to their strong gravitational force but they scatter out almost all the matter information in the form of gravitational or scalar wave radiation so that the final stage is characterized by only three hairs: their mass, angular momentum and charge [11]. The rotation effects induce the ergo region wherein no particle can remain at rest. The ergo region extends to the outside from the horizon, and in this region negative energy of particles can be absorbed into black holes if the particles have angular velocities in the direction opposite to that of the black hole. This effect induces the so called superradiance phenomenon [12], namely, that the energy of the scattered particle can be larger than that of the incident particle, and is generally termed as the Penrose process [13, 14].

Black holes may interact strongly with matter fields and are thought of as thermal objects. From the analogy between the laws of thermodynamics and the laws of black holes, the area law of black holes has been proposed. (A recent article by Bekenstein [15] is also recommended.) Since then, this law (Entropy ∝ Area) has played a pivotal role in the understanding of black hole physics in general [16, 17, 18, 19]. A remarkable consequence of the proportionality is its connection with the holographic principle which produces Einstein’s equations and other thermodynamical relations [20]. The relation of the holographic entropy bound with QFT is investigated by Yurtsever [21]. The area law can be proved by many methods and approximations using the classical black hole solutions of general relativity. The matter field contribution to the black hole entropy has also been studied extensively using some of the methods [22, 23, 24, 25]. Among them, the semiclassical method with the brick wall regularization scheme seems
to be more transparent from the mathematical and physical points of view [24]. These features become evident when the method is applied especially to the well known cases of static Schwarzschild and Reissner-Nordström black holes[26].

However, the statistical mechanics for the matter field contribution in the background of a rotating black hole is somewhat problematic. Calculations involving rotating black holes are relatively few and diverse. Some examples are as follows: The entropy of the Kerr-Newman black hole in (3+1) dimension has been studied by several authors [27, 28, 29, 30] and the extra divergent structures were pointed out to appear due to the superradiant mode. More recent works are found in [31]. A somewhat early work using a complex metric and Jacobi action may be found in [32]. The area law of a rotating black hole emerged only if some additional cutoff parameters were introduced and a special relation between them were imposed. The non-rotating limit of the entropy cannot be taken in these calculations. Likewise, the entropy of BTZ black hole in (2+1) dimension has been studied by several authors [33, 34, 35, 36] . In this case too, the extra divergences appeared due to the superradiant mode and in order to evade these divergences, extra cutoff parameters were introduced. The area law has been derived by fixing the cutoff parameter artificially. Again, the area law does not yield a smooth non-rotating limit. Moreover, the free energy and the entropy from the scalar field contribution in (2+1) dimensional black hole spacetime was reported to depend largely on the specific approach adopted to calculate them [33]. The problems outlined here revealed themselves in the analyses dealing with specific solutions. It is therefore desirable that the calculations be carried out in a sufficiently general framework such that the above difficulties are either removed or at least minimized as far as possible. A solution independent generalized area law is expected to provide a better physical insight in the understanding of rotating black hole thermodynamics.

In this paper, we calculate the scalar field contribution to the black hole entropy in an arbitrary $D$ dimensional rotating black hole spacetime without assuming any particular exact solution. Higher dimensional calculations of this kind may be useful in string theory and M-theory as well [37]. We restrict to treat one parameter rotating black holes in this paper, which includes many essential aspects [38]. The analysis here is carried out under a minimal set of physical requirements: The generic metric components are assumed to be independent of time $t$ and azimuthal angle $\phi$ so that the energy and angular momentum are conserved. The off-diagonal metric component $g_{t\phi}$ is assumed to exist which indicates the rotation of the black hole. We shall first adopt the semiclassical
method for the statistical mechanics of the scalar field and derive the generalized Stefan-Boltzmann’s law with the help of near horizon approximation. In addition, we shall impose the horizon and the temperature conditions on the time and radial components of the metric, restricting to the black hole with simple zero at the horizon and making the temperature well-defined. Under these conditions, we derive the generalized area law of rotating black holes in $D$ dimension. In the derivation, we introduce the zenithal angle dependent cutoff parameter $\epsilon(\theta)$ consistent with the brick wall regularization scheme. We then apply the result to the known rotating black hole solutions: BTZ and Kerr-Newman. Superradiant modes are included and the non-rotating limit can be taken smoothly in our calculation. We shall next adopt the Euclidean path integral method to reconfirm the generalized area law.

This paper is organized as follows. The generalized Stefan-Boltzmann’s law for the massless scalar field will be derived via semiclassical method in section 2.1. The generalized area law will be derived using the brick wall regularization scheme in section 2.2. This law is then applied to (2+1) and (3+1) dimensional black holes in section 3. The area law is reconfirmed by the Euclidean path integral method in section 4. The small scalar field mass contribution to the free energy will be discussed at the end of section 4. In the final section, the results will be summarized and will make some some discussions.

2 Scalar Field Contribution to Rotating Black Hole Entropy by Semiclassical Method

In this section, we study the statistical mechanics for the scalar field in $D$ dimensional one parameter rotating spacetime by the semiclassical method. Our analysis is general in the sense that that the metric does not depend on the explicit black hole solution. For the massless scalar case, we obtain the generalized form of the Stefan-Boltzmann’s law for the free energy, entropy and the internal energy near the rotating black hole horizon. As the real space integral diverges due to the large time delay near the horizon, we introduce the short distance cutoff parameter in the brick wall regularization scheme developed by ’t Hooft [24]. We can then derive the generalized area law for the rotating black holes using the zenithal angle dependent cutoff parameter. We adopt units such that $c = \hbar = k_B = 1$ unless otherwise specified.
2.1 Stephan-Boltzmann’s law in rotating black hole spacetime

In order to study the statistical properties of the black hole entropy in one rotation parameter case, we apply the semiclassical method (or WKB method) and set the $D$ dimensional polar coordinate as

$$x^\rho = (x^0, x^1, x^2, \ldots, x^{D-1}) = (t, r, \phi, \theta^3, \ldots, \theta^{D-1}) ,$$

(2.1)

where the ranges are $t \in (-\infty, \infty)$, $r \in [0, \infty)$, $\phi \in [0, 2\pi]$ and $\theta^3, \ldots, \theta^{D-1} \in [0, \pi]$. The invariant line element is assumed to be of the form

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + g_{\phi\phi} d\phi^2 + 2g_{t\phi} dtd\phi + \sum_{i=3}^{D-1} g_{ii} (d\theta^i)^2 ,$$

(2.2)

where the off diagonal metric $g_{t\phi}$ induces the rotation of the system with the angular velocity $-g_{t\phi}/g_{\phi\phi}$. The metric components in Eq.(2.2) do not depend on $t, \phi$ and can depend on $r$ and $\theta^i$ ($i = 3, \cdots, D-1$). Consequently, two Killing vectors exist:

$$\xi^\rho_t = (1, 0, \cdots, 0) , \quad \xi^\rho_\phi = (0, 0, 1, 0, \cdots, 0) ,$$

(2.3)

which imply the conservations of the total energy $E$ and the azimuthal angular momentum $m$ of a scalar field.

