CSW Diagrams and Electroweak Vector Bosons

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Abstract. Based on the joined work performed together with Z. Bern, D. Forde, and D. Kosower [1], in this talk it is recalled the (twistor-motivated) diagrammatic formalism describing tree-level scattering amplitudes presented by Cachazo, Srček and Witten, and it is discussed an extension of the vertices and accompanying rules to the construction of vector-boson currents coupling to an arbitrary source.


INTRODUCTION

The computation of amplitudes for QCD and mixed electroweak-QCD processes is an important part of a physics program at modern-day colliders, given their important role as experimentally distinctive probes of new physics.

In less than a couple of years, the progress in the evaluation of scattering processes has received a strong boost, due to a better understanding of the analytic structure of scattering amplitudes. Stimulated by Witten’s realization [2] that tree-level gluon amplitudes, once transformed in twistor space [3], have a simple geometrical description, Cachazo, Srček and Witten (CSW) [4] proposed a powerful set of new computational rules to deal with many particles scattering amplitudes in QCD. Only recently, Risager [5] has found that the CSW approach can be understood as a particular class of a more general set of on-shell recurrence relations for amplitudes, which meanwhile had been introduced by Britto, Cachazo, Feng and Witten (BCFW) [6].

The CSW rules are of interest in their own right for tree-level computations. Their efficiency for gluonic amplitudes, was soon extended to account for massless external fermions [7] and Higgs boson coupled to QCD via a massive top-quark loop (in the infinite-mass limit) [8], and improved when recast in a recursive form [9]. They also allow great simplification in loop calculations [10].

In this talk is described the extension of the CSW construction to include building blocks for mixed QCD-electroweak amplitudes, by providing a construction of vector currents, which may in principle be coupled to an arbitrary source. We will focus on the case of coupling a process involving one quark pair and any number of gluons to one colorless off-shell vector boson. The key idea in the construction is to introduce a new set of basic vertices coupling to the off-shell vector boson, having either one or no

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negative-helicity gluons. The rules for combining them into new currents with additional negative-helicity legs are then in fact the same as those of CSW.

**COLOR DECOMPOSITIONS**

Color decompositions \([11, 12]\) allows the disentangling of the the gauge group factors and the pure kinematical terms, namely *partial amplitudes*, in the full momentum-space amplitudes. For example, the tree-level \(n\)-gluon amplitude \(\mathcal{A}_n\) has the color decomposition,

\[
\mathcal{A}_n(1, 2, \ldots, n) = \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_n(\sigma(1), \ldots, \sigma(n)),
\]

where \(S_n/Z_n\) is the group of non-cyclic permutations on \(n\) symbols, and \(j\) denotes the \(j\)-th gluon and its associated momentum. We use the color normalization \(\text{Tr}(T^a T^b) = \delta^{ab}\). Similar decompositions hold for cases involving quarks. In general, it is more convenient to calculate the partial amplitudes than the entire amplitude at once.

The cases in which we are interested here involve colorless vector bosons. Single massive vector boson exchange is easily obtained from pure QCD amplitudes (which are directly calculable from CSW diagrams). For example, for \(e^+ e^- \rightarrow \gamma^* \rightarrow q\bar{q} + n\) gluons, where \(\gamma^*\) represents an off-shell photon, the amplitude reduces to

\[
\mathcal{A}_n(1_{e^+}, 2_{e^-}, 3_q, 4, 5, \ldots, (n-1), n_{\bar{q}}) = -2e^2 Q^2 g^{n-2} \times \sum_{\sigma \in S_{n-4}} (T^{a_{\sigma(4)}} \cdots T^{a_{\sigma(n-1)}})_{13} \times A_n(1_{e^+}, 2_{e^-}, 3_q, \sigma(4), \ldots, \sigma(n-1), n_{\bar{q}}),
\]

where we use an all outgoing momentum convention. The particle labels \(q, \bar{q}, e^-, e^+\) stand for quarks, anti-quarks, electrons and positrons, while legs without labels, for gluons. The off-shell photon, is internal to the amplitude and exchanged between the lepton pair and the quark pair.

To convert the exchanged photon to an electroweak vector boson, for describing \(e^+ e^- \rightarrow Z, \gamma^* \rightarrow q\bar{q} + ng\), one may adjust the coupling and modify the photon kinematic pole to account for an unstable massive particle \([13, 14]\). More generally, one may convert gluons to photons, purely by group theoretic rearrangements \([15]\). However, in general, it is not possible to then convert the photonic amplitudes to ones involving electroweak vector bosons since vector bosons have non-abelian self interactions which photons do not.

