Linear coupling and over-reflection phenomena of magnetohydrodynamic waves in smooth shear flows

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Abstract

Special features of magnetohydrodynamic waves linear dynamics in smooth shear flows are studied. Quantitative asymptotic and numerical analysis are performed for wide range of system parameters when basic flow has constant shear of velocity and uniform magnetic field is parallel to the basic flow. The special features consist of magnetohydrodynamic wave mutual transformation and over-reflection phenomena. The transformation takes place for arbitrary shear rates and involves all magnetohydrodynamic wave modes. While the over-reflection occurs only for slow magnetosonic and Alfvén waves at high shear rates. Studied phenomena should be decisive in the elaboration of the self-sustaining model of magnetohydrodynamic turbulence in the shear flows.

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I. INTRODUCTION

Flows with inhomogeneous velocity profiles are one of the prevalent and still not completely understood macro-systems. They occur in laboratory experiments, industrial applications, the earth’s atmosphere, oceans and many astrophysical objects. We concentrate attention on smooth shear flows (flows without inflection point in the velocity profile) that are linearly stable according to the canonical hydrodynamics. Faults in the comprehension of shear flow dynamics is caused by the incompleteness of the classical method of normal modes. Specifically, method of modes leaves out of account the non-normality of the linear operators that often produces powerful transient development of a subset of perturbations. Indeed, in the case of the non-normality of linear operators, corresponding eigenfunctions are not orthogonal and strongly interfere. Hence, a correct approach should fully analyze the eigenfunction interference, which is feasible in asymptotic: in fact, in modal analysis only the asymptotic stability of flow is studied, while no attention is paid to any particular initial value or finite time period of the dynamics – this period of the evolution is thought to have no significance and is left to speculation.

Recognition of the importance of the non-normality in the linear stability resulted impressive progress in the understanding of the shear flow phenomena in 90s of the last century. The early transient period for the perturbations has been shown to reveal rich and complicated behavior leading to various consequences. It has been grasped phenomena that are overlooked in the framework of the modal analysis.

The progress in the understanding of the shear flow phenomena has been achieved using so-called non-modal approach (see, e.g., Refs. [1, 4, 5]), which implicates a change of independent variables from a laboratory to a moving frame and the study of temporal evolution of spatial Fourier harmonics (SFHs) of perturbations without any spectral expansion in time. Strictly speaking the non-modal approach is applicable to the flows with constant velocity shear. However, it is obvious that the results obtained in the framework of the approach are valid for any shear flows without inflection point, if the wavelength of modes is appreciably shorter than the characteristic length scale of the inhomogeneity.

The progress involves novel results on the time evolution of vortex and wave perturbations; elaboration of a concept of self-sustaining turbulence in the spectrally stable shear flows (that was labelled as bypass concept) [7, 8, 9, 10, 11]. Among the novelties one should
stress (in the context of the present paper) disclosure of the linear mechanisms of resonance transformation of waves at low shear rates \[4, 12, 13\] and non-resonant conversion of vortices to waves at moderate and high shear rates \[14\]. The later phenomena should be inherent to flow systems. This was shown in papers \[13, 15\], where appearances of some waves in solar wind and galactic disks are explained by the linear wave transformations in the magnetohydrodynamic (MHD) flows.

Aim of the present work is quantitative and qualitative study of the linear evolution of three dimensional MHD waves at low and high shear rates. Equilibrium density, pressure and magnetic field are assumed to be homogeneous. Specifically, we perform analytical and numerical study of the important effect caused by linear forces – mutual transformations of MHD waves. The waves are coupled (during a limited time interval) at the specific system parameters and mutual transformations of two, or even all three of them occur. Analytical expressions of transformation coefficients are obtained for wide range of the system parameters. We study the wave over-reflection phenomena at high shear rates: MHD waves extract flow energy, are amplified non-exponentially and are over-reflected.

Mathematical methods used in this paper are similar to ones that were first developed for quantum mechanical problems: non-elastic atomic collisions \[16\] and non-adiabatic transitions in two level quantum systems \[17, 18\]. Later, the same asymptotic methods were successfully applied to various problems (see, e.g., Ref. \[19\]) including interaction of plasma waves in the media with inhomogeneous magnetic field and/or density \[20\].

The paper is organized as follows: employed mathematical formalism is presented in Sec. II. General properties of resonant transformation of wave modes that takes place at small shear rates, are presented in Sec. III. Particular cases of the resonant transformation of MHD wave modes are described in Sec. IV. In Sec. V dynamics of SFHs of the wave modes at high shear rates and over-reflection phenomenon is studied. Conclusions are given in Sec. VI.

II. MATHEMATICAL FORMALISM

Consider compressible unbounded shear flow with constant shear parameter \((U_0(Ay,0,0))\) and uniform density \((\rho_0)\), pressure \((P_0)\) and magnetic field directed along the streamlines \((B_0||U_0)\). Linearized ideal MHD equations governing the evolution of den-
sity ($\rho'$), pressure ($p'$), velocity ($\mathbf{u}'$) and magnetic field ($\mathbf{b}'$) perturbations in the flow are:

\[
(\partial_t + \mathbf{U}_0 \cdot \nabla) \rho' + \rho_0 \nabla \cdot \mathbf{u}' = 0,
\]

(1)

\[
(\partial_t + \mathbf{U}_0 \cdot \nabla) \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{U}_0 + \frac{1}{\rho_0} \nabla p' = -\frac{1}{4\pi \rho_0} \mathbf{B}_0 \times (\nabla \times \mathbf{b}'),
\]

(2)

\[
(\partial_t + \mathbf{U}_0 \cdot \nabla) \mathbf{b}' - (\mathbf{B}_0 \cdot \nabla) \mathbf{u}' + \mathbf{B}_0 (\nabla \cdot \mathbf{u}') = 0,
\]

(3)

\[
\nabla \cdot \mathbf{b}' = 0.
\]

(4)

Further we use the standard technique of the non-modal approach [1], i.e., expand the perturbed quantities as:

\[
\psi'(\mathbf{x}, t) = \psi'(\mathbf{k}, t) \exp \left[i(k_x x + k_y(t)y + k_z z)\right],
\]

(5)

where $\psi' \equiv (u'_x, u'_y, u'_z, \rho', p', b'_x, b'_y, b'_z)$, $k_y(t) = k_y(0) - k_x At$, and $(k_x, k_y(0), k_z)$ are the wave numbers of SFHs at the initial moment of time. It follows from Eq. (5), that $k_y(t)$ varies in time. This fact can be interpreted as a "drift" of SFH in phase space. This circumstance is caused by the fact, that perturbations cannot have a simple plane wave form in the shear flow due to the shearing background [2].

Using the following thermodynamic relation $p' = c_s^2 \rho'$, where $c_s$ is sound speed, and introducing non-dimensional parameters and variables:

\[
S \equiv A/(V_A k_x), \quad \tau \equiv k_x V_A t, \quad \beta \equiv c_s^2/V_A^2,
\]

\[
K_z \equiv k_z/k_x, \quad K_y(\tau) \equiv k_y/k_x - S\tau,
\]

\[
K^2(\tau) = 1 + K_y^2(\tau) + K_z^2, \quad \rho(\mathbf{k}, \tau) \equiv i\rho'(\mathbf{k}, \tau)/\rho_0,
\]

\[
\mathbf{b}(\mathbf{k}, \tau) \equiv i\mathbf{b}'(\mathbf{k}, \tau)/\mathbf{B}_0, \quad \mathbf{v}(\mathbf{k}, \tau) \equiv \mathbf{u}'(\mathbf{k}, \tau)/V_A
\]

(6)

(where $V_A \equiv B_0/\sqrt{4\pi \rho_0}$ is the Alfvén velocity) the set of Eqs. (1)-(4) is reduced to:

\[
\frac{d\rho}{d\tau} = v_x + K_y(\tau)v_y + K_z v_z,
\]

(7)

\[
\frac{dv_x}{d\tau} = -S v_y - \beta \rho,
\]

(8)
\[
\frac{dv_y}{d\tau} = -\beta K_y(\tau)\rho + (1 + K_y^2(\tau))b_y + K_y(\tau)K_z b_z, \quad (9)
\]

\[
\frac{dv_z}{d\tau} = -\beta K_z\rho + (1 + K_z^2)b_z + K_y(\tau)K_z b_y, \quad (10)
\]

\[
\frac{db_y}{d\tau} = -v_y, \quad (11)
\]

\[
\frac{db_z}{d\tau} = -v_z, \quad (12)
\]

\[
b_x + K_y(\tau)b_y + K_z b_z = 0. \quad (13)
\]

SFH energy density in the non-dimensional form may be defined as a sum of non-dimensional kinetic, magnetic and compression energy densities:

\[
E(k, \tau) \equiv E^k(k, \tau) + E^m(k, \tau) + E^c(k, \tau), \quad (14)
\]

where:

\[
E^k(k, \tau) \equiv \sum_{i=1}^{3} |v_i|^2/2, \quad (15)
\]

\[
E^m(k, \tau) \equiv \sum_{i=1}^{3} |b_i|^2/2, \quad (16)
\]

\[
E^c(k, \tau) \equiv \beta|\rho|^2/2, \quad (17)
\]

For further analysis it is convenient to introduce a new variable 
\[d \equiv \rho + K_y(\tau)b_y + K_z b_z\]
and rewrite the set of Eqs. (7)-(13) in the following form \[13\]:

\[
\frac{d^2d}{d\tau^2} + C_{11}d + C_{12}(\tau)b_y + C_{13}b_z = 0, \quad (18)
\]

\[
\frac{d^2b_y}{d\tau^2} + C_{22}(\tau)b_y + C_{21}(\tau)d + C_{23}(\tau)b_z = 0, \quad (19)
\]

\[
\frac{d^2b_z}{d\tau^2} + C_{33}b_z + C_{31}d + C_{32}(\tau)b_y = 0, \quad (20)
\]
where: \( C_{11} \equiv \beta, \ C_{22}(\tau) \equiv 1 + (1 + \beta)K_y^2(\tau), \ C_{33} \equiv 1 + (1 + \beta)K_z^2, \ C_{12}(\tau) = C_{21}(\tau) \equiv -\beta K_y(\tau), \ C_{13} = C_{31} \equiv -\beta K_z \) and \( C_{23}(\tau) = C_{32}(\tau) \equiv (1 + \beta)K_y(\tau)K_z \).