With the metric components in Eq.(2.2), the matter action for the scalar field $\Phi$ of mass $\mu$ in $D$ dimension is

$$I_{\text{matter}}(\Phi) = \int d^Dx \sqrt{-g} \left( -\frac{1}{2} g^{\rho\sigma} \partial_\rho \Phi \partial_\sigma \Phi - \frac{1}{2} \mu^2 \Phi^2 \right) .$$

(2.4)

From this action, the field equation for the scalar field is obtained:

$$\frac{1}{\sqrt{-g}} \partial_\rho \left( \sqrt{-g} g^{\rho\sigma} \partial_\sigma \Phi \right) - \mu^2 \Phi = 0 .$$

(2.5)

We take the ansatz for the scalar field $\Phi$ in the semiclassical method as

$$\Phi(t, r, \phi, \theta^3, \cdots, \theta^{D-1}) \simeq \exp \left( i \sum_{\rho=0}^{D-1} \int_{Q_0} p_\rho dx^\rho \right) ,$$

(2.6)
where the line integral is performed from the fixed point $Q_0$ to an end point $\vec{Q} = (t, \phi, r, \theta^3, \ldots, \theta^{D-1}, r)$ and $p_\rho$ denotes the $D$ dimensional momentum whose components are

$$p_\rho = (-E, p_r, m, p_3, \cdots, p_{D-1}) \, . \quad (2.7)$$

Putting the scalar field function Eq.(2.6) into the field equation Eq.(2.5) and ignoring the derivatives of the metric components, the on-shell energy-momentum relation is obtained

$$-\mu^2 = g^{\rho\sigma} p_\rho p_\sigma = g^{tt} E^2 + g^{rr} p_r^2 + g^{\phi \phi} p_\phi^2 - 2g^{t \phi} E m + \sum_{i=3}^{D-1} g^{ii} p_i^2 \, , \quad (2.8)$$

where the contravariant components of the metric are obtained from the original metric Eq.(2.2) as

$$g^{tt} = g^{\phi \phi} / \Gamma \, , \quad g^{rr} = 1 / g_{rr} \, , \quad g^{\phi \phi} = g_{tt} / \Gamma \, ,$$

$$g^{t \phi} = -g_{t \phi} / \Gamma \, , \quad g^{ii} = 1 / g_{ii} \ (i = 3, \cdots, D - 1) \, , \quad (2.9)$$

with $\Gamma := g_{tt} g^{\phi \phi} - g^{t \phi} g_{t \phi}$.

We note that using the action of a particle picture of mass $\mu$ in rotating spacetime:

$$I_{\text{particle}} = -\mu \int dt \sqrt{-g^{\rho\sigma} \frac{dx^\rho}{dt} \frac{dx^\sigma}{dt}} \, , \quad (2.10)$$

the energy conservation, the angular momentum conservation and the mass shell condition of the particle corresponding to Eq.(2.8) can be derived.

Since the black hole is rotating, its energy can be transferred to scalar particles in the ergo region by the Penrose process [13]. However, not all the black hole energy can be mined out. We can obtain the restriction on energy $E$ and angular momentum $m$ of the scalar particle in the following way [14]. Consider a new Killing vector combining two Killing vectors in Eq.(2.3) linearly as

$$\eta := \xi_t + \xi_\phi \Omega_H \, , \quad (2.11)$$

where the angular velocity on the horizon $r_H$ is defined as

$$\Omega_H := \frac{g^{t \phi}}{g^{tt}} \bigg|_{r_H} = -\frac{g_{t \phi}}{g^{t \phi}} \bigg|_{r_H} \, . \quad (2.12)$$
This vector is light like in future direction on the horizon \(^1\) and the inner product of it with momentum in Eq.(2.7) becomes non-positive, which provides the restriction on the energy:

\[
p \cdot \eta := \sum_{\rho=0}^{D-1} p_\rho \eta^\rho \leq 0 \implies m \Omega_H \leq E .
\] (2.13)

This means that the angular momentum of scalar particle is inverse in sign to the angular velocity of the black hole if the negative energy particle is absorbed into the black hole.

The semiclassical quantization condition is imposed to require the single valueness for the scalar wave function Eq.(2.6) in \((D-1)\) dimensional space, and the number of the quantum state with energy not exceed \(E\) is the sum of phase space \(K\) divided by the unit quantum volume such that

\[
\sum_K \simeq \frac{1}{2\pi^{D-1}} \int d\rho dp_3 d\phi d\theta d\theta_3 \cdots d\theta^{D-1} dp_{D-1} ,
\] (2.14)

where the integration range is the range shown below Eq.(2.1) for angle variables and the range satisfying the energy-momentum condition Eq.(2.8) with the restriction Eq.(2.13) for the momentum variables near the horizon region.

Next we consider the partition function of the scalar field in the rotating black hole geometry, viz.,

\[
Z = \sum_{\{n^{(K)}\}} \exp \left(-\beta \sum_{\{K\}} n^{(K)} (E^{(K)} - m \Omega_H)\right)
\]

\[
= \prod_{\{K\}} \sum_{\{n^{(K)}\}} \exp \left(-\beta n^{(K)} (E^{(K)} - m \Omega_H)\right)
\]

\[
= \prod_{\{K\}} \left[1 - \exp \left(-\beta (E^{(K)} - m \Omega_H)\right)\right]^{-1} ,
\] (2.15)

where \(\beta\) denotes the inverse temperature. The exponent of the Boltzmann factor \(E - m \Omega_H\) is understood taking account of the rotation effect according to the Hartle-Hawking argument \([22]\) and is positive as shown in Eq.(2.13) ensuring that the partition function is well defined. The summation with respect to \(n^{(K)}\)

\(^1\)The light like property of vector \(\eta\) is shown using the horizon condition in subsection 2.2: \(1/g^{tt} = 0\) on the horizon.
and \( K \) are the occupation number sum and the phase space sum in Eq.(2.15) respectively.