A purpose of this talk is to see how constructing an appropriate off-shell continuation so that the CSW diagrammatricks, originary introduced to describe pure gluon scattering, can be applied to such cases as well.

**CSW DIAGRAMS**

The CSW construction \([4]\) builds amplitudes out of vertices which are off-shell continuations of the Parke–Taylor amplitudes \([16, 17]\). These amplitudes, with two negative-helicity gluons and any number of positive-helicity ones, are the maximally helicity-
We follow the standard spinor normalizations where one is proportional to the auxiliary light-cone reference momentum $\eta$ independent of the choice of to obtain an on-shell amplitude. As shown by CSW [4], on-shell amplitudes are in fact factors introduced in this off-shell continuation cancel when sewing together vertices.

The prescription for continuing MHV amplitudes or vertices off shell is to replace, also, if two off-shell vectors sum to zero, $K = K^0 + \zeta(K) \eta$, where $\eta$ is an arbitrary light-like reference vector, in the Parke-Taylor formula. The extra factors introduced in this off-shell continuation cancel when sewing together vertices to obtain an on-shell amplitude. As shown by CSW [4], on-shell amplitudes are in fact independent of the choice of $\eta$, implying that the sum over MHV diagrams is Lorentz invariant.

In our construction we use an alternative, but equivalent way of going off-shell [19, 9]. We instead decompose an off-shell momentum $K$ into a sum of two massless momenta, where one is proportional to the auxiliary light-cone reference momentum $\eta$ (with $\eta^2 = 0$),

$\eta \cdot k^i = 0, \quad (K^0)^2 = 0$.  

The constraint $(K^0)^2 = 0$ yields $\zeta(K) = K^2/(2\eta \cdot K)$. If $K$ goes on shell, $\zeta$ vanishes. Also, if two off-shell vectors sum to zero, $K_1 + K_2 = 0$, then so do the corresponding $k^i$’s. The prescription for continuing MHV amplitudes or vertices off shell is to replace, when $k_i$ is taken off shell. In the on-shell limit, $\zeta(K)$ vanishes and $k_i^0 \rightarrow k_i$. Although equivalent to the original CSW prescription, it is a bit more convenient to implement. In particular, there are no extra factors associated with going off-shell and the MHV vertices carry the same dimensions as amplitudes.
FIGURE 1. The stripped diagrams for 3 minus helicity amplitudes with vector boson exchange between two fermion pairs. Legs 1 and 2 correspond to the leptons and the legs 3 and n to the quarks. Lines with arrows represent quarks and those without arrows represent either vector bosons or gluons.

The CSW construction replaces ordinary Feynman diagrams with diagrams built out of MHV vertices and ordinary propagators. Each vertex has exactly two lines carrying negative helicity (which may be on or off shell), and at least one line carrying positive helicity. The propagator takes the simple form $i/K^2$, because the physical state projector is effectively supplied by the vertices. For example, with this notation an all-gluon vertex would be,

$$V(1^+, \ldots, m^-_1, (m_1+1)^+, \ldots, n, K^-) = i \frac{\langle m_1 K^\circ \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n K^\circ \rangle \langle K^\circ 1 \rangle}. \quad (9)$$

The CSW rules then instruct us to write down all tree diagrams with MHV vertices, subject to the constraints that each vertex has exactly two negative-helicity gluons and at least one positive-helicity gluon attached, and that each propagator connects legs of opposite helicity. For amplitudes with two negative-helicity gluons, the vertex with all legs taken on shell is then the amplitude. For each additional negative-helicity gluon, we must add a vertex and a propagator. The number of vertices is thus the number of negative-helicity gluons, less one.

As a simple example we may use the CSW rules to construct next-to-MHV (NMHV) partial amplitudes needed for the process $e^+ e^- \rightarrow \gamma^*, Z, W \rightarrow q\bar{q} + ng$. The `stripped diagrams’ (where all the positive helicity gluons are not indicated) for this process are shown in fig. FIGURE 1. Dressing the diagrams with the positive helicity gluon legs between $q$ and $\bar{q}$ in the color ordering leads to

$$A(1^-_q, 2^+_q, 3^+_q, 4^+, 5^+, \ldots, (n-1)^-, n^-_{\bar{q}})$$

$$= \sum_{j=4}^{n-1} V(1^-_q, 2^+_q, 3^+_q, 4^+, 5^+, \ldots, (j-1)^+, (-K_{1\ldots(j-1)}^-) \frac{i}{K^2_{1\ldots(j-1)}}$$