Combining Eqs. (18)-(20) it is easy to show that the linear evolution of the perturbations has the following invariant:

\[
d^* \frac{dd}{d\tau} - d^* \frac{dd}{d\tau} + b^* \frac{db_y}{d\tau} - b^* \frac{db_y}{d\tau} + b^* \frac{db_z}{d\tau} - b^* \frac{db_z}{d\tau} = \text{inv},
\]

Here and hereafter asterisk denotes complex conjugated value. Eq. (21) represents conservation of wave action of the system [21, 22, 23].

Eqs. (18)-[20] describe the linear dynamics of MHD waves in the constant shear flow. However, variables \( d, b_y, b_z \) are not normal. This circumstance complicates physical treatment of perturbation dynamics. Therefore, in the next subsection normal variables are introduced in the shearless limit. Afterwards Eqs. (18)-(20) are rewritten in the normal variables and general features of perturbation dynamics as well as particular characteristics of the system evolution for different values of parameters \( \beta, S \) and \( K_z \) are studied in detail.

### A. Shearless limit

For the sake of clearness of further analysis first we shortly discuss the shearless limit. Introducing normal variables \( \Psi_i, \ (i = 1, 2, 3) \) [24]:

\[
\Psi_i = \sum_j Q_{ij} \psi_j,
\]

where

\[
\psi = \begin{pmatrix} d \\ b_y \\ b_z \end{pmatrix},
\]

\[
Q = \begin{pmatrix} \cos \alpha & \sin \alpha \cos \gamma & \sin \alpha \sin \gamma \\ \sin \alpha & -\cos \alpha \cos \gamma & -\cos \alpha \sin \gamma \\ 0 & \sin \gamma & -\cos \gamma \end{pmatrix}.
\]

\( \alpha \) and \( \gamma \) stand for Euler angles in the \( \psi \)-space. After quite long but straightforward algebra it can be shown, that:

\[
\alpha = \arctan \left( \frac{\beta K_i}{\Omega_x^2 - \beta} \right),
\]

\( 6 \)
\[ \gamma = \arctan \left( \frac{K_z}{K_y} \right), \]  

with \( K_\perp \equiv K_y^2 + K_z^2 \).

In the normal variables Eqs. (18)-(20) are decoupled and reduced to

\[ \frac{d^2 \Psi_i}{d\tau^2} + \Omega_i^2 \Psi_i = 0. \]  

These equations describe independent oscillations with so-called fundamental frequencies \( \Omega_i \):

\[ \Omega_{1,2}^2 = \frac{1}{2} (1 + \beta) K^2 \left[ 1 \pm \sqrt{1 - \frac{4\beta}{(1 + \beta)^2 K^2}} \right], \]

\[ \Omega_3^2 = 1, \]  

that are eigen-frequencies of the set of Eqs. (18)-(20) in the shearless limit and can be easily identified as fast and slow magnetoosonic (FMW and SMW) and Alfvén waves (AW), respectively. Hereafter parallel with notations of eigen-frequencies \( \Omega_i \) and eigenfunctions \( \Psi_i \) where \( i \equiv (1, 2, 3) \) we also use the notations associated with corresponding MHD wave modes, i.e., hereafter indexes of \( \Omega_i \) and \( \Psi_i \) vary over \((1, 2, 3)\) or equivalently \((F, S, A)\).

Substituting Eqs. (24)-(26) into (22) one can get the following expressions for the normal variables:

\[ \Psi_1 \equiv \Psi_F = \frac{(\Omega_S^2 - \beta) K_y + \Omega_S^2 (K_y b_y + K_z b_z)}{\sqrt{(\Omega_S^2 - \beta)^2 + \beta^2 K_\perp^2}}, \]  

\[ \Psi_2 \equiv \Psi_S = \frac{\beta K_\perp K_y + (\Omega_S^2 - \beta K^2) (K_y b_y + K_z b_z)}{K_\perp \sqrt{(\Omega_S^2 - \beta)^2 + \beta^2 K_\perp^2}}, \]  

\[ \Psi_3 \equiv \Psi_A = \frac{K_z b_y - K_y b_z}{\sqrt{K_\perp^2}}. \]

**B. Dynamical equations in normal variables**

In shear flows, coefficients in Eqs. (18)-(20) vary in time. Therefore the equations governing the dynamics of SFHs become coupled [in contrast to the shearless equations (27)] and take the following form in the normal variables:

\[ \frac{d^2 \Psi_i}{d\tau^2} + (\Omega_i^2 - \Theta_i) \Psi_i = -\gamma_{ik} \Psi_k - \Lambda_{ik} \frac{d\Psi_k}{d\tau}, \]  

(32)
where:

$$\Theta_i = \sum_j \dot{Q}_{ij}^2 ,$$  \hfill (33)

$$\Upsilon_{ik} = \begin{cases} 
\sum_j Q_{ij} \ddot{Q}_{kj} & i \neq k \\
0 & i = k 
\end{cases} ,$$  \hfill (34)

$$\Lambda_{ik} = \begin{cases} 
\sum_j 2Q_{ij} \dot{Q}_{kj} & i \neq k \\
0 & i = k 
\end{cases} .$$  \hfill (35)

Here and hereafter overdot denotes $\tau$ derivative. $\Omega_i$ and $Q_{ij}$ are defined by the same expressions as in the previous subsection [see Eqs. (24)-(26) and (28)], but with time dependent normalized wave number $K_y(\tau) = K_y - S\tau$. Expressions for the coefficients in Eq. (32) are given in the Appendix A.

There are three main differences between Eqs. (32) and shearless equations (27) caused by the time dependence of $Q_{ij}$ in the shear flow. These differences correspond to three channels of energy exchange processes and are responsible for novel features of linear dynamics of the perturbations in the shear flows:

- time dependence of the eigen-frequencies [$\Omega_i = \Omega_i(K_y(\tau))$] causes adiabatic energy exchange between main flow and perturbations;
- terms on the right hand side of Eq. (32) describe the coupling between different wave modes;
- additional terms ($\Theta_i$) on the left hand side of Eq. (32) describe shear induced modification of frequencies. Influence of these terms become remarkable at high shear rates ($S \gtrsim 1$) and, as it will be described in Sec. V, they are responsible for the over-reflection phenomenon of MHD wave modes.

Our further efforts are focused on the analysis of this novelties of the perturbation dynamics.

**C. Adiabatic evolution of SFHs**

There are two necessary conditions that should be satisfied for the validity of WKB approximation. Firstly, shear modified frequencies $\bar{\Omega}_i^2 \equiv \Omega_i^2 - \Theta_i$ must be slowly varying
functions:
\[ \dot{\Omega}_i \ll \Omega_i^2. \]  
(36)

Second condition implies that coupling terms in Eq. (32) have to be negligible.

Let us first consider the limit \( S \ll 1 \). In this case condition (36) reduces to
\[ \dot{\Omega}_i \ll \Omega_i^2. \]  
(37)

This condition indicates that WKB approximation fails in some vicinities of the turning points where \( \Omega_i(\tau) = 0 \). But direct evaluation of Eqs. (28) indicates that none of the turning points are located near the real \( \tau \)-axis and therefore the condition (37) is satisfied for SFHs of all wave modes at any moment of time for arbitrary values of the parameters \( K_y(\tau), K_z \) and \( \beta \).