The free energy \( F \) is obtained through the partition function:

\[
\beta F = - \log Z = \sum_{\{K\}} \log \left[ 1 - \exp \left( -\beta(E_{(K)} - m\Omega_H) \right) \right].
\]

(2.16)

Using the semiclassical phase space sum in Eq.(2.14) and changing the integration variable from \( p_r \) to \( E \), we obtain the expression of the free energy after the integration by parts in the form

\[
F = - \frac{1}{2\pi^{D-1}} \int_{m\Omega_H}^\infty dE \int d\phi d\theta d^3p_3 \cdots d\theta^{D-1} dp_{D-1} \\
\times \int dr \frac{p_r}{e^{\beta(E - m\Omega_H)}} - 1,
\]

(2.17)

where the radial momentum \( p_r \) is determined by the mass-shell energy-momentum condition Eq.(2.8) as

\[
p_r = \frac{1}{(g^{rr})^{1/2}} \left( -g^{tt} E^2 - g^{\phi\phi} m^2 + 2g^{t\phi} Em - \sum_{i=3}^{D-1} g^{ii} p_i^2 - \mu^2 \right)^{1/2}
\]

\[
= (g^{rr})^{1/2} \left( -g^{tt} (E + \frac{g_{t\phi}}{g_{\phi\phi}} m)^2 - m^2 \frac{g_{\phi\phi}}{g_{\phi\phi}} - \sum_{i=3}^{D-1} \frac{p_i^2}{g_{ii}} - \mu^2 \right)^{1/2}.
\]

(2.18)

The inverse metric relations in Eq.(2.9) have been used in the second equality. 2

In view of the situation of rotating geometry, we introduce a new energy variable as

\[
E' := -p \cdot \eta = E - m\Omega_H.
\]

(2.19)

We now make the near horizon approximation on the angular velocity as

\[-g_{t\phi} / g_{\phi\phi} \simeq \Omega_H \quad \text{for} \quad r \simeq r_H, \]

(2.20)

so that the radial momentum becomes

\[
p_r \simeq (g^{rr})^{1/2} \left( -g^{tt} E'^2 - m^2 \frac{g^{tt}}{g^{\phi\phi}} - \sum_{i=3}^{D-1} \frac{p_i^2}{g_{ii}} - \mu^2 \right)^{1/2}.
\]

(2.21)

\[2\text{Note that the inverse metric component } g^{tt}, \ g^{\phi\phi} \text{ respectively is not equal to } 1/g_{tt}, 1/g_{\phi\phi} \text{ in rotating geometry in general.}\]
The energy and momentum variables are changed to dimensionless ones:

\[ x = \beta E', \quad y_2 = Y \frac{m}{\sqrt{g_{\phi\phi}}}, \]
\[ y_3 = Y \frac{p_3}{\sqrt{g_{33}}}, \quad \cdots, \quad y_{D-1} = Y \frac{p_{D-1}}{\sqrt{g_{D-1D-1}}}, \tag{2.22} \]

with

\[ Y = (-g^{tt}E'^2 - \mu^2)^{-1/2}. \]

Then the free energy is expressed as

\[
F = -\frac{1}{2\pi^{D-1}\beta^D} \int_0^\infty dx \frac{x^{D-1}}{e^x - 1} \int dr \int d\phi d\theta^3 \cdots d\theta^{D-1} (-g^{tt})^{D/2-1/2} \times (g_{rr}g_{\phi\phi}g_{33}\cdots g_{D-1D-1})^{1/2} \left(1 + \frac{\beta^2\mu^2}{g^{tt}x^2}\right)^{D/2-1/2} \frac{v_{\text{unit}}}{2^{D-2}}, \tag{2.23}
\]

where the term \( v_{\text{unit}} \) denotes the \((D - 1)\) dimensional volume of unit sphere from the dimensionless momentum \( y_i \)-integration:

\[
v_{\text{unit}} = 2^{D-2} \int_{-1}^1 dy_2 \int_0^{\sqrt{1-y_2^2}} dy_3 \cdots \int_0^{\sqrt{1-y_2^2-\cdots-y_{D-1}^2}} dy_{D-1} \sqrt{1 - y_2^2 - \cdots - y_{D-1}^2} = \frac{\pi^{D/2-1/2}}{\Gamma(D/2 + 1/2)} = 2^{D-1} \frac{\pi^{D/2-1}}{\Gamma(D)} \tag{2.24}
\]

In the following, we consider the massless scalar field case \((\mu = 0)\), which corresponds to the high temperature case. The mass contribution to the free energy will be discussed in the last part of section 4. The free energy for the massless case is obtained in a compact form

\[
F = -\frac{\zeta(D)\Gamma(D)}{(2\pi)^{D-1}\beta^D} v_{\text{unit}} V_{\text{opt}}, \tag{2.25}
\]

where the dimensionless energy \( x \)-integration is carried out using the formula

\[
\int_0^\infty dx \frac{x^{D-1}}{e^x - 1} = \zeta(D)\Gamma(D), \tag{2.26}
\]
and the optical volume is defined as

\[ V_{\text{opt}} := \int dr d\phi d\theta^3 \ldots d\theta^{D-1} (-g^{tt})^{D/2-1/2} (g_{rr} g_{\phi\phi} g_{33} \ldots g_{D-1D-1})^{1/2}. \] (2.27)

The entropy and the internal energy are then given by

\[ S := \beta^2 \frac{\partial F}{\partial \beta} = \frac{\zeta(D) \Gamma(D + 1)}{(2\pi \beta)^{D-1}} v_{\text{unit}} V_{\text{opt}}, \] (2.28)

\[ U := F + S/\beta = \frac{(D - 1) \zeta(D) \Gamma(D)}{(2\pi)^{D-1} \beta^D} v_{\text{unit}} V_{\text{opt}}. \] (2.29)

The rotation effects are included in \( V_{\text{opt}} \) and may be in \( \beta \).