$$\times V((-K_{j\ldots n})^+_q, j^+, \ldots, (n-2)^+, (n-1)^-, n^-_{\bar{q}})$$

$$+ \sum_{j=4}^{n-2} V(1^-_q, 2^+_q, 3^+_q, 4^+, 5^+, \ldots, (j-1)^+, (-K_{1\ldots(j-1)})^+_q, n^-_{\bar{q}}) \frac{i}{K^2_{n1\ldots(j-1)}}$$
where \( K_{i...j} = k_i + k_{i+1} + \cdots + k_j \). Renaming \( q', q' \rightarrow e^+, e^- \) gives the partial amplitudes appearing in the vector boson exchange amplitudes \( (2) \).

MHV VERTICES FOR VECTOR BOSON CURRENTS

In this section we generalize the CSW construction to allow couplings to arbitrary sources. We focus on the phenomenologically interesting case of vector boson currents, though our construction of currents is applicable more generally.

An important application of these currents is that they allow us to couple the electroweak theory to QCD, while taking full advantage of the CSW formalism on the QCD side. The currents satisfy a similar color decomposition as the photon exchange amplitude \( (2) \),

\[
\mathcal{J}_\mu (1_q, 2, 3, \ldots, (n-1), n_{q'}, \gamma^\nu) = g_V g_n \sum_{\sigma \in S_{n-2}} (T^{a\sigma(2)} T^{a\sigma(3)} \cdots T^{a\sigma(n-1)})_{i_1 i_2}^\dagger \\
\times J^\mu_{(1_q, \sigma(2), \sigma(3), \ldots, \sigma(n-1), n_{q'}, \gamma^\nu)} ,
\]

where \( g_V \) is the appropriate coupling for a vector boson \( V = \gamma^\nu, Z, W \) and \( P_V \) is the momentum carried by the vector boson. We need consider only the partial currents \( J^\mu \) in much the same way that we need only consider color-ordered partial amplitudes.

We start by defining two currents that will serve as new basic vertices for obtaining general vector boson currents:

1. A vector-boson current with \( n \) gluon emissions, all of positive helicity

\[
J^\mu (1_q^-, 2^+, \ldots, (n-1)^+, n_{q'}^+ ; P_V ) = \frac{i}{\sqrt{2}} \frac{\langle(-1)^{n-1} | \gamma^\mu P_V |(-1)^+ \rangle}{\langle(-1)^2 | (23) \cdots (n-1) n \rangle} \\
= c_+ e^{(+)\mu} (P_{V}, \eta) + c_- e^{(-)\mu} (P_{V}, \eta) \\
+ c_L \left( P_{V}^\mu - \frac{P_{V}^2}{\eta \cdot P_V} \eta^\mu \right) ,
\]

where \( P_V = -K_{1...n} \) by momentum conservation, where ‘-1’ as a spinor argument denotes \(-k_1\), and where

\[
c_+ = -V_{\text{MHV}} (1_q^-, \ldots, n_{q'}^+ ; P_{V^-}) ,
\]

\[
c_- = V_{\text{MHV}} (1_q^-, \ldots, n_{q'}^+ ; P_{V^+}) \frac{\langle 1 \eta \rangle^2 P_{V}^2}{\langle \eta P_{V} \rangle^2 \langle 1 P_{V} \rangle^2} ,
\]

\[
c_L = V_{\text{MHV}} (1_q^-, \ldots, n_{q'}^+ ; P_{V^-}) \frac{\sqrt{2} \langle 1 \eta \rangle}{\langle \eta P_{V} \rangle \langle 1 P_{V} \rangle} .
\]
FIGURE 2. The NMHV vector boson current in terms of diagrams where positive helicity gluon lines have been stripped.

The vertex $V_{\text{MHV}}$ is simply a CSW vertex for one photon, one quark pair, and $n-2$ gluons, obtained by fermionic phase adjustments from the amplitude in eq. \((5)\). As with the basic CSW vertices, when any colored leg $j$ is taken off shell, the $k_j$ argument to all spinor products or spinor strings must be replaced by $k_j^\flat$.