If the frequencies of two oscillating modes became equal at some moment of time, \( Q \) becomes degenerated. Equivalently, as it can be seen from direct evaluation of Eqs. (A2)-(A3), coupling coefficients of corresponding oscillations in Eq. (32) which otherwise are of order \( S \) or \( S^2 \) become infinity. So, at \( S \ll 1 \) evolution of SFHs of the wave modes is adiabatic except some vicinities of resonant points where \( \dot{\Omega}_i(\tau) = \dot{\Omega}_j(\tau) \), or equivalently during some time intervals when frequencies of different wave modes are close to each other. Analysis of Eq. (28) yields that all the resonant points are located on the axis \( \text{Re}[K_y(\tau)] = 0 \) in the complex \( \tau \)-plane. Consequently, these time intervals can appear only in a certain \( \Delta K_y \) vicinity of the point \( K_y(\tau) = 0 \) (exact conditions for effective coupling between different modes will be formulated in the next section).

Now consider the case of moderate and high shear rates when condition \( S \ll 1 \) is not satisfied. Again, analysis of turning points where \( \dot{\Omega}_i(\tau) = 0 \) and resonant points where \( \dot{\Omega}_i(\tau) = \dot{\Omega}_j(\tau) \) is important. But in contrast with the limit \( S \ll 1 \), there is no small parameter in the problem. Therefore, in general WKB approximation may fail during the whole evolution. Combining Eqs. (28) and (25)-(26) after long but straightforward calculations one can conclude, that both conditions for the validity of WKB approximation formulated above hold for SFHs of all MHD wave modes for arbitrary \( \beta \) and \( K_z \) at least for high values of \( |K_y(\tau)| \):
\[ |K_y(\tau)| \gg S. \]  
(38)

This circumstance is crucially important for the analysis of the mode coupling at high shear rates presented in Sec. V.
If the conditions for the validity of WKB approximation holds, temporal evolution of SFHs can be described by standard WKB solutions:

\[
\Psi_i^\pm = \frac{D_i^\pm}{\sqrt{\Omega_i(\tau)}} e^{\pm \int \Omega_i(\tau) d\tau},
\]  

(39)

where \(D_i^\pm\) are WKB amplitudes of the wave modes with positive and negative phase velocity along \(X\)-axis respectively. All the physical quantities can be easily found by combining Eqs. (29)-(31). Combining these equations after long but straightforward algebra one can check, that the energies of the wave modes satisfy standard relations of adiabatic evolution:

\[
E_i = \Omega_i(\tau)(|D_i^+|^2 + |D_i^-|^2),
\]  

(40)

so asymptotically \((\tau \to \infty)\) total energy of SFH of SMW and AW become constant whereas the total energy of SFH of FMW increases linearly with \(\tau\). In other words, SFH of FMW can effectively extract energy from the basic flow. It follows from Eq. (40), that \(|D_i^\pm|^2\) can be interpreted as number of wave particles (so called plasmons) in analogy with quantum mechanics.

Using Eq. (39) we can reduce Eq. (21) to the following form:

\[
\sum_i |D_i^+|^2 - \sum_i |D_i^-|^2 = \text{inv.}
\]  

(41)

This equation is crucially important for the study of non-adiabatic processes in the considered flow: as it was mentioned above, the evolution of SFHs of the wave modes is always adiabatic for arbitrary \(K_y(\tau)\) which is outside some \(\Delta K_y\) vicinity of the point \(K_y(\tau_+) = 0\), i.e.:

\[
|K_y(\tau)| > \Delta K_y,
\]  

(42)

where the value of \(\Delta K_y\) depends on the specific parameters of the problem. From the point of view of temporal evolution this means that non-adiabatic processes can be important only during the time interval \(\Delta \tau \equiv \Delta K_y/S\) in the vicinity of \(\tau_+\), where condition (42) fails.

Consequently, the dynamics of SFHs is the following: assume at the initial moment of time \(K_y(0) > \Delta K_y\). According to Eq. (42) the dynamics of SFHs is adiabatic initially. Due to the linear drift in the \(k\)-space, \(K_y(\tau)\) decreases and when condition (42) fails, the dynamics of SFHs become non-adiabatic. Duration of non-adiabatic evolution is \(\Delta \tau\). Afterwards, when \(K_y(\tau) < -\Delta K_y\), the evolution of SFHs becomes adiabatic again.
Denote WKB amplitudes of SFHs of the wave modes on the left and right sides of the area of non-adiabatic evolution (i.e., for \( \tau < \tau_+ - \Delta \tau /2 \) and \( \tau > \tau_+ + \Delta \tau /2 \)) by \( D_{i,L}^\pm \) and \( D_{i,R}^\pm \) respectively. In other words, \( D_{i,L}^\pm \) and \( D_{i,R}^\pm \) are WKB amplitudes before and after non-adiabatic evolution. Eq. (41) provides important relation between WKB amplitudes on the left and right sides of the area of non-adiabatic evolution, independent of the behaviour/dynamics of the system in the non-adiabatic area:

\[
\sum_i |D_{i,L}^+|^2 - \sum_i |D_{i,L}^-|^2 = \sum_i |D_{i,R}^+|^2 - \sum_i |D_{i,R}^-|^2.
\] (43)

### III. GENERAL PROPERTIES OF MODE COUPLING

As it was mentioned above WKB approximation is valid for arbitrary \( K_y(\tau) \) except some vicinity of the point \( K_y(\tau_+) \). In the formal analogy with S-matrix of the scattering theory [25] and transition matrix from the theory of multi-level quantum systems [26, 27], one can connect \( D_{i,R}^\pm \) with \( D_{i,L}^\pm \) by a 6 \times 6 transition matrix:

\[
\begin{pmatrix}
D_{R}^+ \\
D_{R}^-
\end{pmatrix} =
\begin{pmatrix}
\text{T}^{++} & \text{T}^{+-} \\
\text{T}^{-+} & \text{T}^{--}
\end{pmatrix}
\begin{pmatrix}
D_{L}^+ \\
D_{L}^-
\end{pmatrix},
\] (44)

where \( D_{L}^\pm \) and \( D_{R}^\pm \) are 1 \times 3 matrices and \( \text{T}^{\pm\pm} \) are 3 \times 3 matrices.

Due to the fact that all components of the matrix \( C \) in the set of Eqs. (18)-(20) are real and \( C_{ij} = C_{ji} \) [27]:

\[
\text{T}_{ij}^{++} = [\text{T}_{ij}^{-+}]^* \equiv T_{ij},
\] (45)

\[
\text{T}_{ij}^{+-} = [\text{T}_{ij}^{+-}]^* \equiv \overline{T}_{ij}.
\] (46)

Substituting Eq. (44) in Eq. (41) one can get:

\[
\sum_j |T_{ij}|^2 - \sum_j |\overline{T}_{ij}|^2 = 1
\] (47)

In general, the components of transition matrix in Eq. (44) are complex. This means, that the interaction of different wave modes changes not only the absolute values of \( D_{i}^\pm \), but also their phases. We call the absolute value of the transition matrix components \( |T_{ij}| \) and \( |\overline{T}_{ij}| \) the transformation coefficients of corresponding wave modes:

\( |T_{ij}| \) represents the transformation coefficient of \( j \) to \( i \) mode, that has the same sign of the
phase velocity along $X$-axis (i.e., transmitted mode $i$);

$|T_{ij}|$ represents the transformation coefficient of $j$ to $i$ mode, that has the opposite sign of the phase velocity along $X$-axis (i.e., reflected mode $i$).

It has to be noted that in the similar problems of quantum mechanics \[26, 27\], $|T_{ij}|^2$ and $|\overline{T}_{ij}|^2$ represent transitions probabilities between different quantum states.

As it follows from Eqs. (40) and (44), if initially only one, for instance $j$ mode exists, i.e., $D_{\pm i,L} \equiv 0$ for $i \neq j$, the energies of transformed waves do not depend on the phases of transition matrix elements and are entirely determined by $|T_{ij}|$ and $|\overline{T}_{ij}|$. In the presented paper we concentrate attention on transformation coefficients and do not regard the problem of the phase multipliers of transition matrix elements.

Now consider the case $S \ll 1$, for which the condition (36) holds at arbitrary $\tau$. It is well known \[26, 27\], that in this case components of $T$ are exponentially small with respect to the large parameter $1/S$ and can be neglected. Consequently, Eq. (44) decomposes and reduces to:

$$D_R^+ = TD_L^+,$$

$$D_R^- = T^*D_L^-.$$  \hspace{1cm} (48)

$$D_L^+ = TD_R^+,$$

$$D_L^- = T^*D_R^-.$$  \hspace{1cm} (49)

From physical point of view this means, that only wave modes with the same sign of the phase velocity along $X$-axis can effectively interact, i.e., the wave reflection is negligible.

Eq. (47) reduces to the unitary condition for $T$:

$$\sum_j |T_{ij}|^2 = 1.$$  \hspace{1cm} (50)

Due to the fact that condition (36) is satisfied, the only reason for the failure of WKB approximation could be closeness of the wave mode frequencies, i.e., closeness of at least one resonant point (point where two of more fundamental frequencies become equal) to the real $\tau$-axis \[26, 27\]. In the later case coupling of wave modes becomes effective. Analysis of Eq. (28) shows that all the resonant points are located on the axis $\text{Re}[K_y(\tau)] = 0$ in the complex $\tau$-plane. That is why the effective transformation of wave modes can take place only in the vicinity of the moment of time $\tau_+$ where $K_y(\tau_+) = 0$.