In order to confirm these thermodynamical formulae, we calculate the internal energy in \( D = 4 \) flat spacetime:

\[ U_{D=4} = \frac{1}{2} \times \frac{\pi^2}{15} \beta^4 V_{\text{opt}}, \] (2.30)

which agrees with the Stefan-Boltzmann’s law up to a numerical factor if we take into account the photon polarization freedom 2. Therefore, thermodynamical formulae in Eqs.(2.25), (2.28) and (2.29) are recognized as the generalization of the Stefan-Boltzmann’s law to that in the \( D \) dimensional rotating spacetime.

Note also that the non-rotating limit can be taken smoothly and the resultant free energy and entropy Eq.(2.25) and Eq.(2.28) agree with the known non-rotating results (in four dimension, see for example, [26]).

### 2.2 Area law in rotating black hole spacetime

In this section, we apply the generalized Stephan-Boltzmann’s law to the problem under consideration: the rotating black hole entropy. In order to estimate the entropy on the black hole horizon, we should impose the following horizon and temperature conditions on metric components \( g^{tt}, g_{rr} \) and adopt the brick wall regularization scheme [24] to perform the real space integration of \( V_{\text{opt}} \) in Eq.(2.28).

1. **Horizon Condition**

   We require simple zeros for the inverse metric components \( 1/g^{tt} \) and \( 1/g_{rr} \) at the black hole horizon \( r_H \), which is the radius corresponding to the outer
zero of these inverse metric components. We do not treat the extreme case of black holes in this paper, where the above inverse metric components have multi-zeros at the horizon. Then we can expand metric components near horizon as

\[
1/g^{tt} \simeq C_t(\theta)(r - r_H), \quad 1/g^{rr} \simeq C_r(\theta)(r - r_H),
\]

(2.31)

where the coefficient functions \( C_t(\theta), C_r(\theta) \) are defined by

\[
C_t(\theta) := \frac{\partial_r}{g^{tt}} \bigg|_{r_H}, \quad C_r(\theta) := \frac{\partial_r}{g^{rr}} \bigg|_{r_H}.
\]

(2.32)

2. Temperature Condition

As we are considering the thermodynamics for the scalar field around rotating black holes, it is necessary to define the temperature of this system. It is defined by the condition that no conical singularity is required in the Rindler space. This gives the temperature as

\[
\frac{2\pi}{\beta_H} = \frac{-\partial_r(1/g^{tt})}{2\sqrt{-g_{rr}/g^{tt}}} \bigg|_{r_H} = \frac{1}{2} \left( C_t(\theta)C_r(\theta) \right)^{1/2}.
\]

(2.33)

As the temperature on the horizon should not depend on angle variables, we impose the condition that the product of the coefficient functions in Eq.(2.33) depends not on the zenithal angles but only on the horizon radius such that we can state

\[
C_t(\theta)C_r(\theta) = \text{independent function on } \theta.
\]

(2.34)

We call this the horizon temperature condition, which will lead the area law for the entropy in the rotating black hole spacetime.

Under these two conditions, we estimate the radial integration part in the optical volume Eq.(2.27) near the horizon and obtain

\[
\int_{r_H+\epsilon}^{L} dr \left( -g^{tt} \right)^{D/2-1/2} (g_{rr})^{1/2} \simeq C_t(\theta)^{-D/2+1/2}C_r(\theta)^{-1/2} \frac{\epsilon^{-D/2+1}}{D/2 - 1}.
\]

(2.35)
where $\epsilon$ and $L$ are the short distance and large distance regularization parameter respectively in the brick wall regularization scheme with their magnitudes restricted by the relation $0 < \epsilon \ll r_H \ll L < \infty$. Eq.(2.35) is obtained by taking the large $L$ limit and is valid for $3 \leq D$. Instead of $\epsilon$ we introduce a more physical cutoff parameter, the invariant cutoff parameter $\epsilon_{\text{inv}}(\theta)$, as

$$\epsilon_{\text{inv}}(\theta) := \int_{r_H}^{r_H + \epsilon} dr \left( g_{rr} \right)^{1/2} \simeq 2C_r(\theta)^{-1/2} \epsilon^{1/2} .$$ (2.36)

Combining together the radial integration Eq.(2.35) and the invariant cutoff Eq.(2.36) into the expression of optical volume Eq.(2.27), we obtain

$$V_{\text{opt}} = \frac{2^{D-2}}{D/2 - 1} (C_tC_r)^{-D/2 + 1/2} \int d\phi d\theta^3 \cdots d\theta^{D-1} \frac{\left( g_{\phi\phi}g_{33} \cdots g_{D-1D-1} \right)^{1/2}}{(\epsilon_{\text{inv}}(\theta))^{D-2}} \bigg|_{r_H} .$$ (2.37)

The optical volume is divergent as the cutoff parameter $\epsilon$ or $\epsilon_{\text{inv}}(\theta)$ tends to zero, because of the large time dilation effect near the horizon. Combining the near horizon expressions of the temperature Eq.(2.33) with the condition Eq.(2.34), the optical volume Eq.(2.37) and the unit sphere Eq.(2.24), we have a generalized form of the entropy:

$$S = \frac{\zeta(D)D\Gamma(D/2 - 1)}{2^{D-3}D/2 - 1} \int d\phi d\theta^3 \cdots d\theta^{D-1} \frac{\left( g_{\phi\phi}g_{33} \cdots g_{D-1D-1} \right)^{1/2}}{(\epsilon_{\text{inv}}(\theta))^{D-2}} \bigg|_{r_H} .$$ (2.38)

Note that the entropy expression Eq.(2.38) does not depend on the coefficient functions $C_t(\theta)$ and $C_r(\theta)$. The invariant cutoff parameter $\epsilon_{\text{inv}}(\theta)$ depends on zenithal angles $\theta$ and is included in the area integration in Eq.(2.38). In this sense, the obtained entropy could be thought of as a zenithal angle dependent area law of the rotating black holes.