2. A purely bosonic basic current emitting a single vector state,

\[
J_\mu^\mu((-P_V)^-; P_V) = \frac{i}{\sqrt{2}} \frac{\langle \eta^+ | \gamma^\mu | P_V^\flat \rangle}{[P_V^\flat \eta]} P_V^2 = i \varepsilon^{(-)\mu}(P_V^\flat, \eta)P_V^2. \tag{14}
\]

The first of these is the vector-boson current for positive helicity gluons \([13]\). The second is just a negative helicity polarization vector with reference momentum taken to be the CSW reference momentum.

The polarizations in the above equations are defined using the spinor helicity method and are given by \([20]\)

\[
\varepsilon_\mu^{(+)}(k, r) = \frac{1}{\sqrt{2}} \frac{\langle r^- | \gamma^\mu | k^- \rangle}{\langle r k \rangle}, \quad \varepsilon_\mu^{(-)}(k, r) = \frac{1}{\sqrt{2}} \frac{\langle r^+ | \gamma^\mu | k^+ \rangle}{[k r]}, \tag{15}
\]

where $r$ is a null reference momentum.

We take the currents \((12)\) and \((14)\) to act as vertices, using the same CSW prescriptions \((8)\) as used for defining vertices from MHV amplitudes.

To illustrate the construction of a current with more negative helicities, consider the NMHV vector boson current, $J_\mu(1_+^+, 2^+, \ldots, (n-2)^+, (n-1)^-, n_\eta^-; P_V)$ where the negative helicity legs are nearest neighbors in the color ordering. The CSW diagrams for this current may be organized using the four diagrams shown in fig. 2 where the positive helicity gluon legs have all been stripped away. Inserting back the positive helicity gluon legs, leads to the following expression for this NMHV vector boson current,

\[
J_\mu(1_+^+, 2^+, \ldots, (n-2)^+, (n-1)^-, n_\eta^-; P_V)
= \sum_{j=2}^{n-1} J_\mu(1_+^+, 2^+, \ldots, (j-1)^+, (K_{j(n)}^-; P_V) \frac{i}{K_{j(n)}^2}
\times V((-K_{j(n)}^-)^+, j^+, \ldots, (n-2)^+, (n-1)^-, n_\eta^-)
+ \sum_{j=2}^{n-2} J_\mu(1_+^+, 2^+, \ldots, (j-1)^+, (K_{j(n-1)}^-; n_\eta^-; P_V) \frac{i}{K_{j(n-1)}^2}
\times V((-K_{j(n-1)}^-)^-, j^+, \ldots, (n-2)^+, (n-1)^-)
\]
\[ + J_\mu \left( (K_{1\ldots(n-1)})^{+}_{q\bar{q}} : P_V \right) \frac{i}{K^2_{1\ldots(n-1)}} V(1^+, 2^+, \ldots, (n-2)^+, (n-1)^-, (-K_{1\ldots(n-1)})^-) \]
\[ + J_\mu \left( (K_{1\ldots n})^- : P_V \right) \frac{i}{K^2_{1\ldots n}} V(1^+, 2^+, \ldots, (n-2)^+, (n-1)^-, n_{\bar{q}}, (-K_{1\ldots n})^+) \]  

where the momentum of the vector boson is \( P_V = -K_{1\ldots n} \). The explicit values of the current vertices are obtained from eqns. (12) and (14) by relabeling the arguments. Other NMHV helicity configurations are only a bit more complicated. In ref. [9], Bena and two of the authors introduced a recursive reformulation of the CSW rules, useful when increasing the number of negative helicity legs. Indeed, an analogous recurrence relation also applies to the vector-boson currents considered here.

**CONCLUSIONS**

The twistor-inspired computational approach presented by Cachazo, Svrček, and Witten [4], and very recently demonstrated by Risager [5], is among the novel ways of computing tree amplitudes in massless gauge theories, including of course QCD. In this talk, I have discussed the main issue of [1], where, with Z. Bern, D. Forde, and D. Kosower, addressing the question of computing amplitudes containing both colored and non-colored particles, we have shown how to incorporate an additional vector leg coupling to an arbitrary source into the CSW approach. The currents we have constructed can be used directly in the computation of processes producing electroweak vector bosons. The structure of the CSW construction implies that that a similar approach can be used to build multi-Ws currents.

In outlook, novel techniques dealing directly with on-shell objects, like the CSW [4] and the BCFW [6] approaches, relying on general properties of complex analysis, and exploiting the recursive behaviour of scattering amplitudes, are establishing themselves as suitable tools [21, 22] for computing massless and massive multi-legs tree-level [23] and one-loop QCD (and beyond) amplitudes [24].

**REFERENCES**

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