First of all let us discuss some general properties of wave resonant interaction (mathematical details are discussed in the Appendix B):
(i) for effective coupling between different (for instance $i$ and $j$) wave modes there should exist a time interval (so-called resonant interval) where \[26, 28\]:

$$|\Omega_i^2 - \Omega_j^2| \lesssim |\Lambda_{ij}\Omega_i|. \quad (51)$$

If this condition is not satisfied transformation coefficients are exponentially small with respect to the large parameter $1/S$, namely \[26, 27\]:

$$T_{ij} \sim \exp \left( - \left| \text{Im} \int_{\tau_0}^{\tau_{ij}} (\Omega_i - \Omega_j)d\tau \right| \right). \quad (52)$$

Here and hereafter the signs of absolute magnitude for transformation coefficients are omitted, i.e., under $T_{ij}$ we mean $|T_{ij}|$. In Eq. (52), $\tau_0$ is arbitrary point on the real $\tau$-axis and $\tau_{ij}$ is the nearest to the real $\tau$-axis resonant point where $\Omega_i(\tau_{ij}) = \Omega_j(\tau_{ij})$.

The characteristic equation of the set (18)-(20) is real and symmetric with respect to the transform $K_y(\tau) \rightarrow -K_y(\tau)$. Therefore, resonant points always appear as complex conjugated pairs: if $\tau_{ij}$ is a resonant point, so is its complex conjugated one $\tau_{ij}^*$. 

(ii) the following equation holds for resonant interaction of two wave modes, e.g., $i$ and $j$ (i.e., when condition (51) is satisfied only for two wave modes):

$$T_{ij} = T_{ji}. \quad (53)$$

This symmetry property, which follows from unitarity property (50), holds for resonant interaction of two wave modes only. If at the same interval of time there is effective coupling of more then two wave modes, then Eq. (53) fails.

(iii) if in the neighborhood of the real $\tau$-axis only a pair of complex conjugated first order resonant points $\tau_{ij}$ and $\tau_{ij}^*$ exists [the resonant point is called of order $n$ if $(\Omega_i^2 - \Omega_j^2) \sim (\tau - \tau_{ij})^{n/2}$ in the neighborhood of $\tau_{ij}$], the transformation coefficients are \[19, 26\]:

$$T_{ij} = \exp \left( - \left| \text{Im} \int_{\tau_0}^{\tau_{ij}} (\Omega_i - \Omega_j)d\tau \right| \right) [1 + O(S^{1/2})]. \quad (54)$$

There are two remarks about this equation. Firstly, it shows, that only dispersion equation is needed to derive the transformation coefficient with accuracy $S^{1/2}$ in the case of the first order resonant points. In other words, only the solution of the characteristic equation of the governing set of Eqs. (7)-(13) is needed to derive the transformation coefficient. Secondly, Eq. (54) is valid also in the case of strong wave interaction. For example, if
complex conjugated resonant point of the first order tends to the real $\tau$-axis, then $T_{ij} \to 1$. According to Eq. (50), this means that one wave mode is fully transformed into another.

(iv) for the second or higher order resonant points analytical expression of transformation coefficient can be derived only in the case of weak interaction ($T_{ij} \ll 1$, $i \neq j$). Namely, if there exist a pair of complex conjugated second order resonant points and condition (51) is not satisfied, transformation coefficients are:

$$T_{ij} = \frac{\pi}{2} \exp \left( - \left| \text{Im} \int_{\tau_0}^{\tau_{ij}} (\Omega_i - \Omega_j) d\tau \right| \right) \left[ 1 + O(S^{1/2}) \right].$$

Let us note once again, that this equation is valid only for $T_{ij} \ll 1$. Derivation of this formula is presented in Appendix B.

In earlier studies, an attention was always paid to the resonant points of the first order [19, 26, 27]. As it will be shown later, all the resonant points are of the second order in the case of resonant coupling between SFHs of AW and the magneto-sonic wave modes. This is not an unique property of the evolution of MHD wave modes. It can be readily shown that these type of resonances naturally appear in systems of three or more coupled oscillators.

(v) from Eq. (51) it can be shown (see Appendix B), that the time scale of the resonant interaction is as follows:

$$\Delta \tau_{ij} \sim S^{-1+1/n},$$

where $n$ is the order of the resonant point. Whereas the time scale of adiabatic evolution $\Delta \tau \sim 1/S$. Therefore if $S \ll 1$

$$\Delta \tau_{ij} \ll \Delta \tau,$$

i.e., resonant interaction of waves is much faster process then energy exchange between background flow and wave modes. Consequently, conservation of wave action (47) reduces to energy conservation during the resonant interaction of wave modes.

IV. SPECIFIC LIMITS OF THE RESONANT TRANSFORMATION OF MHD WAVE MODES

In this section we study specific cases of MHD wave coupling at $S \ll 1$:

(A) Two dimensional (2D) problem when $K_z \equiv b_z \equiv 0$. In this case Alfvén waves are absent, and obtained set of equations describes the coupling between FMW and SMW.
(B) $\beta \ll 1$. In this limit $\Omega_{F,A} \gg \Omega_S$ and only mutual transformation of FMW and AW is possible.

(C) $\beta \gg 1$. In this limit $\Omega_{A,S} \ll \Omega_F$ and mutual transformation of SMW and AW can be effective.

(D) $\beta \sim 1$. In this case frequencies of all the MHD waves can be of the same order and mutual transformations of all the modes is possible.

As it was mentioned above resonant transformations of MHD wave modes in shear flows are investigated recently \[4, 12, 13\]. The content of this section is concerned on detailed quantitative analysis of the problem. In particular - derivation of analytical expressions of transformation coefficients.

A. 2D problem

To derive equations in 2D case one has to assume $K_z \equiv b_z \equiv 0$. This limit excludes Alfvén waves and Eqs. (18)-(20) reduce to:

\[
\frac{d^2d}{d\tau^2} + C_{11}d + C_{12}(\tau)b_y = 0,
\]

\[(58)\]

\[
\frac{d^2b_y}{d\tau^2} + C_{22}(\tau)b_y + C_{21}(\tau)d = 0,
\]

\[(59)\]

where: $C_{11} \equiv \beta$, $C_{22}(\tau) \equiv 1 + (1 + \beta)K_y^2(\tau)$, $C_{12}(\tau) = C_{21}(\tau) \equiv -\beta K_y(\tau)$. Eqs. (58)-(59) describe coupled evolution of SFHs of FMW and SMW. Frequencies of the modes are given by Eq. (28) where now $K^2 \equiv 1 + K_y^2(\tau)$. Normal variables are defined by Eqs. (29)-(30) with substitution $K_z = b_z = 0$.

Solving the equation $\Omega_F = \Omega_S$ one can easily obtain that there is only a pair of complex conjugated resonant points of the first order in the complex $\tau$-plane:

\[K_y(\tau_{FS}) = i\frac{\beta - 1}{\beta + 1}, \quad K_y(\tau_{FS}^*) = -i\frac{\beta - 1}{\beta + 1}.\]

\[(60)\]

Noting, that in the neighborhood of resonant points:

\[\Omega_F - \Omega_S \approx \left(\frac{\beta + 1}{2}\right)^{1/2} \left[K_y^2(\tau) + \left(\frac{\beta - 1}{\beta + 1}\right)^2\right]^{1/2},\]

\[(61)\]

from Eq. (54) one can easily obtain:

\[T_{FS} \approx \exp \left[-\frac{\pi \sqrt{1 + \beta}}{4\sqrt{2S}} \left(\frac{\beta - 1}{\beta + 1}\right)^2\right].\]

\[(62)\]
It is seen from Eq. (60), that if $\beta \rightarrow 1$, the resonant points tends to the real $\tau$-axis. Then from Eq. (62) it follows that $T_{FS} \rightarrow 1$. According to Eq. (50), it means that one wave mode totally transforms into another.

Dependence of transformation coefficient on $\beta$ is presented in Fig. 1 for $S = 0.05$ and $S = 0.02$. Dotted line shows transformation coefficient obtained by numerical solution of the set of Eqs. (18)-(20), where initial conditions are chosen by WKB solutions (39) far on the left hand side of resonant time interval ($K_y(0) \ll -1$) and solid line is the curve of analytical solution (62).