However, we change the idea from the $\theta$ independent cutoff parameter $\epsilon$ to the $\theta$ dependent cutoff parameter $\epsilon(\theta)$ defined through the integration

$$\epsilon_{\text{inv}} := \int_{r_H}^{r_H + \epsilon(\theta)} dr \sqrt{g_{rr}} ,$$ (2.39)

which is solved near the horizon as

$$\epsilon(\theta) = \frac{C_r(\theta)}{4} \epsilon_{\text{inv}}^2 .$$ (2.40)
where the invariant cutoff parameter $\epsilon_{\text{inv}}$ is fixed to be constant. Using this constant invariant cutoff parameter, the entropy can be expressed as

$$S = \frac{\zeta(D) \Gamma(D/2 - 1)}{2^D \pi^{3D/2 - 1}} \left( \frac{A}{(\epsilon_{\text{inv}})^{D-2}} \right),$$

where $A$ is the surface area of the rotating black holes on the horizon

$$A := \int d\phi d\theta^3 \cdots d\theta^{D-1} (g_{\phi\phi} g_{33} \cdots g_{D-1D-1})^{1/2} \bigg|_{r_H}.$$  \hspace{1cm} (2.42)

Eq.(2.41) exhibits the desired area law of the entropy in $D$ dimensional rotating black hole spacetime. Note that we have derived the result without using any explicit expression for the metric solutions but using only two conditions; Horizon Condition Eq.(2.31) and Temperature Condition Eq.(2.34). Note also that we can take the smooth non-rotating limit of the generalized area law, which then reproduces the known expressions. This is the main result of this paper.

At this stage, we comment on the superradiant modes which are characteristic of rotation black holes [12]. The horizon radius $r_H$ is defined to occur at the outer zero of the metric $1/g^{tt}$ and the ergosphere radius $r_{\text{ergo}}$ is defined to occur at the outer zero of $g_{tt}$. The difference between these outer radii $r_{\text{ergo}} - r_H \geq 0$ is due to the rotating geometry, and the region between them is the ergosphere. In the momentum picture, the situation corresponds to the energy restriction in Eq.(2.13): $E - m\Omega_H \geq 0$. As azimuthal momentum of the scalar field $m$ can be negative as well as positive, the energy of the scalar fields can also be negative within the energy restriction, which correspond to the superradiant modes. We stress that the superradiant modes are included in our integration regions on radial coordinate $r \in [r_H + \epsilon, L]$ and energy $E \in [\rho \Omega_H, \infty)$.

3 Application

In this subsection, we apply the generalized area law of the entropy Eq.(2.38) or Eq.(2.41) to some special rotating black hole solutions.

3.1 BTZ black hole in (2+1) dimension

This is the rotating black hole solution with a negative cosmological term $\Lambda(< 0)$ in (2+1) dimension. The geometry is asymptotically anti-de Sitter spacetime. The unit of the gravitational constant is set to $G_{D=3} = 1/8$. 

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The line element of the BTZ black hole solution is given by
\[ ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + g_{\phi\phi} d\phi^2 + 2g_{t\phi} dt d\phi , \] (3.43)
with metric components
\[ g_{tt} = M - r^2|\Lambda| , \quad g_{t\phi} = \frac{J}{2} , \]
\[ g_{\phi\phi} = r^2 , \quad g_{rr} = \left(-M + \frac{J^2}{4r^2} + r^2|\Lambda|\right)^{-1} , \] (3.44)
where \( M \) and \( J \) are the mass and the angular momentum of the \((2+1)\) dimensional black hole. Note that all metric components depend only on the radial coordinate \( r \). The contravariant time component of the metric is given by
\[ g^{tt} = -g_{rr} . \] (3.45)
The metric component \( g^{tt} \) has two zeros at the radii given by
\[ r_{\pm} = \left[ \frac{M}{2|\Lambda|} \left(1 \pm (1 - \frac{|\Lambda|J^2}{M^2})^{1/2}\right)\right]^{1/2} , \] (3.46)
which are valid for the range: \( 0 < M \) and \( |J| \leq M/\sqrt{|\Lambda|} \). The event horizon is defined by \( r_H = r_+ \) and the radius of the ergosphere is given by the outer zero of \( g_{tt} \) as, \( r_{\text{erg}} = \sqrt{M/|\Lambda|} \), which is larger than \( r_H \). The off-diagonal component of the metric induces the rotating effect and the angular velocity on the horizon is given by
\[ \Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}} \bigg|_{r_H} = \frac{J}{2r_H^2} . \] (3.47)
Due to the metric relation Eq.(3.45) the temperature condition is satisfied and the temperature on the horizon is given by
\[ \frac{2\pi}{\beta_H} = |\Lambda| \frac{(r_H^2 - r^2)}{r_H} . \] (3.48)
As the metric components are the function of \( r \) only, the invariant cutoff parameter \( \epsilon_{\text{inv}} \) does not dependent on angle variable and is given, as a function of the cutoff parameter \( \epsilon \), by
\[ \epsilon_{\text{inv}} = \int_{r_H}^{r_H+\epsilon} dr \sqrt{g_{rr}} \simeq \left(\frac{2r_H\epsilon}{|\Lambda|(r_H^2 - r_-^2)}\right)^{1/2} . \] (3.49)
The area on the horizon in (2+1) dimension is the perimeter

\[ \mathcal{A} = \int_{0}^{2\pi} \phi \sqrt{g_{\phi\phi}} \bigg|_{r_H} = 2\pi r_H, \quad (3.50) \]

and the entropy of the BTZ black hole from the generalized formula Eq.(2.38) or Eq.(2.41) is obtained as

\[ S_{D=3} = \frac{3\zeta(3)\mathcal{A}}{(2\pi)^3 \epsilon_{\text{inv}}}. \quad (3.51) \]

The entropy of the BTZ black hole Eq.(3.51) includes the rotation effect through the perimeter \( \mathcal{A} \) and the invariant cutoff \( \epsilon_{\text{inv}} \) via horizon \( r_H = r_+ \) in Eq.(3.46). The zero rotation limit can be taken smoothly.

### 3.2 Kerr-Newman black hole in (3+1) dimension

Next we treat the Kerr-Newman black hole as an application to the (3+1) dimensional spacetime. This black hole solution is interesting because the metric components depend on zenithal angles \( \theta \) as well as on the radial coordinate \( r \). The unit of the gravitational constant is set to \( G_{D=4} = 1 \).

The line element of the Kerr-Newman black hole is given by

\[ ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dt\,d\phi + g_{\theta\theta}d\theta^2, \quad (3.52) \]

with the metric components

\[ g_{tt} = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}, \quad g_{t\phi} = -\frac{a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma}, \]

\[ g_{\phi\phi} = \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta, \quad g_{\theta\theta} = \Sigma, \]

\[ g_{rr} = \frac{\Sigma}{\Delta}, \quad (3.53) \]

where

\[ \Sigma(r, \theta) := r^2 + a^2 \cos^2 \theta, \quad \Delta(r) := r^2 - 2Mr + a^2 + e^2. \quad (3.54) \]

Here \( M, a \) and \( e \) denote the three hairs of the black holes: mass, angular momentum per unit mass and charge respectively. The time component of the inverse metric is

\[ g^{tt} = -\frac{g_{\phi\phi}}{\Delta \sin^2 \theta}. \quad (3.55) \]
The inverse metric components $1/g_{tt}$ and $1/g_{rr}$ have two zeros for $\Delta = 0$ at the radii
\[ r_{\pm} = M \pm (M^2 - a^2 - e^2)^{1/2}, \quad (a^2 + e^2 \leq M^2), \quad (3.56) \]
among which horizon is $r_H = r_+$. The radius of the ergoregion is the outer zero of $g_{tt}$ and is given by $r_{\text{erg}} = M + (M^2 - a^2 \cos^2 \theta - e^2)^{1/2}$, which is larger than $r_H$. The angular velocity on the horizon is defined by the ratio of $g_{t\phi}$ and $g_{\phi\phi}$ and is given
\[ \Omega_H = \frac{a}{r_H^2 + a^2}. \quad (3.57) \]
The temperature condition Eq.(2.34) is satisfied due to the expressions of $g_{rr}$ in Eq.(3.53) and $g_{tt}$ in Eq.(3.55) and the temperature on the horizon is given by
\[ \frac{2\pi}{\beta_H} = \frac{r_H - r_-}{2(r_H^2 + a^2)}. \quad (3.58) \]
The invariant cutoff parameter near the horizon is given by Eq.(2.36) as a function of the original cutoff parameter $\epsilon$, viz.,
\[ \epsilon_{\text{inv}}(\theta) = 2\sqrt{\frac{r_H^2 + a^2 \cos^2 \theta}{r_H - r_-}} \sqrt{\epsilon}. \quad (3.59) \]
The entropy of the Kerr-Newman black hole is obtained from the general expression in Eq.(2.38) as
\[ S = \frac{1}{360\pi} \int_{-\pi}^{\pi} d\phi \int_{0}^{\pi} d\theta \left( \frac{(g_{\phi\phi} g_{\theta\theta})^{1/2}}{(\epsilon_{\text{inv}}(\theta))^2} \right)_{r_H} - \frac{A_{\text{rea}}}{720\pi} \int_{0}^{\pi} d\theta \frac{\sin \theta}{(\epsilon_{\text{inv}}(\theta))^2}, \quad (3.60) \]
in which the area of the Kerr-Newman black hole is given by
\[ A = 4\pi(r_H^2 + a^2). \quad (3.61) \]

Instead of using the constant cutoff parameter, the $\theta$ dependent cutoff parameter Eq.(2.40) is used, which, in this case, is
\[ \epsilon(\theta) = \frac{(r_H - r_-)^2}{4(r_H^2 + a^2 \cos^2 \theta)} \epsilon_{\text{inv}}, \quad (3.62) \]
and the generalized area law in the Kerr-Newman case is obtained

\[ S = \frac{1}{360\pi} \frac{A}{\epsilon_{\text{inv}}^2}. \]  

(3.63)

The final form Eq.(3.63) is the same form as the non-rotating black hole cases, but the rotating effects are included in \( A \) and implicitly in \( \epsilon_{\text{inv}} \) through \( \theta \) dependent \( \epsilon(\theta) \) parameter.

The results above show that the generalized area law works well enough giving known results. The non-rotating limit is straightforward.

4 Another Derivation: Euclidean Path Integral Method

In this section, we try to derive the entropy formula of the scalar field in the rotating black hole spacetime by the Euclidean path integral method. This is a check of our result whether it depends on the calculation method or the approximation method.

Using the Euclidean time \( \tau := it \), we first set the Euclidean \( D \) dimensional polar coordinate as

\[ x^a = (x^1, x^2, x^3, \cdots, x^D) = (r, \phi, \theta^3, \cdots, \theta^{D-1}, \tau) , \]  

(4.64)

and rewrite the line element of Eq.(2.2) in Euclidean form as

\[ ds^2 = g_{\tau \tau} d\tau^2 + g_{rr} dr^2 + g_{\phi \phi} d\phi^2 + 2g_{\tau \phi} d\tau d\phi + \sum_{a=3}^{D-1} g_{aa} (d\theta^a)^2 , \]  

(4.65)

with \( g_{\tau \tau} = -g_{tt} \) and \( g_{\tau \phi} = -ig_{t\phi} \). Just as in the previous section, the metric components are assumed to be functions of \( \tau \) and \( \phi \) for the rotating spacetime and then two Killing vectors exist:

\[ \xi^a_\tau = (0, \cdots, 0, 1) , \quad \xi^a_\phi = (0, 1, 0, \cdots, 0) , \]  

(4.66)

which imply the energy and angular momentum conservations. The Euclidean action \( I_{\text{matter}} \) and the Lagrangian density \( \mathcal{L} \) of the scalar field of mass \( \mu \) are written
as
\[ \begin{align*}
I_{\text{matter}} &= \int d^Dx \sqrt{g_E} \mathcal{L}, \\
\mathcal{L} &= -\sum_{a,b=1}^D \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi - \frac{1}{2} \mu^2 \Phi^2,
\end{align*} \tag{4.67} \]

where \( g_E \) denotes the determinant of the Euclidean metric. The canonical
momentum of the scalar field is given by
\[ \Pi := \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = -g^{\tau \tau} \dot{\Phi} - g^{\tau \phi} \partial_\phi \Phi , \tag{4.68} \]

with the notation \( \dot{\Phi} := \partial \Phi / \partial \tau \). The quantization condition is given by
\[ \left[ \Phi (\tau, x), \Pi (\tau, y) \right] = \delta^{D-1}(x - y) \sqrt{g_E} . \tag{4.69} \]