**B. Low $\beta$ regime**

In this case the frequency of SMW is far less then frequencies of FMW and AW ($\Omega_S \ll \Omega_F, \Omega_A$). Therefore, the coupling of SMW with other MHD modes is exponentially small with respect to the parameter $1/S$ and can be neglected. Consequently, in the set of Eqs. (18)-(20) equation for $d$ decouples and equations for $b_y$ and $b_z$ describes coupled evolution of SFHs of AW and FMW:

$$\frac{d^2 b_y}{d\tau^2} + (1 + K_y^2(\tau)) b_y = -K_y(\tau) K_z b_z,$$

$$\frac{d^2 b_z}{d\tau^2} + (1 + K_z^2) b_z = -K_y(\tau) K_z b_y.$$  

Normalized frequencies of the coupled wave modes are:

$$\Omega^2_F(\tau) = 1 + K_z^2 + K_y^2(\tau), \quad \Omega^2_A = 1.$$  

From this equation it follows that there are two complex conjugated second order resonant points:

$$K_y(\tau_{FA}) = iK_z, \quad K_y(\tau_{FA}^*) = -iK_z.$$  

Necessary condition for effective coupling expressed by Eq. (51) now takes the form:

$$|K_z^2| \leq S.$$  

Thus, the critical parameter is $\delta \equiv K_z/S^{1/3}$. 
Consider the case $\delta \gg 1$. For the calculation of the transformation coefficients one can use general analysis presented in Sec. III and Appendix B. Specifically, Eq. (68) yields:

$$T_{FA} = \frac{\pi}{2} \exp \left( -\frac{\phi_{FA}(K_z)}{2S} \right),$$  \hspace{1cm} (68)

where:

$$\phi_{FA}(K_z) = (1 + K_z^2) \arctan(K_z) - K_z.$$  \hspace{1cm} (69)

If in addition $K_z \ll 1$, then Eq. (68) reduces to:

$$T_{FA} \approx \frac{\pi}{2} \exp \left( -\frac{\delta^3}{3} \right).$$  \hspace{1cm} (70)

Analytical expression for the transformation coefficients can be derived also in the opposite limit $\delta \ll 1$, $(K_z \ll S^{1/3})$. Taking into account the relation $\Omega_S^2 \approx \beta$, one can readily obtain from Eqs. (29) and (31) that $b_y$ and $b_z$ coincide with the eigenfunctions of SFHs of FMW and AW respectively accurate to the terms of order $K_z^2$. Consequently, terms on the right hand sides of Eqs. (63)-(64) represent the coupling terms with accuracy $K_z^2$. Since $K_z \ll S^{1/3}$, coupling is weak and feedback in the set of Eqs. (63)-(64) strongly depends on the amplitudes of the wave modes. Then it follows that if initially only AW exists, feedback of FMW on AW is negligible and the set of Eqs. (63)-(64) reduces to:

$$\frac{d^2 \Psi_F}{d\tau^2} + \left[ 1 + K_y^2(\tau) \right] \Psi_F = -K_y(\tau)K_z \Psi_A.$$  \hspace{1cm} (71)

$$\frac{d^2 \Psi_A}{d\tau^2} + \Psi_A = 0.$$  \hspace{1cm} (72)

Using the solution of Eq. (72) and well known expressions for the solution of linear inhomogeneous second order differential equation, in the considered limit $(\delta \ll 1)$, we obtain:

$$T_{FA} \approx 2^{2/3} \delta \int_0^\infty x \sin \left( \frac{x^3}{3} - \frac{\delta^2}{2^{2/3}} x \right) dx.$$  \hspace{1cm} (73)

Note that:

$$\int_0^\infty x \sin \left( \frac{x^3}{3} - \gamma x \right) dx \equiv \pi \frac{\partial}{\partial\gamma} Ai(-\gamma)$$  \hspace{1cm} (74)

and using the expansion of Airy function $Ai(\gamma)$ into power series we finally obtain:

$$T_{FA} \approx \frac{2^{2/3} \pi}{3^{1/3} \Gamma \left( \frac{1}{3} \right)} \delta \left( 1 - \frac{\Gamma \left( \frac{1}{3} \right)}{27/43^{1/3} \Gamma \left( \frac{5}{3} \right)} \delta^4 \right).$$  \hspace{1cm} (75)

Results of numerical solution of the initial set of equations (7)-(13) (solid line) as well as analytical expressions (70) (dash-dotted line) and (75) (dashed line) are presented in Fig. 2. It shows that the transformation coefficient reaches its maximal value \((T_{FA})_{max} = 1/2\) at \(\delta^{cr}\) that can be found numerically or alternatively by finding the maximum of the analytical expression presented by Eq. (75):

\[
\delta^{cr} = \left(\frac{2^{7/4}3^{1/3}\Gamma\left(\frac{2}{3}\right)}{5\Gamma\left(\frac{1}{3}\right)}\right)^{1/4}.
\] (76)

Eq. (76) is in perfect accordance with numerically calculated \(\delta^{cr}\) (see Fig. 2) despite the failure of Eq. (75) at \(\delta \sim 1\). This fact can be explained as follows: the only reason of failure of Eq. (75) is the neglect of the feedback in Eq. (72). The feedback changes the value of the transformation coefficient but does not affect on the value of \(\delta^{cr}\).

\((T_{FA})_{max} = 1/2\) means that only half of the energy of FMW can be transformed into AW and vice versa even in the optimal regime. It has to be noted, that Landau-Zener theory \([17, 18]\) provides the same maximum value for the transition probability in two-level quantum mechanical systems.

C. High \(\beta\) regime

In this case \(\Omega_S, \Omega_A \ll \Omega_F\). Consequently, the coupling of AW and SMW with FMW are exponentially small with respect to the parameter \(1/S\) and can be neglected.

Analysis of Eq. (28) provides that for the coupling of AW and SMW there exist two complex conjugated second order resonant points (solutions of the equation \(\Omega_S = \Omega_A\)), that are also given by Eq. (66). The condition of effective coupling (51) takes the form:

\[
\frac{|K_z|^3}{\sqrt{1 + K_z^2}} \leq \beta S. \tag{77}
\]

If this condition fails, then Eq. (55) yields exponentially small transformation coefficient:

\[
T_{SA} = \frac{\pi}{2} \exp \left(-\frac{|\phi_{SA}(K_z)|}{2\beta S}\right), \tag{78}
\]

where

\[
\phi_{SA}(K_z) = K_z - \frac{\text{arcsinh}(K_z)}{\sqrt{1 + K_z^2}}. \tag{79}
\]
If additionally $K_z \ll 1$, then Eq. (78) reduces to:

$$ T_{SA} \approx \frac{\pi}{2} \exp \left( -\frac{|K_z|^3}{3\beta S} \right). $$

(80)

Consider the transformation process when the condition of effective coupling Eq. (77) is satisfied. Similar to the low $\beta$ regime, in this case it is also possible to derive the set of two second order coupled equations that describe the linear coupling of AW and SMW. Expressing the density perturbation $\rho$ from the condition $\Psi_F \equiv 0$ (thus eliminating the terms that describe interaction of AW and SMW with FMW) and combining Eqs. (18)-(20) one can obtain:

$$ \frac{d^2 b_y}{d\tau^2} + \left( 1 + \theta K_y^2 (\tau) \right) b_y = -\theta K_y (\tau) K_z b_z, $$

(81)

$$ \frac{d^2 b_z}{d\tau^2} + \left( 1 + \theta K_z^2 \right) b_z = -\theta K_z (\tau) K_y b_y, $$

(82)

where:

$$ \theta \equiv 1 - \Omega_S^2 \frac{\beta}{\beta - \Omega_S^2} \approx -\frac{1}{\beta K^2 (\tau)}. $$

(83)

As it follows from Eqs. (81)-(82) $T_{AS} \equiv 0$ at $K_z = 0$, i.e., there is no interaction between AW and SMW.

In contrast with low $\beta$ limit, there is no unique parameter in high $\beta$ limit that totally describes the transformation process. In this limit there are two such parameters $K_z$ and $\beta S$.

First consider the case $\beta S \ll 1$. Then the condition (77) reduces to $\delta_1 \equiv |K_z|/(\beta S)^{1/3} \leq 1$. Note, that the sign of $\theta$ does not affect the transformation coefficient and $K^2 (\tau) \approx 1$ in the resonant area. Thus we conclude that the properties of the wave transformation is the same as in the case of transformation of AW and FMW. Namely, if $\delta_1 \ll 1$, then the leading terms of asymptotic expressions of transformation coefficient is given by Eq. (75) with $\delta$ replaced by $\delta_1$:

$$ T_{AS} \approx \frac{2^{2/3} \pi}{3^{1/3} \Gamma \left( \frac{1}{3} \right)} \delta_1 \left( 1 - \frac{\Gamma \left( \frac{1}{3} \right)}{2^{7/4} 3^{1/3} \Gamma \left( \frac{2}{3} \right)} \delta_1^4 \right). $$

(84)

At $\beta S \ll 1$, the transformation coefficient reaches its maximum $(T_{AS}^2)_{max} = 1/2$ at $\delta_1^{cr}$ that coincides with $\delta_1^{cr}$ defined by Eq. (76).

Dependence of the transformation coefficient on $K_z$ obtained by numerical solution of the initial set of equations (18)-(20) for $\beta S = 0.025$, $\beta S = 0.5$ and $\beta S = 1$ are presented in Fig. 19.
Consider the case when $\beta S$ is not small. The numerical study shows that in this case the properties of transformation process is totally different (see Fig. 3):

- transformation coefficient $T_{AS}$ does not depend on $\beta S$ at $K_z \ll 1$

$$T_{AS} \approx 2.05 K_z,$$ (85)

- $(T_{AS})_{max} = 1$, i.e., total transformation of one wave mode into another is possible.