The partition function with inverse temperature \( \beta \) is
\[ Z = \text{Tr} \left[ e^{-\beta (H - \Omega H P_\phi)} \right] , \tag{4.70} \]

where \( \Omega_H \) denotes the angular velocity of the black hole on the horizon. The
total energy \( H \) and the azimuthal angular momentum \( P_\phi \) of the scalar field are
given by
\[ \begin{align*}
H &= \int_\Sigma (\xi_\tau) a T^{a \tau} d\Sigma_\tau = \int d^{D-1}x \sqrt{g_E} \left( \Pi \dot{\Phi} - \mathcal{L} \right), \\
P_\phi &= \int_\Sigma (\xi_\phi) a T^{a \phi} d\Sigma_\tau = -\int d^{D-1}x \sqrt{g_E} \Pi \partial_\phi \Phi , \tag{4.71} \end{align*} \]

which involve the energy-momentum tensor
\[ T^{ab} := -\frac{2}{\sqrt{g_E}} \frac{\delta I_{\text{matter}}}{\delta g_{ab}} , \tag{4.72} \]

and the Killing vectors \( \xi \) as in Eq.(4.66). The exponent of the Boltzmann factor
is the linear combination of these conserved quantities:
\[ H - \Omega_H P_\phi = \int_\Sigma (\xi_\tau + \Omega_H \xi_\phi) a T^{a \tau} d\Sigma_\tau = \int d^{D-1}x \sqrt{g_E} H' , \tag{4.73} \]
which should be non-negative according to the same argument as in Eq. (2.13). The newly defined Hamiltonian density $\mathcal{H}'$ is given by

\[
\mathcal{H}' = -\frac{1}{2}g_{\tau \tau} \Pi^2 + \sum_{a=1}^{D-1} \frac{1}{2g_{aa}} (\partial_a \Phi)^2 + (\Omega_H + \frac{g_{r\phi}}{g_{\phi \phi}})\Pi \partial_\phi \Phi + \frac{\mu^2}{2} \Phi^2
\]

\[
\simeq -\frac{1}{2}g_{\tau \tau} \Pi^2 + \sum_{a=1}^{D-1} \frac{1}{2g_{aa}} (\partial_a \Phi)^2 + \frac{\mu^2}{2} \Phi^2 , \quad (4.74)
\]

where sum runs $a = r, \phi, 3, \cdots, D - 1$. The near horizon approximation has been used in Eq. (4.74), corresponding to Eq. (2.20), such that

\[
-\frac{g_{r\phi}}{g_{\phi \phi}} \simeq \Omega_H \quad \text{for} \quad r \simeq r_H . \quad (4.75)
\]

The contravariant $\tau$ component of the metric in Eq. (4.74) is $g_{\tau \tau} = g_{\phi \phi}/(g_{r\tau}g_{\phi \phi} - g_{r\phi}^2)$, which does not equal to $1/g_{\tau \tau}$ for rotating case. Turning back to the calculation of the partition function, we express it in the Euclidean path integral form

\[
Z = \int [D\Phi \ D\Pi \ g_{EE}^{1/2}] \exp \left( \int d^Dx \sqrt{g_E} (\Pi \dot{\Phi} - \mathcal{H}') \right) . \quad (4.76)
\]

We perform the momentum integration and obtain after the integration by parts as

\[
Z = \int [D\Phi g_{EE}^{1/4} (g_{\tau \tau})^{1/2}] \exp \left( -\int d^Dx \sqrt{g_E} \frac{g_{\tau \tau}}{2} \Phi \ K \ K \Phi \right) , \quad (4.77)
\]

where $K$ denotes the kernel in the optical space given by

\[
\bar{K} := -\partial^2 - \frac{1}{g_{\tau \tau}} (\frac{1}{\sqrt{g_E}} \sum_{a=2}^{D} \partial_a \sqrt{g_E} \partial_a - \mu^2) . \quad (4.78)
\]

After the Gaussian integration with respect to $\Phi$, we obtain the free energy using the heat kernel representation [39] as

\[
\beta F = \frac{1}{2} \ln \det K = \frac{1}{2} \text{Tr} \ln K
\]

\[
= -\frac{1}{2} \text{Tr} \int_0^\infty \frac{ds}{s} \exp (-s \bar{K}) . \quad (4.79)
\]
The trace of the heat kernel is divided into two parts: the Euclidean time part and the space part. The Euclidean time part is evaluated by using the one dimensional eigenfunction of $-i\partial_{\tau}$ as

$$\text{Tr} \exp(s \partial^2_{\tau}) = \int_0^\beta d\tau \sum_{\ell=\infty}^{\infty} \frac{1}{\beta} \exp(-s \frac{2\pi \ell}{\beta})$$

$$= \sum_{\ell=-\infty}^{\infty} \frac{\beta}{(4\pi s)^{1/2}} \exp\left(-\frac{\beta^2 n^2}{4s}\right). \quad (4.80)$$

The trace of the trace parts is evaluated by the asymptotic expansion method

$$\text{Tr} \exp\left(\frac{s}{g^{\tau\tau}} \left(\frac{\partial^2_{\phi}}{g_{\phi\phi}} + \sum_{a=1}^{D-1} \frac{\partial^2_{\tau}}{g_{a a}} + \frac{\partial^2_{\tau}}{g_{\tau\tau}} - \mu^2\right)\right)$$

$$= \frac{1}{(4\pi s)^{(D-2)/2}} \sum_{k=0}^{\infty} \bar{B}_k(-s)^k \exp\left(-\frac{s\mu^2}{g^{\tau\tau}}\right)$$

$$\simeq \frac{1}{(4\pi s)^{(D-2)/2}} \bar{B}_0 \exp\left(-\frac{s\mu^2}{g^{\tau\tau}}\right). \quad (4.81)$$

where $\bar{B}_k$'s are the coefficient functions of the asymptotic expansion and only the lowest contribution $\bar{B}_0$ is taken account, which is the integration function in the optical space, given by

$$\bar{B}_0 = \int d^{D-1}x (g^{\tau\tau})^{(D-1)/2} (g_{\tau\tau} g_{\phi\phi} g_{33} \cdots g_{D-1,D-1})^{1/2}. \quad (4.82)$$

In the asymptotic expansion, the derivative terms of the metric are considered to be small and neglected in the lowest contribution. The free energy is expressed by multiplying the two trace parts and we obtain

$$F = - \int_0^\infty ds \frac{1}{s (4\pi s)^{D/2}} \sum_{n=1}^{\infty} \exp\left(-\frac{\beta^2 n^2}{4s}\right) \bar{B}_0 \exp\left(-\frac{\mu^2 s}{g^{\tau\tau}}\right)$$

$$= - \frac{1}{\beta^D (4\pi)^{D/2}} \int_0^\infty \frac{dt}{t^{D/2} e^{-t}} \sum_{n=1}^{\infty} \frac{1}{n^D} \bar{B}_0 \exp\left(-\frac{\mu^2 \beta^2 n^2}{4t g^{\tau\tau}}\right). \quad (4.83)$$

\(^3\text{Note that the lowest order contribution of the asymptotic expansion in the trace calculation Eq.}(4.81)\text{ can also be derived by using the semiclassical momentum eigenfunction (Eq.}(2.6)\text{) with normalization factor and by the Gaussian integration over momenta. This confirms the result of the lowest contribution from another side.}\)
where the integration variable in second equality has been changed according to

\[ t = \frac{\beta^2 n^2}{4s} . \]  

The temperature independent term \((n = 0)\) is subtracted in the sum of Eq.(4.83). The massless case \((\mu = 0)\) may be important and in this case we have a compact expression for the free energy

\[ F = -\zeta(D)\Gamma(D/2) \frac{\pi^{D/2} \beta^D}{\Gamma(D)} V_{\text{opt}} , \]  

where we have used the equality relation \(\bar{B}_0 \times 1 = V_{\text{opt}}\) for the massless case, where \(V_{\text{opt}}\) is defined in Eq.(2.27) in the previous section. The massless expression for the free energy obtained by the Euclidean path integral method in Eq.(4.85) completely coincides with that obtained by the semiclassical method in Eq.(2.25) with the use of the expression of \(v_{\text{unit}}\) in Eq.(2.24). Non-rotating limit of the free energy can be taken smoothly and agrees with the previous result [23].

Scalar field mass contribution

So far we have studied the massless scalar case using two methods, semiclassical and the Euclidean path integral method, and find that the thermodynamic quantities are the same. How about the mass contribution? In general, it is not so easy to estimate this. Only the small mass contribution to the free energy can be studied by the perturbation method in four dimension. The zeroth order free energy in \(D = 4\) is given, from Eq.(2.25) or Eq.(4.85), by

\[ F_0^{(4-\text{dim.})} = -\frac{\pi^2}{90\beta^4} \int drd\phi d\theta (g^{r\tau} g^{r\phi} g^{\phi\theta})^{1/2} , \]  

which is the same for two methods, of course. In right hand side, the space integration part is the four dimensional optical volume itself. The next order with respect to the scalar field mass contribution to the free energy is calculated from Eq.(2.23) or Eq.(4.83) as

\[ F_1^{(4-\text{dim.})} = \frac{\mu^2}{24\beta^2} \int drd\phi d\theta (g^{r\tau} g^{r\phi} g^{\phi\theta})^{1/2} , \]  

which again happens to coincide with for two methods. It is worthwhile noting that the next order mass contribution does not show the Stephan-Boltzmann's
law. The short distance singular behavior is the logarithmic divergent with respect to the cutoff parameter $\epsilon$ and is less singular than the first order contribution, and does not show the area law as well.

Here we comment on the allowed range of energy. This is determined by two restrictions: the threshold energy from (2.8), $E \geq -\mu/\sqrt{g^{tt}}$, and the rotating effect on the energy in (2.13), $E \geq m\Omega_H$. The threshold effect to the free energy is shown to be negligible for $D \geq 3$ in the small mass case in the semiclassical method. The situation is similar in the Euclidean path integral method. Further we can show that the small mass contribution to the free energy in arbitrary $D$ dimension is also the same in both methods:

$$F_1^{(D-dim.)} = \frac{\mu^2}{4\pi^{D/2}\beta_H^{D-2}} \Gamma\left(\frac{D}{2} - 1\right)\zeta(D-2) \bar{B}_0 \frac{1}{g^{rr}},$$

(4.88)

which is less singular than zeroth order free energy, where $\bar{B}_0$ is given in (4.82) and does not show the area law as well.

5 Summary and Discussions

We have studied the statistical mechanics of the scalar field in $D$ dimensional rotating spacetime. We have imposed the following physically admissible minimal set of conditions on the one parameter black hole metric.

a1. Metric components are assumed not to be functions of $t$ and $\phi$ which ensure the energy and angular momentum conservations.

a2. Off-diagonal component of the metric, viz., $g_{t\phi}$ is assumed to exist which indicates rotation with the angular velocity $-g_{t\phi}/g_{\phi\phi}$.

a3. Horizon condition is imposed on the inverse of metric components, that is, $1/g^{tt}$ and $1/g_{rr}$ to have simple zeros at $r_H$ respectively.

a4. Temperature condition is also imposed on $\partial_r(1/g^{tt}) \times \partial_r(1/g_{rr})$ at $r_H$ to be $\theta$ independent. This condition leads the $\theta$ independent horizon temperature.

Further, we have used some approximations:

b1. Semiclassical method is adopted for the calculation.

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b2. Near horizon approximation is adopted. The implication of this approximation is that the angular velocity of the scalar field near the horizon is the same as that of the black hole on the horizon.

b3. Mass contribution of the scalar field is neglected because the high temperature limit is considered mainly.

Under these conditions and approximations, we derived the generalized Stefan-Boltzmann’s law and the generalized area law of the rotating black holes in arbitrary $D$ dimensional spacetime. One of the key point is the introduction of the zenithal angle dependent cutoff parameter $\epsilon(\theta)$, which leads to the constant invariant cutoff parameter $\epsilon_{\text{inv}}$ and the generalized area law in a compact form for rotating black holes. The generalized area law is applied to the BTZ black hole in (2+1) dimension and the Kerr-Newman black hole in (3+1) dimension. Non-rotating limit of these thermal quantities can be taken smoothly and they straightforwardly reproduce the known results.

We also adopted the Euclidean path integral method for the quantized scalar field with the asymptotic expansion and derived results for the free energy in rotating black hole spacetime that are consistent with the semiclassical method. Small scalar field mass contribution has been assessed both in the semiclassical and in the Euclidean methods and the same result followed. The mass contribution to the entropy is less singular than the massless case and doesn’t show the area law.

There are other methods to study the statistical mechanics of the scalar field in black hole spacetime. Among them, the quasi-normal mode approach [40] and the Green’s function approach are interesting because more rigorous treatments than the present methods could be possible in those approaches, especially in the (2+1) dimension [33, 41]. Another interesting problem would be to study the thermodynamics of the multi-parameter rotating black hole with the cosmological term in multi-dimension [8, 9, 10] because the existence of the cosmological term is suggested in the observations and the recent development in the view of AdS/CFT correspondence is interesting. These are problems we intend to work in the future [38].
References


[38] M. Kenmoku, Many parameter cases of rotating black holes will be studied in a separate paper.