D. $\beta \sim 1$ regime

In the case of $\beta \sim 1$, the frequencies of all MHD wave modes are of the same order and no simplification of the set of Eqs. (18)-(20) is possible. Analysis of Eq. (28) shows that there exist the pair of complex conjugated first order resonant points:

$$K_y(\tau_{1,2}) = \pm i \sqrt{K_z^2 + \left(\frac{\beta - 1}{\beta + 1}\right)^2}$$ (86)

and the pair of complex conjugated second order resonant points:

$$K_y(\tau_{3,4}) = \pm i K_z.$$ (87)

No analytical expressions can be obtained for transformation coefficients if more than two wave modes are effectively coupled.

Numerical study of the set of Eqs. (18)-(20) is performed as follows: WKB solutions are used to obtain initial values of $d, b_y, b_z$ and their first derivatives for different wave modes separately far on the left side of the resonant interval ($K_y(0) \gg 1$). After passing through the resonant interval (i.e., for any $\tau$, for which $K_y(\tau) \ll -1$), WKB solutions were used again to determine the intensities of the transformed wave modes.

Figure 4 shows results obtained by numerical solution of the set of Eqs. (18)-(20), for initial SMW. Namely, transformation coefficients $(T_{FS}, T_{AS}, T_{SS})$ vs $K_z$ are presented for different values of $\beta$ and $S$.

According to our numerical study, qualitative character of wave transformation process is similar to the cases described in the previous sections. However, there are some differences. The most interesting is the failure of symmetry property (53). The presence of third effectively interacting wave mode leads to the asymmetry of two wave mode interactions, e.g., intensity of AW generated by SMW differs from the intensity of the inverse process.
V. HIGH SHEAR REGIME: WAVE OVER-REFLECTION

Let us consider the dynamics of SFHs of MHD wave modes in the flow with moderate and high shear rates ($S \gtrsim 1$). In this case the dynamics is strongly non-adiabatic $[\Theta_i$ terms can not be neglected in the left hand side of Eqs. (32)] except time intervals when $|K_y(\tau)| \gg S$. The existence of these adiabatic intervals permits to study wave interaction based on the asymptotic analysis presented in Sec III.

The main novelty that appears in the flow at high shear rates is that the dynamics involves wave reflection/over-reflection phenomena. Specifically, $T_{ij}$ can not be neglected and becomes important in Eq. (47). Physically it means that initial $\Psi_i^+$ mode can be effectively transformed into $\Psi_j^-$ modes (wave modes with phase velocity directed opposite to phase velocity of initial wave mode with respect to X-axis). Thus, at high shear rates, the wave dynamics represents an interplay of transformation and (over)reflection phenomena that are described by $T_{ij}$ and $\overline{T}_{ij}$.

Dependence of $T_{ij}$ and $\overline{T}_{ij}$ on $K_z$ obtained by numerical solution of the initial set of equations (18)-(20) for different values of parameters $\beta$ and $S$ are presented in Figs. 5-8. Figs. 5-7 show the cases when initially only SMW with positive phase velocity along X-axis exists, i.e., only $D_{S,L}^+ \neq 0$. Whereas in Fig. 8 initially only AW with positive phase velocity along X-axis exists, i.e., only $D_{A,L}^+ \neq 0$.

In Fig. 5 $\beta = 2$ and $S = 1$ (dashed lines) and $S = 4$ (solid lines). Both, wave transformation and reflection phenomena appear and enhance with increase of $S$. Coupling of SMW with FMW is dominant at $K_z \ll 1$, while coupling of SMW with AW is dominant at $K_z \gtrsim 1$. Figure 6 shows the same plots for high $\beta$ and moderate $S$. Particularly, dashed lines correspond to $\beta = 50$ and $S = 1$, whereas solid lines correspond to $\beta = 50$ and $S = 2$. For this range of parameters $\beta^{1/2} \gg S$ and according to Eq. (62) $T_{FS}, \overline{T}_{FS} \approx 0$. Hence, SMW and FMW are not coupled. Fig. 6 shows one more new feature of the wave dynamics. Namely, $\overline{T}_{SS} > 1$ at $S = 2$ and $K_z \ll 1$. Physically it means that the amplitude of the reflected SMW is greater than amplitude of incident SMW. This is the over-reflection phenomenon first discovered by Miles for acoustic waves (30). This phenomenon becomes dominant at $S \gg 1$.

Figures 7 and 8 show the case when $\beta = 100$ and $S = 8, 12, 16, 20$. The dashed lines correspond to $S = 8$. For this range of the parameters all the wave modes are coupled. The
wave over-reflection phenomenon is more profound. Particularly, in Fig. 7 $T_{SS}, \overline{T}_{SS} \approx 7$ when $K_z \ll 1$. In other words, the energy density of the reflected (as well as transmitted) SMW is about 50 times greater that the energy density of the incident SMW. When $K_z \sim 1$, then $T_{AS}, \overline{T}_{AS} \approx 5$, i.e., the reflected and transmitted AW are substantially greater than incident SMW. Figure 8 represents the transformation coefficients if only AW exists initially. In this case significant growth of energy density of perturbations takes place at $K_z \gtrsim 1$.

For more detailed analysis of the over-reflection phenomena consider the dynamics of SMW at $K_z = 0$, $\beta \gg 1$ and $S \ll \beta^{1/2}$. Due to the condition $K_z = 0$ there is no coupling between SMW and AW. Condition $S \ll \beta^{1/2}$ provides that the coupling between SMW and FMW is also negligible. Combining Eqs. (32), and (A1)-(A3) one can obtain following second order equation for the evolution of SMW:

$$\frac{d^2 \Psi_S}{d\tau^2} + f(S, \tau) \Psi_S = 0.$$  \hspace{1cm} (88)

where:

$$f(S, \tau) \equiv 1 - \frac{S^2}{[1 + K_y^2(\tau)]^2}$$  \hspace{1cm} (89)

Asymptotic analysis of this kind of equations is well known in quantum mechanics [25, 26] and the theory of differential equations [27, 31]. From mathematical point of view the reflection is caused due to the cavity of $f(S, \tau)$ that appears in the vicinity of the point $\tau_+$ where $K_y(\tau_+) = 0$.

In accordance with quantum mechanics we define reflection and transmission coefficients as $R \equiv \overline{T}_{SS}^2$ and $T \equiv T_{SS}^2$ respectively. It is well known [26, 27, 31] that wave action conservation provides that:

$$1 = T_{SS}^2 - \overline{T}_{SS}^2 \equiv T - R.$$  \hspace{1cm} (90)

So we can conclude that the total energy of SMW always increase during the reflection/over-reflection process:

$$\frac{E}{E_0} = T + R.$$  \hspace{1cm} (91)

The dependence of reflection coefficient $R$ on $S$ at $\beta = 400$ obtained by numerical solution of Eqs. (18)-(20) is presented in Fig. 9. Initial conditions for numerical solution are chosen by WKB solutions (39) far on the left side of resonant area ($K_y(0) \gg 1, S$). For high shear rates ($S \sim \beta^{1/2}$) SMW is partially transformed to FMW, that is why the growth rate of the reflection coefficient decreases (dashed part of the graph in Fig. 9). According to
the numerical study of Eq. (88), the amplitude of reflected SFH exceeds the amplitude of incident SFH if:

\[ S^2 \equiv \left( \frac{A}{k_x V_A} \right)^2 > 2. \]  

(92)

This condition indicates that the phenomenon of over-reflection can take place in plasma with \( \beta \gg 1 \) even for small values of the shear parameter \( A \). The amplification of the energy density of perturbations is always finite, but it can become arbitrarily large under due increase of the shear rate.

VI. SUMMARY AND DISCUSSIONS

The linear dynamics of MHD wave modes in constant shear flows is studied qualitatively and quantitatively in the framework of non-modal approach. Usage of asymptotic analysis familiar in quantum mechanics allows to study the physics of the (over)reflection and mutual transformation phenomena of MHD wave modes. Quantitative asymptotic and numerical analysis is performed for wide range of the system parameters: relation of spanwise and streamwise wave numbers \( (K_z \equiv k_z/k_x) \), plasma beta \( (\beta \equiv c_s^2/V_A^2) \) and the shear rate \( (S \equiv A/(k_x V_A)) \). The transformation takes place at small as well as at high shear rates and involves all MHD wave modes. The over-reflection becomes apparent only for slow magnetosonic and Alfvén wave modes at high shear rates.

At \( S \ll 1 \), the transformation has resonant nature and different wave modes are coupled at different \( \beta \).

- In high \( \beta \) regime Alfvén and slow magnetosonic waves are coupled and the transformation is effective at \( K_z \simeq 1 \). At \( \beta S \ll 1 \), for the maximum transformation coefficient we have \( (T_{\text{AS}}^2)_{\text{max}} = 1/2 \). If \( \beta S \) is not small, \( (T_{\text{AS}})_{\text{max}} = 1 \), i.e., the total transformation of one wave mode into another is possible.

- In low \( \beta \) regime Alfvén and fast magnetosonic waves are coupled. The transformation is effective also at \( K_z \simeq 1 \). \( (T_{\text{FA}}^2)_{\text{max}} = 1/2 \), i.e., even in the optimal regime only half of the energy of SFH of FMW can be transformed into AW and vice versa.

- At \( \beta \sim 1 \) all of the MHD wave modes are coupled: at \( K_z \ll 1 \) mainly are coupled slow and fast magnetosonic waves. Whereas at \( K_z \simeq 1 \) the coupling of these waves with Alfvén wave is dominant.

At moderate and high shear rates \((S)\), the wave mutual transformation is accompanied by
wave (over)reflection phenomenon. The amplification of the energy density of perturbations is always finite. However, the ratio of final to initial energies of waves can become arbitrarily large with increase of the shear rate.

Described phenomena could have substantial applications in the theory of MHD turbulence that is intensively developing recently (see, e.g., Ref. [32] and references therein). This concerns both the fundamental problem of MHD turbulence and wave modes participation in the turbulent processes.

Fundamental problem is the elaboration of the concept of self-sustaining MHD turbulence. None of the previous studies were concerned on energy source of the turbulence. The usual procedure is the following: some energy source of perturbations is assumed a priori and the evolution of energy spectrum is studied. On the other hand, in the framework of canonical MHD theory, there is no linear instability in the flows (they are spectrally stable). Therefore, existence of an energetic source for the permanent forcing of turbulence is problematic in the canonical/spectral theory. Nevertheless, one can apply the concept of bypass turbulence elaborated by hydrodynamic community for spectrally stable shear flows.

The role of the energy source in the MHD flows could play waves on-exponential growth (over-reflection).

Due to the importance of over-reflection phenomenon as a possible energy source for MHD turbulence in spectrally stable flows, detailed analysis of evolution of SFHs of MHD wave modes in flows with high shear rate will be treated more extensively elsewhere.

As for ingredients of the turbulence: the study of homogeneous MHD turbulence provides that in the case of $\beta \gg 1$, the turbulence is mainly Alfvénic (i.e., mainly consists of AW) and SMW plays a role of passive admixture. Studied in this paper linear coupling of MHD wave modes can significantly change this scenario even in the presence of small velocity shear ($S \ll 1$). Thus for the wide range of the system parameters all of the wave modes (at list two of them) should be the ingredients of the real MHD turbulence – the turbulence should be of a "mixed" type.

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**APPENDIX A: COUPLING COEFFICIENTS**

Substituting Eq. (24) into Eqs. (33)-(35) after straightforward calculations one can obtain:

\[
\Theta = \begin{pmatrix}
\dot{\alpha}^2 + \dot{\gamma}^2 \sin^2 \alpha \\
\dot{\alpha}^2 + \dot{\gamma}^2 \cos^2 \alpha \\
\dot{\gamma}^2
\end{pmatrix},
\]

(A1)

\[
\Lambda = \begin{pmatrix}
0 & \dot{\alpha} & \dot{\gamma} \sin \alpha \\
-\dot{\alpha} & 0 & -\dot{\gamma} \cos \alpha \\
-\dot{\gamma} \sin \alpha & \dot{\gamma} \cos \alpha & 0
\end{pmatrix},
\]

(A2)

and for non-zero components of \( \Upsilon \):

\[
\Upsilon_{12} = \ddot{\alpha}^2 + \dot{\gamma} \sin \alpha \cos \alpha,
\]

\[
\Upsilon_{13} = \dot{\gamma}^2 \sin \alpha,
\]

\[
\Upsilon_{21} = -\ddot{\alpha} + \dot{\gamma}^2 \sin \alpha \cos \alpha,
\]

\[
\Upsilon_{23} = -\dot{\gamma} \cos \alpha,
\]

\[
\Upsilon_{31} = -2 \dot{\alpha} \dot{\gamma} \cos \alpha - \ddot{\gamma} \sin \alpha,
\]

\[
\Upsilon_{32} = -2 \dot{\alpha} \dot{\gamma} \sin \alpha + \ddot{\gamma} \cos \alpha,
\]

(A3)

where \( \alpha(\tau) \) and \( \gamma(\tau) \) are defined by Eqs. (25) and (26) with \( K_y(\tau) \equiv K_y - S \tau \).

**APPENDIX B: SOME MATHEMATICAL ASPECTS OF LINEAR TRANSFORMATIONS**

To avoid the complication of mathematical formalism first consider the case of two coupled oscillators:

\[
\frac{d^2 \psi_1}{d\tau^2} + G_{11} \psi_1 = G_{12} \psi_2,
\]

(B1)

\[
\frac{d^2 \psi_2}{d\tau^2} + G_{22} \psi_2 = G_{21} \psi_1,
\]

(B2)

with \( G_{12} = G_{21} \) and eigen-frequencies:

\[
\Omega_{1,2}^2 = \frac{1}{2} \left[ G_{11} + G_{22} \pm \sqrt{(G_{11} - G_{22})^2 + 4G_{12}^2} \right],
\]

(B3)
that are slowly varying functions of $\tau$:

$$\dot{\Omega}_i \ll \Omega_i^2, \quad i = 1, 2.$$  \hfill (B4)

$\tau$ is normalized so that $\Omega_i(S\tau) \sim 1$ and $S \ll 1$. In addition, let us assume that all eigen-frequencies are real during the evolution. It can be easily seen from Eqs. (B1)-(B2) that this condition implies:

$$G_{11}G_{22} > G_{12}^2. \hfill (B5)$$

From this condition it follows that neither turning points (where $\Omega_i = 0$) nor resonant points ($\Omega_i^2 = \Omega_2^2$) can be located on the real $\tau$ axis. Transformation matrix to the normal variables, similar to Eq. (24), is:

$$Q = \begin{pmatrix} \cos \lambda & \sin \lambda \\ \sin \lambda & -\cos \lambda \end{pmatrix}, \hfill (B6)$$

with:

$$\lambda \equiv \arctan \left( \frac{G_{12}}{\Omega_2^2 - G_{11}} \right). \hfill (B7)$$

In the normal variables the set of equations (B1)-(B2) reduces to:

$$\ddot{\Psi}_1 + \left( \Omega_1^2 - \dot{\lambda}^2 \right) \Psi_1 = -2\dot{\lambda}\dot{\Psi}_2 - \ddot{\lambda}\Psi_2, \hfill (B8)$$

$$\ddot{\Psi}_2 + \left( \Omega_2^2 - \dot{\lambda}^2 \right) \Psi_2 = 2\dot{\lambda}\dot{\Psi}_1 + \ddot{\lambda}\Psi_1, \hfill (B9)$$

where:

$$\dot{\lambda} \equiv \frac{(\Omega_2^2 - G_{11})dG_{12}/d\tau - G_{12}(\Omega_2^2 - G_{11})/d\tau}{(\Omega_2^2 - G_{11})(\Omega_2^2 - \Omega_1^2)}. \hfill (B10)$$

According to Eq. (B3) condition $\Omega_2^2 = G_{11}$ leads to $G_{12} = 0$. Consequently, as it was mentioned above, only singular points of coupling coefficient are resonant points, where $\Omega_1^2 = \Omega_2^2$.

Let $\Psi_1^\pm$ and $\Psi_2^\pm$ be linear independent solutions of corresponding homogeneous equations (Eqs. (B8)-(B9) with right hand sides equal to zero), normalized such that asymptotically they converge to WKB solutions:

$$\Psi_{1,2}^\pm \approx \frac{1}{\sqrt{\Omega_{1,2}}} \exp \left( \pm i \int \Omega_{1,2} d\tau \right). \hfill (B11)$$

The general solution of the set of Eqs. (B8)-(B9) is ($i = 1, 2$):

$$\Psi_i = D_i^+ \Psi_i^+ + D_i^- \Psi_i^-. \hfill (B12)$$
For simplicity let us assume, that far on the left side of the resonant area only one, for example \( \Psi_2^+ \) mode exists, i.e., \( D_2^+ = 1 \) and all other WKB amplitudes are zero. Considering terms on the right hand side of Eqs. (B8)-(B9) as external source, using well known expressions for the solution of linear inhomogeneous second order differential equation and integrating by parts one can estimate the amplitude of generated wave \( \Psi_1^+ \) as:

\[
D_1^+ \sim \int_{-\infty}^{\infty} \dot{\lambda} \left( \Psi_1^+ \frac{d\Psi_2^+}{d\tau} - \Psi_2^+ \frac{d\Psi_1^-}{d\tau} \right) d\tau. \tag{B13}
\]

Necessary condition for effective coupling implies:

\[
D_1^+ \sim 1. \tag{B14}
\]

Presenting \( \Psi_1^- \) and \( \Psi_2^+ \) in the form of formal series, substituting them into Eq. (B13) and taking into account that obtained series of integrals converges rapidly, condition (B14) takes the form:

\[
\int \dot{\lambda} \exp \left( i \int (\Omega_1 - \Omega_2) d\tau \right) d\tau \sim 1. \tag{B15}
\]

If \( S \ll 1 \), then phase multiplier in (B13) is rapidly oscillating. Condition (B14) can be satisfied only if there exists an interval where time scale of the changing \( \dot{\lambda} \) and the phase multiplier in the integrand of Eq. (B15) are at least the same. This lead to the condition:

\[
|\Omega_1^2 - \Omega_2^2| \leq |\dot{\lambda} \Omega_2|, \tag{B16}
\]

at some interval along the path of integration. On the other hand one can note that only singular points of \( \dot{\lambda} \) in the complex \( \tau \)-plane are resonant points (where \( \Omega_1 = \Omega_2 \)). If so, Landau’s rule provides that if \( S \ll 1 \), then leading term of asymptotics of transformation coefficient is:

\[
T_{12} \sim \exp \left( - \left| \text{Im} \int_{\tau_0}^{\tau_{12}} (\Omega_1 - \Omega_2) dt \right| \right), \tag{B17}
\]

where \( \tau_0 \) is arbitrary point on the real \( \tau \)-axis and \( \tau_{12} \) is the nearest to the real \( \tau \)-axis resonant point. Eq. (B17) represents an alternative necessary condition for effective coupling of wave modes. Namely, resonant point \( \tau_{12} \) must be close to the real \( \tau \)-axis in such a way that the exponent in right hand side of Eq. (B17) has to be of the order of unity. Surely this condition is equivalent to Eq. (B16). On the other hand, it shows that if the condition of effective coupling is not satisfied, then the transformation coefficient is exponentially small with respect to the large parameter \( 1/S \).
According to Eq. (B10), in the neighborhood of \(n\)th order resonant point:

\[
\dot{\lambda} \sim \frac{S}{(S\tau_{12})^{n/2}}. \tag{B18}
\]

Eq. (B16) yields following estimation for the effective coupling time scale:

\[
\Delta\tau_{12} \sim S^{-1+1/n}. \tag{B19}
\]

1. Resonant points of the first order

Consider the case when only a pair of complex conjugated first order resonant points exists close to the real \(\tau\)-axis. This means that in the neighborhood of \(\tau_{12}^\prime\):

\[
|\Omega_1^2 - \Omega_2^2| \sim (\tau - \tau_{12})^{1/2}. \tag{B20}
\]

In this case, the leading term of asymptotic of transformation coefficient can be found exactly by the method first used by Landau [26].

Consider Eqs. (B8)-(B9) along the path \(\zeta\) that turns over \(\tau_{12}\) in the complex \(\tau\)-plane (Fig. 10). Considering Eqs. (B8)-(B9) in the neighborhood of \(\tau_{12}\), where \(\dot{\lambda} \sim (\tau - \tau_{12})^{-1/2}\), one can obtain that Eqs. (B8) and (B9) exchange after turning over resonant point. This means that \(\Psi_2\) became \(\Psi_1\) and vice versa. After this kind of analysis it is usually concluded (see e.g., Ref. [26]), that the transformation coefficient is:

\[
T_{12} = \exp \left( -\mathcal{I} \int_{\tau_0}^{\tau_{12}} (\Omega_1 - \Omega_2) d\tau \right) \left[ 1 + O(S^{1/2}) \right]. \tag{B21}
\]

From our point of view this kind of analysis needs to be commented. When one considers the set of Eqs. (B8)-(B9) along the path \(\zeta\), except mentioned above exchange of \(\Psi_1\) and \(\Psi_2\), there are also terms

\[
\int_{\zeta} \dot{\lambda} \left( \frac{d\Psi_1^{\dagger}}{d\tau} - \frac{d\Psi_2^{\dagger}}{d\tau} \right) d\tau. \tag{B22}
\]

In the case of the first order resonant points it can be shown that according Watson’s lemma contribution of these terms are of order \(S^{1/2} \exp(-\mathcal{I} \int_{\tau_0}^{\tau_{12}} (\Omega_1 - \Omega_2) d\tau)\) and (B22) is valid. However, in the case of the second or higher order resonant points, the contribution of terms like (B22) is at least of the same order as (B21). That is why this asymptotic method works only in the case of first order resonant points.
2. Resonant points of the second order

Unfortunately the transformation coefficient cannot be obtained analytically in the case of the second or higher order resonant points. Exact asymptotic expressions can be obtained only in the case of weak interaction. Let us consider the case when \( \tau_{12} \) and \( \tau^{*}_{12} \) are complex conjugated second order resonant points. Assume again, that far away on the left side of the resonant area only one, say, \( \Psi_{+}^{1} \) mode exists. In the case of a weak interaction feedback of \( \Psi_{+}^{1} \) to \( \Psi_{+}^{2} \) in Eqs. (B8)-(B9) can be neglected, so these equations decouple and there remains only inhomogeneous second order differential equation. We skip some details of calculations noting only main steps. In the case of negligible feedback, for the transformation coefficient one can obtain:

\[
T_{12} = \frac{1}{2} \int_{-\infty}^{\infty} \lambda \left( \bar{\Psi}_{1} \frac{d\bar{\Psi}_{2}}{d\tau} - \bar{\Psi}_{2} \frac{d\bar{\Psi}_{1}}{d\tau} \right) d\tau.
\]

In the neighborhood of the resonant point one can use the formal asymptotic series for \( \Psi_{+}^{\pm} \), considering \( \Omega_{1,2}(\tau) = \Omega_{1,2}(\tau_{12}) \) as constants. Consequently, the last equation takes the form:

\[
T_{12} = \left| \sum_{n} \int_{-\infty}^{\infty} \chi_{n} \lambda^{n} \exp \left( i \int_{\tau}^{\tau_{12}} (\Omega_{1} - \Omega_{2}) d\tau \right) d\tau \right|,
\]

where \( \chi_{n} \) are regular functions in the neighborhood of \( \tau_{12} \) and:

\[
\chi_{1} = \frac{\Omega_{1} + \Omega_{2}}{2\sqrt{\Omega_{1}\Omega_{2}}}.
\]

To evaluate the integrals in (B24), one can use Van der Wearden’s method [?]. It is not difficult to show that each term in Eq. (B24) are of order \( S^{1/2} \) with respect to the previous term. Consequently, in the case of the second order resonant point, leading term of the asymptotics is defined by the first term, for which we have:

\[
T_{12} = \frac{\pi}{2} \exp \left( - \left| \text{Im} \int_{\tau_{0}}^{\tau_{12}} (\Omega_{1} - \Omega_{2}) d\tau \right| \right).
\]


Figure captions

FIG. 1 Transformation coefficient $T_{FS}$ vs $\beta$ for $S = 0.05$ and $S = 0.02$ obtained by numerical (dotted line) and analytical solutions (solid line), see Eq. 62.

FIG. 2 Transformation coefficient $T_{AF}$ vs $\delta$. Dash-dotted line and dashed line represent analytical expressions (70) and (75), respectively. Solid line is obtained by numerical solution of Eqs. (63)-(64).

FIG. 3 Transformation coefficient $T_{AS}$ vs $K_z$ for $S = 0.005$ and $\beta = 5$ ($\beta S = 0.025$), $\beta = 100$ ($\beta S = 0.5$) and $\beta = 200$ ($\beta S = 1$).

FIG. 4 Transformation coefficients $T_{FS}$, $T_{AS}$ and $T_{SS}$ vs $K_z$ for $\beta = 0.8$ (left column), $\beta = 1.0$ (center column) and $\beta = 1.2$ (right column) for different normalized shear parameters: $S = 0.005$ (dash-dotted lines), $S = 0.02$ (dashed lines) and $S = 0.08$ (solid lines). Note, that according to the conservation of wave action $T_{FS}^2 + T_{AS}^2 + T_{SS}^2 = 1$ (see Eq. 41).

FIG. 5 Transformation coefficients ($T_{iS}$ and $\bar{T}_{iS}$ respectively) vs $K_z$ at $\beta = 2$ at different shear rates: $S = 1$ (dashed lines) and $S = 4$ (solid lines).

FIG. 6 Transformation coefficients $T_{AS}$, $\bar{T}_{AS}$, $T_{SS}$ and $\bar{T}_{SS}$ vs $K_z$ for $\beta = 50$ at different shear rates: $S = 1.0$ (dotted lines) and $S = 2.0$ (solid lines). $T_{FS}, \bar{T}_{FS} \approx 0$.

FIG. 7 Transformation coefficients ($T_{iA}$ and $\bar{T}_{iA}$ respectively) vs $K_z$ at $\beta = 100$ at different shear rates: $S = 8, 12, 16, 20$. Dashed lines correspond to $S = 8$.

FIG. 8 Transformation coefficients ($T_{iA}$ and $\bar{T}_{iA}$ respectively) vs $K_z$ at $\beta = 100$ and different shear rates: $S = 8, 12, 16, 20$. Dashed lines correspond to $S = 8$.

FIG. 9 Reflection coefficient of SMW $R(S)$ vs normalized shear parameter $S$. Here $\beta = 400$. For high shear parameters ($S \sim \beta^{1/2}$) SMW is partially converted to FMW, that is why the slope of the refraction coefficient graph decreases (dashed part of the graph).

FIG. 10 Resonant points and path $\zeta$ in the complex $\tau$ plane.
FIG. 1:

FIG. 2:
FIG. 3:

FIG. 4:
FIG. 5:

FIG. 6: