Stationary Spacetime from Intersecting M-branes

Kei-ichi Maeda\(^1,2,3\)\(^*\) and Makoto Tanabe\(^1\)\(^†\)

\(^1\)Department of Physics, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan
\(^2\)Advanced Research Institute for Science and Engineering, Waseda University, Shinjuku, Tokyo 169-8555, Japan and
\(^3\)Waseda Institute for Astrophysics, Waseda University, Shinjuku, Tokyo 169-8555, Japan

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We study a stationary "black" brane in M/superstring theory. Assuming BPS-type relations between the first-order derivatives of metric functions, we present general stationary black brane solutions with a traveling wave for the Einstein equations in \(D\)-dimensions. The solutions are given by a few independent harmonic equations (and plus the Poisson equation). General solutions are constructed by superposition of a complete set of those harmonic functions. Using the hyperspherical coordinate system, we explicitly give the solutions in 11-dimensional M theory for the case with M2 \(\perp\) M5 intersecting branes and a traveling wave. Compactifying these solutions into five dimensions, we show that these solutions include the BMPV black hole and the Brinkmann wave solution. We also find new solutions similar to the Brinkmann wave. We prove that the solutions preserve the 1/8 supersymmetry if the gravi-electromagnetic field \(F_{ij}\), which is a rotational part of gravity, is self-dual. We also discuss non-spherical "black" objects (e.g., a ring topology and an elliptical shape) by use of other curvilinear coordinates.

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I. INTRODUCTION

Black holes are now one of the most important subjects in string theory. The Beckenstein-Hawking black hole entropy of an extreme black hole is obtained in string theory by statistical counting of the corresponding microscopic states.\(^1\) While, we have found several interesting black hole solutions in supergravity theories, which are obtained as an effective theory of a superstring model in a low energy limit. We also know black hole solutions in a higher-dimensional spacetime, which play a key role in a unified theory such as string theory. In higher dimensions, because there is no uniqueness theorem of black holes, we have a variety of “black” objects such as a black brane.\(^1,2,3,4,5,6,7\) One of the most remarkable solutions is a black ring, which horizon has a topology of \(S^1 \times S^2\).\(^8\) Among such “black” objects, supersymmetric ones are very important. The black hole solutions in a supergravity include the higher-order effects of a string coupling constant, although these are solutions in a low energy limit. On the other hand, the counting of states of corresponding branes is performed at the lowest order of a string coupling. The results of these two calculations need not coincide each other. However, if there is supersymmetry, these should be the same because the numbers of dynamical freedom cannot be different in these BPS representations. Therefore, supersymmetric black hole (or black ring) solutions are often discussed in many literature.\(^1,2,3,4,5,6,7,8,9\)

The classification of supersymmetric solutions in minimal \(\mathcal{N} = 2\) supergravity in \(D = 4\) was first performed by a time-like or null Killing spinor.\(^9\) Recently, solutions in minimal \(\mathcal{N} = 1\) supergravity in \(D = 5\) have been classified into two classes by use of G-structures analysis.\(^10,11,12\) The six-dimensional minimal supergravity has also been discussed.\(^13\)

However, the fundamental theory is constructed in either ten or eleven dimensions. When we discuss the entropy of black holes, we have to show the relation between those supersymmetric black holes and more fundamental “black” branes in either \(D = 10\) or 11, from which we obtain “black” holes (or rings) via compactification. The entropy is microscopically described by the charges of branes.\(^14\) A supersymmetric rotating solution is obtained by compactification from M or type II supergravity.\(^15\) The supersymmetric rotating black ring solution is found.\(^16,17,18,19,20\) Such solutions are obtained also in lower dimensions. These solutions are in fact new classes of rotating solutions in four- or five-dimensional supergravity. The existence of such solutions suggests that the uniqueness theorem of black holes is no longer valid even in supersymmetric spacetime if the dimension is five or higher.\(^21\) Thus we may need to construct more generic “black” brane solutions in the fundamental theory and the
black holes by some compactification. M-theory is the best candidate for such a unified theory. Since its low energy limit coincides with the eleven-dimensional supergravity, it provides a natural framework to study “black” brane or BPS brane solutions.

In this paper we study a class of intersecting brane solutions in $D$-dimensions with a $(d - 1)$-dimensional transverse conformally flat space. We start with a generic form of the metric and solve the field equations of the supergravity (the Einstein equations and the equations for form fields). Assuming the intersection rule for the intersecting branes, which is the same as that derived in a spherically symmetric case \[30, 41, 42\], we derive the equations for each metric. We find that most metric components are described by harmonic functions, which are independent. One metric component $f$, which corresponds to a traveling wave, is usually given by the Poisson equation, which source term is given by the quadratic form of the “gravit-electromagnetic” field $F_{ij}$. In some configuration of branes, e.g., for two intersecting charged branes (M2⊥M5), the source term vanishes. As a result, we find only independent harmonic functions. Hence, we can easily construct arbitrary solution by superposing those harmonics. In order to preserve 1/8 supersymmetry, we have to impose that $F_{ij}$ is self-dual.

In §II, we consider the dilaton coupling gauged supergravity actions in $D$ dimensions, and derive the basic equations for a stationary “black” brane, which extra $(D - d)$-dimensions are filled by several branes, with a traveling wave. The solutions are given by harmonic functions on the $(d - 1)$-dimensional Euclidian space. In §III we construct the solution in eleven (or ten ) dimensions. In §IV, we present the explicit solutions for $d = 5$ by use of a hyperspherical coordinate system. We recover the BMPV solution \[32, 33\] and the Brinkmann wave \[43\] as a special case. The concluding remarks follow in §V. In Appendix A, we prove the 1/8 supersymmetry is preserved in our stationary “black” brane solutions, if $F_{ij}$ is self-dual. We also present some explicit solutions by use of different coordinate systems (hyperelliptic and hyperpolorical coordinates) in Appendix B.

II. BASIC EQUATIONS FOR A STATIONARY SPACETIME WITH BRANES

We first present the basic equations for a stationary spacetime with intersecting branes and describe how to construct generic solutions. We consider the following bosonic sector of a low energy effective action of superstring theory or M-theory in $D$ dimensions ($D ≤ 11$):

$$S = \frac{1}{16\pi G_D} \int d^D X \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \varphi)^2 - \sum_A \frac{1}{2 \cdot n_A} e^{a_A \varphi} F_{nA}^2 \right],$$  \hspace{1cm} (2.1)

where $R$ is the Ricci scalar of a spacetime metric $g_{\mu\nu}, F_{nA}$ is the field strength of an arbitrary form with a degree $n_A(\leq D/2)$, and $a_A$ is its coupling constant with a dilaton field $\varphi$. Each index $A$ describes a different type of brane. Although we leave the spacetime dimension $D$ free, the present action is most suitable for describing the bosonic part of $D = 10$ or $D = 11$ supergravity.

The equations of motion are written in the following forms:

$$R_{\mu\nu} = \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi + \sum_A \Theta_{nA\mu\nu},$$

$$\nabla^2 \varphi = \sum_A \frac{a_A}{2 \cdot n_A} e^{a_A \varphi} F_{nA}^2,$$

$$\partial_{\mu_1} (\sqrt{-g} e^{a_A \varphi} F_{nA}^{\mu_1 \cdots \mu_{nA}}) = 0,$$  \hspace{1cm} (2.2)

where $\Theta_{nA\mu\nu}$ is the stress-energy tensor of the $n_A$-form, which is given by

$$\Theta_{nA\mu\nu} = \frac{1}{2 \cdot n_A} e^{a_A \varphi} \left[ n_A F_{nA\mu\rho \cdots \sigma} F_{nA\nu\rho \cdots \sigma} - \frac{n_A}{D - 2} F_{nA}^2 g_{\mu\nu} \right].$$

We also have an additional equation, which is the Bianchi identity for the $n_A$-form, i.e.,

$$\partial_{[\mu} F_{nA\nu_1 \cdots \nu_{nA}] = 0.$$  \hspace{1cm} (2.3)

This is automatically satisfied if we introduce the potentials of $n_A$-form.

As for a metric form for a spacetime with intersecting branes, we assume the following metric form \[38\]:

$$ds^2 = 2\theta^\delta \delta + \sum_{i=1}^{d-1} (\theta^i)^2 + \sum_{\alpha=2}^{p} (\theta^\alpha)^2,$$  \hspace{1cm} (2.5)
where \( D = d + p \) and the dual basis \( \theta^\alpha \) are given by

\[
\theta^{\bar{u}} = e^\xi du, \quad \theta^{\bar{v}} = e^\xi \left( dv + f du + \frac{A}{\sqrt{2}} \right), \quad \theta^{\bar{v}} = e^{\eta} dv, \quad \theta^{\bar{a}} = e^{\zeta_a} dy^a.
\] (2.6)

Here we have used light-cone coordinates; \( u = -(t - y_1)/\sqrt{2} \) and \( v = (t + y_1)/\sqrt{2} \). This metric form includes rotation of spacetime and a traveling wave. Since we are interested in a stationary solution, we assume that the metric components \( f, A = A_i dx^i, \xi, \eta \) and \( \zeta_a \) depend only on the spatial coordinates \( x^i \) in \( d \)-dimensions, which coordinates are given by \( \{t, x^i(i = 1, 2, \cdots, d - 1)\} \). In this setting, we set each brane \( A \) in a submanifold of \( p \)-spatial dimensions, which coordinates are given by \( \{y_\alpha(\alpha = 1, 2, \cdots, p)\} \). Note that the solution in this metric form is invariant under the gauge transformation, \( A \to A + d \Lambda, \quad v \to v - \Lambda/\sqrt{2} \).

As for the \( n_A \)-form field with a \( q_A \)-brane, we assume that the source brane exists in the coordinates \( \{y_1, y_{\alpha_2}, \cdots, y_{\alpha_A}\} \). The form field generated by an “electric” charge is given by the following form:

\[
F_{n_A} = \partial_j E_A dx^j \wedge du \wedge dv \wedge dy_2 \wedge \cdots \wedge dy_{q_A} + \frac{1}{\sqrt{2}} \partial_j B_j^A dx^j \wedge du \wedge dv \wedge dy_2 \wedge \cdots \wedge dy_{q_A},
\] (2.7)

where \( n_A = q_A + 2 \) and \( E_A \) and \( B_j^A \) are scalar and vector potentials. This setting automatically guarantees the Bianchi identity (2.4).

We can also discuss the form field generated by a “magnetic” charge by use of a dual *-\( n_A \)-field with *-\( q_A \)-brane, which is obtained by a dual transformation of the \( n_A \)-field with a \( q_A \)-brane (*\( n_A \equiv D - n_A \), *\( q_A \equiv n_A - 2 \)). In other words, the field components of \( F_{n_A} \) generated by a “magnetic” charge are described by the same form of the dual field *\( F_{n_A} = F_{*n_A} \). We then treat \( F_{n_A} \), which is generated by a “magnetic” charge, as another independent form field with a different brane from \( F_{n_A} \), which is generated by an “electric” charge, when we sum up by the types of branes \( A \).

Setting

\[
H_A = \exp \left[ - \left( 2 \xi + \sum_{\alpha=2}^p \zeta_\alpha - \frac{1}{2} \theta_A a_A \varphi \right) \right]
\] (2.8)

\[
V = \exp \left[ 2 \xi + (d - 3) \eta + \sum_{\alpha=2}^p \zeta_\alpha \right],
\] (2.9)

where

\[
\theta_A = \begin{cases} +1 & \text{for } n_A \text{-form field } (F_{n_A}) \\ -1 & \text{for } \text{dual field } (*F_{n_A}) \end{cases}
\] (2.10)

we find the basic equations as follows:

\[
\partial^2 f + \partial_j \partial^j \ln V = \frac{1}{8} e^{2(\xi - \eta)} \left[ F_{ij}^2 - \frac{1}{2} \sum_A \left( F_{ij}^{(A)} \right)^2 \right],
\] (2.11)

\[
\partial^2 \xi + \partial_\xi \partial^j \ln V = \frac{1}{2(D - 2)} \sum_A (D - q_A - 3) H_A^2(\partial E_A)^2;
\] (2.12)

\[
\partial^i F_{ij} + F_{ij} \partial^j \left[ 2 (\xi - \eta) + \ln V \right] = \sum_A H_A F_{ij}^{(A)} \partial^i E_A;
\] (2.13)

\[
(\partial^2 \eta + \partial_\eta \partial^j \ln V) \delta^j_i + 2 \partial_\xi \partial^i \xi + (d - 3) \partial_\xi \partial^j \eta + \sum_{\alpha=2}^p \partial_\xi \zeta_\alpha \partial^j \zeta_\alpha + \partial_\xi \partial^j \ln V - (\partial_\eta \partial^j \ln V + \partial^j \eta \partial_\eta \ln V)
\]

\[
= -\frac{1}{2} \partial_i \varphi \partial^i \varphi + \frac{1}{2} \sum_A H_A^2 \left[ \partial_i E_A \partial^j E_A - \frac{q_A + 1}{(D - 2)} (\partial E_A)^2 \delta^j_i \right],
\] (2.14)

\[
\partial^2 \zeta_\alpha + \partial_\xi \zeta_\alpha \partial^i \ln V = \frac{1}{2(D - 2)} \sum_A \delta_\alpha A H_A^2(\partial E_A)^2;
\] (2.15)

\[
\partial^2 \varphi + \partial_j \varphi \partial^j \ln V = -\frac{1}{2} \sum_A \epsilon_A \partial_\alpha H_A^2(\partial E_A)^2;
\] (2.16)

\[
\partial_j (H_A^2 V \partial^j E_A) = 0,
\] (2.17)

\[
\partial^j \left( \varphi F_{ij}^{(A)} \right) = 0.
\] (2.18)
where $\partial_i$ is a partial derivative $\partial/\partial x^i$ in a flat $(d - 1)$-space, $\partial^2 \equiv \partial_i \partial^i$, and $F_{ij}$, $F^{(A)}_{ij}$, and $\delta_{\alpha A}$ for each coordinate $\alpha$ are defined by

$$
F_{ij} \equiv \partial_i A_j - \partial_j A_i
$$

$$
F^{(A)}_{ij} \equiv 2H_A \left( A_{ij} \partial_j - \partial_i B^A_{ji} \right)
$$

$$
\delta_{\alpha A} \equiv \begin{cases} D - q_A - 3 & \alpha = \alpha_2, \ldots, \alpha_q \ A \\
-(q_A + 1) & \text{otherwise}
\end{cases}
$$

(2.19)

The square bracket denotes the anti-symmetrization of indices, i.e., $X[i]Y[j] \equiv \frac{1}{2}(X[i]Y[j] - Y[j]X[i])$

Since $E_A$ appears just with a spatial derivative $\partial_i$, we can replace it with $E\left( A \right) = E_A - E_A^{(0)}$, where $E_A^{(0)}$ is a constant, which is fixed by a boundary condition. Using Eqs. (2.17) and (2.18), we obtain from Eqs. (2.12), (2.14), (2.16) and (2.18),

$$
\partial^i \left[ V \left( \partial_j \xi - \frac{1}{2(D-2)} \sum_A (D - q_A - 3)H_A^2 \tilde{E}_A \partial_j \tilde{E}_A \right) \right] = 0,
$$

(2.20)

$$
\partial^i \left[ V \left( \partial_j \zeta_\alpha - \frac{1}{2(D-2)} \sum_A \delta_{\alpha A} H_A^2 \tilde{E}_A \partial_j \tilde{E}_A \right) \right] = 0,
$$

(2.21)

$$
\partial^i \left[ V \left( \partial_j \varphi + \frac{1}{2} \sum_A \epsilon_{AaA} H_A^2 \tilde{E}_A \partial_j \tilde{E}_A \right) \right] = 0.
$$

(2.22)

This set of equations is a coupled system of elliptic-type differential equations, for which it is very difficult to find general solutions. Hence, in this paper, we assume the following special relations:

$$
\partial_j \xi = \frac{1}{2(D-2)} \sum_A (D - q_A - 3)H_A^2 \tilde{E}_A \partial_j \tilde{E}_A,
$$

(2.23)

$$
\partial_j \zeta_\alpha = \frac{1}{2(D-2)} \sum_A \delta_{\alpha A} H_A^2 \tilde{E}_A \partial_j \tilde{E}_A,
$$

(2.24)

$$
\partial_j \varphi = -\frac{1}{2} \sum_A \epsilon_{AaA} H_A^2 \tilde{E}_A \partial_j \tilde{E}_A,
$$

(2.25)

which guarantee Eqs. (2.20), (2.21) and (2.22) to be correct.

These equations are relations between the first-order derivatives of variables just as the BPS conditions. The existence of supersymmetry in the obtained solutions is shown in the Appendix. Hence, these relations may be related to a BPS state, or an extremal “black” brane solution in supergravity. In fact, if $F^{(A)}_{ij}$ is proportional to $F_{ij}$ and $F_{ij}$ is self-dual, we prove that 1/8 supersymmetry remains in the solutions with M2 ⊥ M5 branes for $D = 11$ supergravity theory.

$\eta$ is obtained from $\xi$, $\zeta_\alpha$ and $\varphi$ as

$$
\partial^i \eta = -\frac{1}{d-3} \partial^i \left( 2\xi + \sum_{\alpha=2}^p \zeta_\alpha - \ln V \right)
$$

$$
= -\frac{1}{2(D-2)} \sum_A (q_A + 1)H_A^2 \tilde{E}_A \partial^i \tilde{E}_A + \frac{1}{(d-3)} \partial^i \ln V.
$$

(2.26)

This gives

$$
\partial^2 \eta + \partial_j \eta \partial^i \ln V = \frac{1}{V} \partial_j (V \partial^i \eta)
$$

$$
= -\frac{1}{2V(D-2)} \sum_A (q_A + 1)\partial_j (H_A^2 V \tilde{E}_A \partial^i \tilde{E}_A) + \frac{1}{(d-3)V} \partial^2 V
$$

$$
= -\frac{1}{2(D-2)} \sum_A (q_A + 1)H_A^2 (\partial \tilde{E}_A)^2 + \frac{1}{(d-3)V} \partial^2 V.
$$

(2.27)
We have, however, another equation for $\eta$, i.e., Eq. (2.14), which should be satisfied as well. We have to find a solution which satisfies both equations. This consistency gives two conditions for $\tilde{E}_A$. In order to derive them, we first take a trace of Eq. (2.14), which leads to

\[(d-1)(\partial^2 \eta + \partial \eta \partial^j \ln V) + 2(\partial \xi)^2 + (d-3)(\partial^2 \eta) + \sum_{\alpha=2}^{p} (\partial \zeta^\alpha)^2 + \partial^2 \ln V - 2\partial_\eta \partial^j \ln V\]

\[= -\frac{1}{2} (\partial \varphi)^2 + \frac{1}{2(D-2)} \sum_A [D - 2 - (q_A + 1)(d-1)] \tilde{H}_A^2 (\partial \tilde{E}_A)^2.\]  

(2.28)

Substituting Eqs. (2.26), (2.27), (2.30) and (2.31) into Eq. (2.26), we find the first condition:

\[\frac{1}{2} \sum_{A,B} M_{AB} \tilde{H}_A^2 \tilde{H}_B^2 \tilde{E}_A \tilde{E}_B (\partial \tilde{E}_A)(\partial \tilde{E}_B) - \sum_A \tilde{H}_A^2 (\partial \tilde{E}_A)^2 + 4 \left( \frac{d-2}{d-3} \right) V^{-1/2} \partial^2 V^{1/2} = 0,\]  

(2.29)

where

\[M_{AB} = \frac{1}{(D-2)^2} \left[ 2 (D - q_A - 3)(D - q_B - 3) + (d-3)(q_a + 1)(q_b + 1) + \sum_{\alpha=2}^{p} \delta_{\alpha A} \delta_{\alpha B} \right] + \frac{1}{2} \epsilon_A \epsilon_B a_A a_B.\]  

(2.30)

We have also a traceless part of Eq. (2.14), which is written as

\[2 \left( \partial_\xi \partial^j \xi - \frac{1}{d-1} (\partial \xi)^2 \delta^j_i \right) + (d-3) \left( \partial_\eta \partial^j \eta - \frac{1}{d-1} (\partial \eta)^2 \delta^j_i \right) + \sum_{\alpha=2}^{p} \left( \partial_\alpha \zeta^\alpha \partial^j \zeta^\alpha - \frac{1}{d-1} (\partial \zeta^\alpha)^2 \delta^j_i \right)\]

\[+ \frac{1}{2} \left( \partial_\varphi \partial^j \varphi - \frac{1}{d-1} (\partial \varphi)^2 \delta^j_i \right) - \frac{1}{2} \sum_A \tilde{H}_A^2 \left( \partial_\xi \tilde{E}_A \partial^j \tilde{E}_A - \frac{1}{d-1} (\partial \tilde{E}_A)^2 \delta^j_i \right)\]

\[+ \partial_\eta \partial^j \ln V - \frac{1}{d-1} \partial^2 \ln V \delta^j_i - \left( \partial_\eta \partial^j \ln V + \partial^j \eta \partial_i \ln V - \frac{2}{d-1} (\partial \eta)(\partial \ln V) \delta^j_i \right) = 0.\]  

(2.31)

This equation gives the second condition:

\[\frac{1}{2} \sum_{A,B} M_{AB} \tilde{H}_A^2 \tilde{H}_B^2 \tilde{E}_A \tilde{E}_B (\partial \tilde{E}_A)(\partial \tilde{E}_B) - \sum_A \tilde{H}_A^2 (\partial \tilde{E}_A)^2\]

\[- \frac{1}{d-1} \delta^j_i \left[ \frac{1}{2} \sum_{A,B} M_{AB} \tilde{H}_A^2 \tilde{H}_B^2 \tilde{E}_A \tilde{E}_B (\partial \tilde{E}_A)(\partial \tilde{E}_B) - \sum_A \tilde{H}_A^2 (\partial \tilde{E}_A)^2 \right]\]

\[-2(d-3) V \left[ \partial_\eta \partial^j \left( V^{-1/4} \right) - \frac{1}{d-1} \delta^j_i \partial^2 \left( V^{-1/4} \right) \right] = 0,\]  

(2.32)

We have to find a solution for two conditions (2.29) and (2.32). Here we shall assume $V = \text{constant}$. We shall also impose the condition $M_{AB} = 0$ for $A \neq B$, which is called the “intersection” rule [40, 41, 42]. This rule is derived in the case of “spherically symmetric” spacetime from the condition that each $E_A$ is independent. Our case is just an ansatz.

Suppose that the $q_A$-brane and $q_B$-brane are filled in different spatial dimensions, but those branes are crossing on $\tilde{q}_{AB}$ dimensions ($\tilde{q}_{AB} < q_A, q_B$). Calculating (2.31), we obtain

\[M_{AB} = \tilde{q}_{AB} + 1 - \frac{(q_A + 1)(q_B + 1)}{D - 2} + \frac{1}{2} \epsilon_A \epsilon_B a_A a_B.\]  

(2.33)

Since we assume that it vanishes for $A \neq B$, we obtain the crossing dimensions $\tilde{q}_{AB}$ as

\[\tilde{q}_{AB} = \frac{(q_A + 1)(q_B + 1)}{D - 2} - 1 - \frac{1}{2} \epsilon_A \epsilon_B a_A a_B.\]  

(2.34)

Eqs. (2.29) and (2.32) are then reduced to

\[\sum_A \left[ \frac{1}{2} M_{AA} H_A^2 \tilde{E}_A^2 - 1 \right] H_A^2 (\partial \tilde{E}_A)^2 = 0,\]  

(2.35)

\[\sum_A \left[ \frac{1}{2} M_{AA} H_A^2 \tilde{E}_A^2 - 1 \right] H_A \left[ \partial_\xi \tilde{E}_A \partial^j \tilde{E}_A - \frac{1}{d-1} (\partial \tilde{E}_A)^2 \delta^j_i \right] = 0.\]  

(2.36)
Hence, if

\[ \frac{1}{2} M AA H^2 \tilde{E}_A^2 = 1, \quad \text{or} \quad \tilde{E}_A = \text{const}, \]

Eqs. (2.37) and (2.38) are satisfied.

Since

\[ M AA = \frac{(q_A + 1)(D - q_A - 3)}{D - 2} + \frac{1}{2} q_A^2 \equiv \frac{\Delta_A}{D - 2}, \]

from Eq. (2.37), we find \( \tilde{E}_A \) as

\[ \tilde{E}_A = \sqrt{\frac{2(D - 2)}{\Delta_A}} \frac{1}{H_A}, \quad \text{or} \quad \tilde{E}_A = \text{const}. \]  

If we impose that a spacetime is asymptotically flat (i.e., \( H_A \to 1 \) as \( r \to \infty \)) and the potential \( E_A \) vanishes at infinity, we find that

\[ E_A = -\sqrt{\frac{2(D - 2)}{\Delta_A}} \left( 1 - \frac{1}{H_A} \right), \quad \text{or} \quad E_A = 0. \]  

Inserting this relation into Eq. (2.17), we obtain the equation for \( H_A \) as

\[ \partial^2 H_A = 0, \]

which means that \( H_A \) is a harmonic function on \( \{x^i\} \in \mathbb{E}^{d-1} \). From the relation (2.41) with Eqs. (2.12), (2.15) and (2.16), we then obtain the solutions for metric functions in terms of the harmonic functions \( H_A \):

\[ \xi = -\sum_A \frac{D - q_A - 3}{\Delta_A} \ln H_A, \quad \eta = \sum_A \frac{q_A + 1}{\Delta_A} \ln H_A, \]

\[ \zeta_\alpha = -\sum_A \frac{\delta_{\alpha A}}{\Delta_A} \ln H_A, \quad \varphi = (D - 2) \sum_A \frac{\epsilon^A q_A}{\Delta_A} \ln H_A. \]  

We have two remaining equations (2.13) and (2.18) for \( A_i (\mathcal{F}_{ij}) \) and one Poisson equation (2.11) for \( f \). In order to solve the former two equations, we classify \( n_A \)-form field into the following three cases:

(1) Charged Branes

We expect that each brane \( A \) has a charge \( Q_H^{(A)} \) (either electric or magnetic type), and then \( E_A \) becomes non-trivial, i.e., \( H_A \neq 1 \). In this case, if we set

\[ B_i^A = -\tilde{E}_A A_i = -\sqrt{\frac{2(D - 2)}{\Delta_A}} \frac{A_i}{H_A}, \]

we have

\[ \mathcal{F}_{ij}^{(A)} = \sqrt{\frac{2(D - 2)}{\Delta_A}} \mathcal{F}_{ij}. \]

Inserting Eqs. (2.42), we can show that two equations (2.13) and (2.18) are reduced to the following one Laplace equation:

\[ \partial^2 \mathcal{F}_{ij} = 0. \]  

\( B_i^A \) describes a magnetic-type field produced by a current appearing through rotation of a charged brane.

It turns out that the condition (2.41) plays a key role for the system to keep supersymmetry (see Appendix A).
(2) Neutral Branes with Currents

If $H_A = 1$ (i.e., $E_A = 0$), which is a trivial solution of Eq. (2.41), we find that there is no electric type field, and then zero charge on the brane. This brane does not make any contribution to velocities. This situation is similar to the conventional electric current in a metal. The negative charges of electrons current may appear in a system consisting of the same numbers of branes and anti-branes, which move with different motion of electrons.

In this case ($H_A = 1$), the equations for $A_i$ and $B_i^A$ become independent as

$$\partial^j F_{ij} = 0, \quad \partial^j F_{ij}^{(A)} = 0.$$  

(2.46)

Since these two equations are exactly the same, we can adopt the same solution with different amplitudes, i.e.,

$$B_i^A = -\lambda_A \sqrt{\frac{2(D-2)}{\Delta_A}} A_i, \quad F_{ij}^{(A)} = \lambda_A \sqrt{\frac{2(D-2)}{\Delta_A}} F_{ij},$$  

(2.47)

where $\lambda_A$ is an arbitrary constant, which corresponds to the strength of a current, i.e., numbers of branes and anti-branes and its relative velocity. This relation not only makes the equation for $f$ simple (see below) but also keeps supersymmetry of the system (see Appendix A).

(3) Charged Branes with Currents

If numbers of branes and anti-branes are different, such a system has a net charge. Then the magnetic field may be divided into two parts:

$$B_i^A = -\lambda_A \sqrt{\frac{2(D-2)}{\Delta_A}} \frac{A_i}{H_A} + B_i^{A(N)},$$  

(2.48)

where the first term is produced by a current appearing through the motion of a net charge, while $B_i^{A(N)}$ is produced by a current even in the case of zero net charge (just as in Case (2)). From Cases (1) and (2), we expect that $A_i$ and $B_i^{A(N)}$ are arbitrary harmonic functions (just as Eqs. (2.46)). However, since we have two equations (2.48) and (2.49) for $A_i$, it is not trivial whether our expectation is the case. In fact, we have to impose the following condition

$$\partial_i B_j^A \cdot \partial^j \tilde{E}_A = 0,$$  

(2.49)

in order for $A_i$ and $B_i^A$ to be a solution.

If we can impose the relation of $B_i^{A(N)} \propto A_i$ just as that for $B_i^A$ in Case (2), the equation for $f$ becomes simple, but this relation is not consistent with the condition (2.50). As a result, the equation for $f$ becomes very complicated, although we can solve it in principle because it is the Poisson equation in a flat space. We also find that this condition breaks the supersymmetry of the system. Therefore, in what follows, we will discuss only Cases of (1) and (2).

Finally we discuss the last equation (2.41) for $f$. Here we assume we have $N_{A'}$ charged branes (Case (1)) and $N_{A''}$ neutral branes with currents (Case (2)). $N_A = N_{A'} + N_{A''}$ is the total number of branes. Then, as for the metric $f$, we find

$$\partial^2 f = \frac{\beta}{2} \prod_A H_A^{2(D-2)} \left( \partial_{ij} A_j \right)^2,$$  

(2.50)

where

$$\beta = \left[ 1 - (D-2) \left( \sum_{A'} \frac{1}{\Delta_A} + \sum_{A''} \frac{\lambda^2_{A''}}{\Delta_{A''}} \right) \right],$$  

(2.51)

is just a constant. $A'$ describes charged branes which provide non-trivial potentials $E_A$ (2.39), while $A''$ gives a contribution from the neutral branes with currents.

If the following condition is satisfied:

$$(D-2) \left( \sum_{A'} \frac{1}{\Delta_A} + \sum_{A''} \frac{\lambda^2_{A''}}{\Delta_{A''}} \right) = 1,$$  

(2.52)

then
\( \beta \) vanishes, and then \( f \) is given by an arbitrary harmonic function on \( \mathbb{E}^{d-1} \). In general, however, since \( \lambda_{AB} \) is free, \( \beta \) can have any sign (either positive, zero or negative). Thus, we have to solve the Poisson equation (2.50).

The solution obtained in this section is summarized as follows:

\[
\begin{aligned}
    ds^2 &= \prod_A H_A^{2q_A - 1} \left[ 2 \prod_B H_B^{-2 \frac{q_A}{a_A}} \left( dv + f du + \frac{A}{\sqrt{2}} \right) + \sum_{\alpha=2}^p \prod_B H_B^{-2 \frac{q_A}{a_A}} dy_\alpha^2 + \sum_{i=1}^{d+1} dx_i^2 \right], \\
    \varphi &= (D - 2) \sum_A \frac{\gamma_{\alpha A}}{\Delta_A} \ln H_A,
\end{aligned}
\]

where

\[
\begin{aligned}
    \Delta_A &= (q_A + 1)(D - q_A - 3) + \frac{D - 2}{2} a_A^2, \\
    \gamma_{\alpha A} &= \delta_{\alpha A} + q_A + 1 = \begin{cases} 
        D - 2 & \alpha = \alpha_2, \cdots, \alpha_{q_A} \\
        0 & \text{otherwise}
    \end{cases}.
\end{aligned}
\]

\( H_A \) for each \( q_A \)-brane and \( A = A_i dx^i \) are arbitrary harmonic functions, while the vector potential \( B_i^A \) can be chosen either as \( B_i^A \propto A_i / H_A \) (when \( H_A \neq 1 \)), or an arbitrary harmonic function (when \( H_A = 1 \)). The “wave” metric \( f \) usually satisfies the Poisson equation (2.50) with some source term originated by the “rotation”-induced metric \( A_i \), although it can be also an arbitrary harmonic function for some specific configuration of branes (\( \beta = 0 \)).

It is worth noticing that we have independent Laplace equations for \( H_A \) and \( A_i \) (and \( f \) when \( \beta = 0 \)). This makes the construction of solutions very easy. The superposition of any solutions also provides us an exact solution. Hence we can construct an infinite number of solutions. We can also show that a part of supersymmetry is preserved in Cases (1) and (2) if \( F_{ij} \) is self-dual (see Appendix A).

### III. “BLACK” BRANE SOLUTIONS IN M-THEORY AND IN TYPE IIB SUPERSTRING THEORY

In what follows, we shall show how to construct the exact stationary solutions in M-theory (or in type IIB superstring theory), and present concrete examples. We do not know very much about M-theory or a superstring theory except for the fact that, at the low energy scale, it is described by eleven-dimensional or ten-dimensional supergravity. Then in this section, we discuss “black” brane solutions in the eleven-dimensional and ten-dimensional supergravity. Here we use the phrase “black” brane for a stationary and asymptotically flat spacetime solution with intersecting branes, although it may contain a naked singularity or it will be a BPS solution for appropriate values of parameters.

#### A. M2 and M5-brane solutions in M-theory

In 11-dimensional supergravity, we have a 4-form field \( (n_A = 4) \) and no dilaton \( \varphi \) \( (a_A = 0) \). Setting \( D = 11 \) and \( a_A = 0 \), we have

\[
\Delta_A = (q_A + 1)(8 - q_A).
\]

The form field produced by an “electric” charge is related to the M2-brane, i.e., \( q_A = n_A - 2 = 2 \). This gives \( \Delta_A = 18 \). The “black” brane solution in this case is written as

\[
\begin{aligned}
    ds_{11}^2 &= H_2^{1/3} \left[ 2 H_2^{-1} du \left( dv + f du + A / \sqrt{2} \right) + H_2^{-1} dy_6^2 + \sum_{i=1}^8 dx_i^2 \right], \\
    F_4 &= d(1/H_2) \wedge du \wedge dv \wedge dy_6 + \frac{1}{\sqrt{2}} dB_2 \wedge du \wedge dy_6,
\end{aligned}
\]

where \( H_2 \) is a harmonic function on \( \mathbb{E}^8 \).

Similarly, the field with a “magnetic” charge is related to the M5-brane because \( *q_A = *n_A - 2 = D - n_A - 2 = 5 \). This also gives \( \Delta_A = 18 \). The solution is described by

\[
\begin{aligned}
    ds_{11}^2 &= H_5^{2/3} \left[ 2 H_5^{-1} du \left( dv + f du + A / \sqrt{2} \right) + H_5^{-1} \sum_{\alpha=2}^5 dy_\alpha^2 + \sum_{i=1}^5 dx_i^2 \right], \\
    *F_4 &= d(1/H_5) \wedge du \wedge dv \wedge dy_2 \wedge dy_3 \wedge dy_4 \wedge dy_5 + \frac{1}{\sqrt{2}} dB_5 \wedge du \wedge dy_2 \wedge dy_3 \wedge dy_4 \wedge dy_5,
\end{aligned}
\]
where $H_5$ is a harmonic function on $\mathbb{E}^5$. In both cases, $A_i$ is also a vector harmonic function, while $f$ is given by the Poisson equation \[2.51\] with $\beta = 1/2$ because of Eq. \[2.51\] (see the exact solution in §IV).

These two branes (M2 and M5) can intersect if and only if
\[ M2 \cap M2 \rightarrow q_{22} = 0, \quad M2 \cap M5 \rightarrow q_{25} = 1, \quad M5 \cap M5 \rightarrow q_{55} = 3. \] (3.4)

The crossing rule leads that there exists a four-dimensional (4D) “black” object with four independent branes (or three M5 branes and one wave), or a five-dimensional (5D) “black” object with three independent M2 branes (or two branes and one wave) (see Table I). The 4D “black” object with $M2 \parallel M2 \parallel M5 \parallel M5$ branes and the 5D object with $M2 \parallel M2 \parallel M5$ have no traveling wave. While, the 4D “black” object with M5 branes have no traveling wave. We shall discuss the details of the 5D “black” object with $M2 \parallel M5 \parallel W$ branes in the next section.

<table>
<thead>
<tr>
<th>$d = 4$</th>
<th>$d = 4$</th>
<th>$d = 5$</th>
<th>$d = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M2</td>
<td>M2</td>
<td>M2</td>
<td>M2</td>
</tr>
<tr>
<td>M5</td>
<td>M5</td>
<td>M5</td>
<td>M5</td>
</tr>
<tr>
<td>M2</td>
<td>M2</td>
<td>M2</td>
<td>W</td>
</tr>
<tr>
<td>M5</td>
<td>M5</td>
<td>M5</td>
<td>M5</td>
</tr>
</tbody>
</table>

Table I: Some examples of intersecting branes for $d = 4$ and $5$. M2, M5 and W denote the location where the M2 brane, the M5 brane and a wave exist, respectively.

B. D1 and D5-brane solutions in type IIB superstring theory

In the case of $N = 2$, $D = 10$ type IIB supergravity theory, the action in the Einstein frame is given by
\[ S = \frac{1}{2\kappa^2} \int d^{10}X \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2 \cdot n_A} \varphi^{5-n} F_{nA}^2 \right]. \] (3.5)

The coupling constant $a_A$ in the previous action \[2.41\] is given by $a_A = (5 - n_A)/2$. The three-form field with an “electric” charge is related to the D1-brane, i.e., $q_A = n_A - 2 = 1$. Then we find $\Delta_A = (q_A + 1)(7 - q_A) + 4a_A^2 = 16$, which does not depend on the type of branes ($A$ or $n_A$). This gives the same value of $(D - 2)/\Delta_A = 1/2$ as that in the case of eleven-dimensional supergravity. Then we find solutions in type IIB supergravity similar to those in eleven-dimensional supergravity. In fact a black brane solution in this case is written as
\[ ds_{10}^2 = H_1^{1/4} \left[ 2H_1^{-1}du \left( dv + fdu + \frac{A}{\sqrt{2}} \right) + \sum_{i=1}^{8} dx_i^2 \right], \] (3.6)

where $H_1$ is a harmonic function on $\mathbb{E}^8$. The form field with a “magnetic” charge is related to the D5-brane because $* q_A = * n_A - 2 = D - n_A - 2 = 5$. This also gives $\Delta_A = 16$. The solutions is described by
\[ ds_{10}^2 = H_5^{1/4} \left[ 2du \left( dv + fdu + \frac{A}{\sqrt{2}} \right) + H_5 \sum_{i=1}^{4} dx_i^2 + \sum_{\alpha=2}^{5} dy_\alpha^2 \right], \] (3.7)

where $H_5$ is a harmonic function on $\mathbb{E}^4$. The solutions with two intersecting branes (D1 and D5) are also given just as the previous subsection.

C. The Compactification of a Black Brane

The critical dimension for M-theory (or a superstring ) is eleven (or ten). We then have to compactify extra dimensions to obtain an effective $d$-(four or five) dimensional world.
Rewriting the following part of the metric as

\[ 2du \left( dv + fdu + \frac{A}{\sqrt{2}} \right) = (1 + f) \left[ dy_1 - \frac{1}{(1 + f)} \left( fdt - \frac{A}{2} \right)^2 \right] - \frac{1}{(1 + f)} \left( dt + \frac{A}{2} \right)^2, \quad (3.8) \]

we obtain our metric in \( D \)-dimensions as

\[ ds_{D}^2 = \prod_A H_A^{\frac{2q_A+1}{\Delta A}} \left[ - \prod_A H_A^{-\frac{2q_A}{\Delta A}} \frac{1}{(1 + f)} \left( dt + \frac{A}{2} \right)^2 + \sum_{i=1}^{d-1} dx_i^2 \right] \]

\[ + \prod_A H_A^{-\frac{2q_A}{\Delta A} - \frac{2}{\Delta A}} (1 + f) \left[ dy_1 - \frac{1}{(1 + f)} \left( fdt - \frac{A}{2} \right)^2 \right]^2 + \sum_{\alpha=2}^{p} \prod A H_A^{-\frac{2q_A}{\Delta A}} dy_{\alpha}^2. \quad (3.9) \]

Introducing the conformal factors \( \Omega_1, \Omega_{\alpha} \) and \( \Omega \) by

\[ \Omega_1^2 = (1 + f) \prod_A H_A^{-\frac{2q_A}{\Delta A} - \frac{2}{\Delta A}}, \]

\[ \Omega_{\alpha}^2 = \prod_A H_A^{-\frac{2q_A}{\Delta A}} \quad (\alpha = 2, \cdots, p) \]

\[ \Omega^2 = \prod_{\alpha=1}^{p} \Omega_{\alpha}^2 = (1 + f) \prod_A H_A^{-\frac{2q_A}{\Delta A} \left[ D - q_A(d-2) \right]}, \quad (3.10) \]

we perform a conformal transformation of our metric as

\[ ds_{D}^2 = \Omega^{-\frac{1}{2d}} ds_{d}^2 + \Omega_1^2 \left[ dy_1 - \frac{1}{(1 + f)} \left( fdt - \frac{A}{2} \right)^2 \right]^2 + \sum_{\alpha=2}^{p} \Omega_{\alpha}^2 dy_{\alpha}^2. \quad (3.11) \]

With this conformal transformation, we obtain the Einstein gravity in \( d \)-dimensions, which metric is given by

\[ ds_{d}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\Xi^{d-3} \left( dt + \frac{A}{2} \right)^2 + \Xi^{-1} \sum_{i=1}^{d-1} dx_i^2, \]

\[ \Xi = (1 + f)^{-1/(d-2)} \prod_A H_A^{-\frac{2d-2}{(d-2)\omega_{d-2}}}, \quad (3.12) \]

where \( \mu, \nu, \cdots \) are coordinate indices for \( d \)-dimensional spacetime. If the compactified space is sufficiently small, we find the effective \( d \)-dimensional world with the metric (3.12).

If this spacetime is asymptotically flat, which we impose, it may describe a “black” object in \( d \)-dimensions. From the asymptotic form of the metric, we can define the ADM mass \( M_{\text{ADM}} \) as

\[ g_{00} \sim -1 + \frac{16\pi G_d}{(d-2)\omega_{d-2}} \frac{M_{\text{ADM}}}{r^{d-3}}, \quad (3.13) \]

where

\[ \omega_{d-2} = \frac{2\pi^{d-1}}{\Gamma\left(\frac{d-1}{2}\right)}, \quad \text{and} \quad r^2 = \sum_{i=1}^{d-1} x_i^2. \quad (3.14) \]

Assuming

\[ H_A \rightarrow 1 + \frac{Q_H(A)}{r^{d-3}}, \]

\[ f \rightarrow \frac{Q_0}{r^{d-3}}, \quad (3.15) \]
we obtain

\[ M_{\text{ADM}} = \left( d - 3 \right) \frac{\pi^{d-3}}{8G_d \Gamma \left( \frac{d-1}{2} \right)} \left[ Q_0 + \sum_A \frac{2(D - 2)\xi^{(A)}}{\Delta_A} \right] . \]  

(3.16)

For the case of eleven-dimensional M theory (and ten-dimensional type IIB string theory), we find

\[ \Xi = \left( 1 + f \right) \prod_A H_A \]  

\[ M_{\text{ADM}} = \left( d - 3 \right) \frac{\pi^{d-3}}{8G_d \Gamma \left( \frac{d-1}{2} \right)} \left[ Q_0 + \sum_{A'} \xi^{(A')} \right] , \]  

(3.17)

where \( A' \) denotes charged branes.

Once we find solutions described by the above set of equations, we have to study a spacetime structure. In particular, the horizon and the singularity of a spacetime are important geometrical objects. We then have to evaluate the curvature invariant of the metric \( XX \). We calculate the Kretschmann invariant, which is given by

\[ \mathcal{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \mathcal{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = \frac{1}{128} \Xi^{2d-1} \left[ 3F_{ij}^4 + 5(F_{ij}F_{kl}k)^2 \right] - \frac{1}{16} \Xi^d \left[ 4(\partial_i F_{kl})^2 + 6(d - 2)\partial^j X \partial_i (F_{kl}^2) - 4d\partial^j X F_{ik}F_{j}^k + 2(3d^2 - 18d + 22)(\partial^j X F_{ij})^2 + (4d - 9)(\partial X)^2 F_{ij}^2 \right] + \frac{1}{8} \Xi^2 \left[ 8(d - 2)(d - 3)(\partial_i \partial_j X)^2 + 8(\partial^2 X)^2 + 8(d - 2)^2(d - 3)\partial^2 X \partial^j X \partial_i \partial_j X \right] - 8(d - 2)(d - 3)\partial^2 X (\partial X)^2 + (d - 2)(d - 3)(2d^2 - 8d + 7)(\partial X)^4 \right] , \]  

(3.18)

where \( X \equiv \ln \Xi \).

For \( d = 5 \), we have

\[ \mathcal{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \mathcal{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = \frac{1}{128} \Xi^{2d-1} \left[ 3F_{ij}^4 + 5(F_{ij}F_{kl}k)^2 \right] - \frac{1}{16} \Xi^d \left[ 4(\partial_i F_{kl})^2 + 18(\partial^i X \partial_i (F_{kl}^2) \right] - 20d\partial^j X F_{ik}F_{j}^k + 14(\partial^j X F_{ik})^2 + 11(\partial X)^2 F_{ij}^2 \right] + \frac{1}{4} \Xi^2 \left[ 24(\partial_i \partial_j X)^2 + 4(\partial^2 X)^2 + 72(\partial^j X \partial^i X \partial_i \partial_j X - 24(\partial X)^2 \partial^2 X + 51(\partial X)^4 \right] . \]  

(3.19)

In what follows, we present the exact solutions for \( D = 11 \) and \( d = 5 \). For \( D = 10 \), the construction of solutions is almost the same as that of \( D = 11 \).

IV. “BLACK” BRANE SOLUTIONS WITH M2-M5 (D1-D5) BRANES : THE CASE OF \( d = 5 \)

We consider solutions in five-dimensions. There are two branes (M2 and M5). Then \( N_{A'} + N_{A''} = 2 \). In the ten-dimensional type IIB case, we find the exactly the same as what we show below, when we replace M2 and M5 with D1 and D5 (the indices \( A = 2, 5 \) with the indices \( A = 1, 5 \)).

The metric in five-dimensions is written by

\[ ds_5^2 = -\Xi^2 \left( dt + \frac{A}{2} \right)^2 + \Xi^{-1} ds_{S^4}^2 , \]  

(4.1)

where \( \Xi = [H_2 H_5 (1 + f)]^{-1/3} \). The unknown functions \( H_A (A = 2, 5) \), \( A_i \) and \( f \) satisfy the following equations:

\[ \partial^2 H_A = 0 \quad (A = 2, 5) \]  

(4.2)

\[ \partial^i F^{ij} = 0 \]  

(4.3)

\[ \partial^2 f = S = \frac{\beta}{8H_2 H_5 F^{ij} F_{ij}} , \]  

(4.4)
where

\[ F_{ij} = \partial_i A_j - \partial_j A_i \]

\[ \beta = 1 - \frac{1}{2} \left( N_A + \sum_{A'} \lambda_{A'}^2 \right), \]  

(4.5)

which value is explicitly given in Table II.

<table>
<thead>
<tr>
<th>M_2</th>
<th>M_5</th>
<th>type of source branes</th>
<th>( \beta )</th>
<th>( H_A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>C</td>
<td>C</td>
<td>0</td>
<td>( H_2, H_5 ) : harmonic functions</td>
</tr>
<tr>
<td>(2a)</td>
<td>C</td>
<td>( N' )</td>
<td>( \frac{1}{2} (1 - \lambda_2^2) )</td>
<td>( H_2 ) : a harmonic function, ( H_5 = 1 )</td>
</tr>
<tr>
<td>(2b)</td>
<td>( N' )</td>
<td>C</td>
<td>( \frac{1}{2} (1 - \lambda_2^2) )</td>
<td>( H_2 = 1, H_5 ) : a harmonic function</td>
</tr>
<tr>
<td>(2c)</td>
<td>( N' )</td>
<td>( N' )</td>
<td>( 1 - \frac{1}{2} (\lambda_2^2 + \lambda_5^2) )</td>
<td>( H_2 = H_5 = 1 )</td>
</tr>
</tbody>
</table>

TABLE II: The type of source branes and the value of \( \beta \). There are two branes (M2 and M5). \( C \) and \( N' \) denote a charged brane and a neutral brane with a current. \( \lambda_2 \) and \( \lambda_5 \) are arbitrary parameters, which correspond to current strength.

In order to find the exact solutions, we assume that the 4-dimensional x-space has two rotation symmetries which Killing vectors \( (\xi_{(\phi)} \) and \( \xi_{(\psi)} \) commute each other. In this case, Eq. (4.3) is reduced to two uncoupled equations for two scalar fields, \( A_{\phi} = A_{(\xi_{(\phi)})} \) and \( A_{\psi} = A_{(\xi_{(\psi)})} \), as

\[ \partial^2 A_{\phi} - \partial_i \ln (\xi_{(\phi)} \cdot \xi_{(\phi)}) \partial^i A_{\phi} = 0, \]  

(4.6)

\[ \partial^2 A_{\psi} - \partial_i \ln (\xi_{(\psi)} \cdot \xi_{(\psi)}) \partial^i A_{\psi} = 0. \]  

(4.7)

Here we have assumed that the other components of \( A_i \) vanish.

We now have the Laplace equations or similar equations (the Poisson equation or Eqs. (4.6) and (4.7)) for several scalar functions \( (H_A, A_{\phi}, A_{\psi}, \text{and } f) \). Each equation is linear and uncoupled except for the equation for \( f \) with \( \beta \neq 0 \). Hence it is very easy to find general solutions because the Laplace-Beltrami operator is defined on the flat Euclidian space. Once we obtain a complete set of solutions in an appropriate curvilinear coordinate system, we can construct any solutions by superposing them.

Giving an explicit form of a solution, we obtain the properties of a “black” object. For example, assuming the asymptotic behaviors for \( H_A \) and \( f \) as

\[ H_A \to 1 + \frac{Q_H^{(A)}}{r^2}, \]

\[ f \to \frac{Q_0}{r^2}, \]  

(4.8)

as

\[ r \equiv \left( \sum_{i=1}^{4} x_i^2 \right)^{1/2} \to \infty, \]  

(4.9)

we find

\[ M_{ADM} = \frac{\pi}{4G_5} \left( Q_0 + Q_H^{(2)} + Q_H^{(5)} \right). \]  

(4.10)

The entropy of a black hole, if it exists, is defined by

\[ S = \frac{A_h}{4G_5}, \]  

(4.11)

where \( A_h \) is the area of horizon.

In what follows, adopting the hyperspherical coordinates as a curvilinear coordinate system, we show explicitly how to construct the exact solutions.
A. hyperspherical coordinates

We adopt the hyperspherical coordinates:

\[
x_1 + ix_2 = r \cos \theta e^{i \phi}, \quad x_3 + ix_4 = -r \sin \theta e^{i \psi},
\]

(4.12)

where \(0 \leq \phi, \psi < 2\pi\) and \(0 \leq \theta \leq \pi/2\). The line element of 4D flat space is

\[
d s_{\text{FL}}^2 = dr^2 + r^2 (d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2).
\]

(4.13)

The symmetric axis is described by \(\theta = 0\) and \(\pi/2\), and the infinity corresponds to \(r = \infty\).

From regularity conditions on the symmetric axis (\(\theta = 0\) where \(Q\)

where \(M \leq 0\) where 0

\[
\text{Eq. (4.2) in this coordinate system is}
\]

\[
\frac{1}{r} \frac{d}{dr} (r^3 \partial_r H_A) + \frac{1}{\sin \theta \cos \theta} \partial_\theta (\sin \theta \cos \theta \partial_\theta H_A) = 0.
\]

(4.14)

Setting \(H_A = h_A(r) j_A(\theta)\), we separate the variables and obtain two ordinary differential equations:

\[
\frac{1}{r} \frac{d}{dr} \left( r^3 \frac{d}{dr} h_A \right) - M h_A = 0,
\]

(4.15)

\[
\frac{1}{\sin \theta \cos \theta} \frac{d}{d\theta} \left( \sin \theta \cos \theta \frac{d}{d\theta} j_A \right) + M j_A = 0,
\]

(4.16)

where \(M\) is a separation constant. Eq. (4.16) with \(\mu = \cos 2\theta\) is just the Legendre equation as

\[
\frac{d}{d\mu} \left( 1 - \mu^2 \right) \frac{d j_A}{d\mu} + \frac{M}{4} j_A = 0.
\]

(4.17)

From regularity conditions on the symmetric axis (\(\theta = 0, \pi/2\)), we obtain \(j_A = P_\ell(\cos 2\theta)\) by setting \(M = 4\ell(\ell + 1)\) (\(\ell = 0, 1, 2, \cdots\)). Eq. (4.16) is easily solved as \(h_A = r^{2\ell} \) or \(r^{-2(\ell+1)}\). The general solution for \(H_A\) is then

\[
H_A = \sum_{\ell=0}^\infty \left[ g_\ell^{(A)} r^{2\ell} + h_\ell^{(A)} r^{-2(\ell+1)} \right] P_\ell(\cos 2\theta),
\]

(4.18)

where \(g_\ell^{(A)}\) and \(h_\ell^{(A)}\) are arbitrary constants.

From the asymptotically flatness condition, the solution is given by

\[
H_A = 1 + \sum_{\ell=0}^\infty h_\ell^{(A)} r^{-2(\ell+1)} P_\ell(\cos 2\theta).
\]

(4.19)

The spherically symmetric solution (\(\ell = 0\)) is given by

\[
H_A = 1 + \frac{Q_H^{(A)}}{r^2},
\]

(4.20)

where \(Q_H^{(A)}\) is a constant, which corresponds to a conserved charge.

Next, we discuss Eqs. (4.6) and (4.7), which are written as

\[
r \partial_r \left( r \partial_r A_\phi \right) + \cot \theta \partial_\theta \left( \tan \theta \partial_\theta A_\phi \right) = 0,
\]

(4.21)

\[
r \partial_r \left( r \partial_r A_\psi \right) + \tan \theta \partial_\theta \left( \cot \theta \partial_\theta A_\psi \right) = 0.
\]

(4.22)

Setting \(A_\phi = a_\phi(r) b_\phi(\theta)\) and \(A_\psi = a_\psi(r) b_\psi(\theta)\), we have the following ordinary differential equations:

\[
r \frac{d}{dr} \left( r \frac{d}{dr} a_\phi \right) - K a_\phi = 0
\]

(4.23)

\[
\frac{d^2 b_\phi}{d\mu^2} - \frac{1}{1 - \mu^2} \frac{d b_\phi}{d\mu} + \frac{K}{4(1 - \mu^2)} b_\phi = 0,
\]

(4.24)

\[
r \frac{d}{dr} \left( r \frac{d}{dr} a_\psi \right) - L a_\psi = 0
\]

(4.25)

\[
\frac{d^2 b_\psi}{d\mu^2} + \frac{1}{1 - \mu^2} \frac{d b_\psi}{d\mu} + \frac{L}{4(1 - \mu^2)} b_\psi = 0,
\]

(4.26)
where $\mu = \cos 2\theta$, and $K$ and $L$ are separation constants. The solutions of Eqs. \ref{4.21} and \ref{1.26} are described by Gauss’s hypergeometric functions as $b_\phi(\mu) = F(-\sqrt{K}/2, \sqrt{K}/2, 1, (1 - \mu)/2)$ and $b_\psi(\mu) = F(-\sqrt{L}/2, \sqrt{L}/2, 1, (1 + \mu)/2)$. The Gauss’s hyper geometrical function $F(\alpha, \beta, \gamma, z)$ is defined by

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma) \Gamma(\beta + n) \Gamma(\alpha + n)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma + n) n!} \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$ \hspace{1cm} (4.27)

From regularity conditions, we have to impose that $F = \frac{J}{{\sqrt{K}/2, \sqrt{K}/2, 1, (1 - \mu)/2}}$, and $a_\psi(\mu) = F(-n, n, 1, \cos^2 \theta)$. The explicit forms of this hypergeometric function with $m, n = 1, 2$ are given by

$$F(-1, 1, 1, z) = 1 - z$$
$$F(-2, 2, 1, z) = 1 - 4z + 3z^2.$$ \hspace{1cm} (4.28)

The equations for $a_\phi$ and $a_\psi$ are easily solved, i.e., $a_\phi = r^{2m}, r^{-2m}$ and $a_\psi = r^{2n}, r^{-2n}$. We then obtain a general solution for $A_i$ as

$$A_\phi = \sum_{m=1}^{\infty} \left[ a_m(\phi) r^{2m} + b_m(\phi) r^{-2m} \right] F(-m, m, 1, \sin^2 \theta)$$
$$A_\psi = \sum_{n=1}^{\infty} \left[ a_n(\psi) r^{2n} + b_n(\psi) r^{-2n} \right] F(-n, n, 1, \cos^2 \theta),$$ \hspace{1cm} (4.29)\hspace{1cm} (4.30)

where $a_m(\phi), b_m(\phi), a_n(\psi)$ and $b_n(\psi)$ are arbitrary constants.

Assuming asymptotically flatness, the solution for $A_i$ is given by

$$A_\phi = \sum_{m=1}^{\infty} \frac{b_m(\phi)}{r^{2m}} F(-m, m, 1, \sin^2 \theta),$$
$$A_\psi = \sum_{n=1}^{\infty} \frac{b_n(\psi)}{r^{2n}} F(-n, n, 1, \cos^2 \theta).$$ \hspace{1cm} (4.31)\hspace{1cm} (4.32)

If we take the first two terms in the general solution, we obtain a simple solution as

$$A_\phi = \frac{\cos^2 \theta}{r^2} \left[ J_1^{(\phi)} + \frac{J_2^{(\phi)}}{r^2} (1 - 3 \sin^2 \theta) \right]$$
$$A_\psi = \frac{\sin^2 \theta}{r^2} \left[ J_1^{(\psi)} + \frac{J_2^{(\psi)}}{r^2} (1 - 3 \cos^2 \theta) \right],$$ \hspace{1cm} (4.33)\hspace{1cm} (4.34)

where $J_1^{(\phi)}, J_1^{(\psi)}, J_2^{(\phi)}$ and $J_2^{(\psi)}$ are constants. The first two constants describe angular momenta of a “black” object. As we show in Appendix A, if $F_{ij}$ is self-dual, the spacetime is supersymmetric. This condition implies $J_1^{(\phi)} = -J_1^{(\psi)}$ and $J_2^{(\phi)} = J_2^{(\psi)}$.

Finally we discuss Eq. \ref{4.14}:

$$\frac{1}{r} \partial_r (r^3 \partial_r f) + \frac{1}{\sin \theta \cos \theta} \partial_\theta (\sin \theta \cos \theta \partial_\theta f) = S(r, \theta)$$
$$\equiv \frac{\beta}{4H_2 H_5} \left[ \frac{1}{\cos^2 \theta} \left( (\partial_r A_\phi)^2 + \frac{1}{r^2} (\partial_\theta A_\phi)^2 \right) + \frac{1}{\sin^2 \theta} \left( (\partial_r A_\psi)^2 + \frac{1}{r^2} (\partial_\theta A_\psi)^2 \right) \right].$$ \hspace{1cm} (4.35)

**B. BMPV type solutions: $\beta = 0$**

If $\beta = 0$, i.e., Case (1) (two charged brane) or Case (2a-2c) (neutral branes) with appropriately chosen current strength $\lambda_{A^\mu}$, we find the Laplace equation for $f$, which gives us a simple solution:

$$f = \sum_{\ell=0}^{\infty} Q_\ell r^{-2(\ell+1)} P_\ell (\cos 2\theta),$$ \hspace{1cm} (4.36)
where \( Q_r \)'s are constants.

In this case, the solution with the lowest multipole moment is given by
\[
H_A = 1 + \frac{Q_H^{(A)}}{r^2}, \quad (A = 2, 5),
\]
\[
f = \frac{Q_0}{r^2},
\]
\[
A_\phi = \frac{J_\phi \cos^2 \theta}{r^2},
\]
\[
A_\psi = \frac{J_\psi \sin^2 \theta}{r^2}.
\]

The mass and the entropy of this spacetime are
\[
M_{\text{ADM}} = \frac{\pi}{4G_5} (Q_0 + Q_H^{(2)} + Q_H^{(5)}), \quad (4.38)
\]
\[
S = \frac{A_h}{4G_5} = \frac{\pi^2}{3G_5} \frac{\Lambda_+^2 + \Lambda_+ \Lambda_- + \Lambda_-^2}{\Lambda_+^{3/2} + \Lambda_-^{3/2}}, \quad (4.39)
\]

where
\[
\Lambda_+ = Q_0 Q_H^{(2)} Q_H^{(5)} - \frac{J^2}{8} + \frac{\Delta J^2}{16}, \quad (4.40)
\]
\[
\Lambda_- = Q_0 Q_H^{(2)} Q_H^{(5)} - \frac{J^2}{8} - \frac{\Delta J^2}{16}. \quad (4.41)
\]

\( J^2 \) and \( \Delta J^2 \) are defined by \( J^2 \equiv (J_\phi^2 + J_\psi^2)/2 \) and \( \Delta J^2 \equiv J_\phi^2 - J_\psi^2 \), respectively.

Fixing \( J^2 \), if we maximize entropy \( S \), we find the maximum entropy with
\[
S = S_{\text{max}} = \frac{\pi^2}{2G_5} \sqrt{Q_0 Q_H^{(2)} Q_H^{(5)}} - \frac{J^2}{8}, \quad (4.42)
\]

if \( \Delta J^2 = 0 \), i.e., \( J_\phi^2 = J_\psi^2 = J^2 \). Note that supersymmetry implies \( J_\phi = -J_\psi = J \), which corresponds to the BMPV solution \( ^{32,33} \). If \( J_\phi \neq -J_\psi \), the above solution describes a regular rotating non-BPS black hole spacetime in five dimensions.

**C. Brinkmann wave type solutions: \( \beta \neq 0 \)**

When \( \beta \neq 0 \), since the source term is quadratic with respect to \( A_i \), it is not so easy to find a general solution. However, once we know the explicit form of the source term, expanding \( f(r, \theta) \) and the source term \( S(r, \theta) \) by the Legendre functions as
\[
f(r, \theta) = \sum_{\ell=0}^{\infty} f_\ell(r) P_\ell(\cos 2\theta),
\]
\[
S(r, \theta) = \sum_{\ell=0}^{\infty} S_\ell(r) P_\ell(\cos 2\theta), \quad (4.43)
\]

we find the ordinary differential equation for each moment \( \ell \) as
\[
\frac{1}{r} \frac{d}{dr} \left( r^3 \frac{df_\ell(r)}{dr} \right) - 4\ell(\ell + 1)f_\ell(r) = S_\ell(r). \quad (4.44)
\]

If we can integrate this equation, we find an analytic solution.

Here we give one simple example, i.e., \( H_2 = H_5 = 1 \) [Case (2c) in Table II] with Eqs. (4.33) and (4.34). We find the solutions as
\[
f = f_0(r) + f_1(r) P_1(\cos 2\theta) + f_2(r) P_2(\cos 2\theta), \quad (4.45)
\]
with

\[ f_0 = \frac{Q_0}{r^2} + \beta \left( \frac{J_1^2}{12r^6} + \frac{J_2^2}{20r^{10}} \right) \]

\[ f_1 = \frac{Q_1}{r^4} + \beta \frac{J_1^{(\phi \psi)}}{40r^8} \]

\[ f_2 = \frac{Q_2}{r^6} + \beta \frac{J_2^2}{14r^{10}}, \]

(4.46)

where \( Q_0, Q_1, Q_2 \) are integration constants and \( 2J_1^2 = (J_1^{(\phi)})^2 + (J_1^{(\psi)})^2, 2J_2^2 = (J_2^{(\phi)})^2 + (J_2^{(\psi)})^2, \) and \( \Delta J_{12}^{(\phi \psi)} = J_1^{(\phi)} J_2^{(\phi)} - J_1^{(\psi)} J_2^{(\psi)}. \)

If we set \( J_1^{(\phi)} = -J_1^{(\psi)} = J \) and \( J_2^{(\phi)} = J_2^{(\psi)} = 0, \) we find

\[ H_2 = H_5 = 1 \]

\[ f = \frac{Q_0}{r^2} + \beta \frac{J_2^2}{12r^6} \]

\[ A_\phi = \frac{J}{r^2} \cos^2 \theta, \]

\[ A_\psi = -\frac{J}{r^2} \sin^2 \theta. \]

(4.47)

We then recover the Brinkmann solution by setting \( \beta = 1 \) (i.e., \( B^2 = 0 \)) [43]. If \( \beta < 0, \) \( 1 + f \) vanishes at \( r = r_S(>0), \) which is a singularity. For the case of \( \beta > 0, \) this solution is similar to the Brinkmann wave.

We can extend the above solution to Case (2a) in Table II: \( H_2 \neq 1 \) and \( H_5 = 1 \) (or (2b): \( H_2 = 1 \) and \( H_5 \neq 1)). We find the following new solution. Supposing that \( H_2 \) depends only on \( r \) as Eq. (4.20) and the lowest moment for \( A_i, \)

we can obtain the exact solution:

\[ H_2 = 1 + \frac{Q_H^{(2)}}{r^2} \]

(4.48)

\[ f = \frac{Q_0}{r^2} + \beta \frac{J_2^2}{12r^6} \left[ H_2^2 - 1 - 2H_2 \ln H_2 \right] \]

(4.49)

\[ A_\phi = \frac{J_1^{(\phi)}}{r^2} \cos^2 \theta \]

(4.50)

\[ A_\psi = \frac{J_1^{(\psi)}}{r^2} \sin^2 \theta, \]

(4.51)

where \( 2J^2 = \left[ (J_1^{(\phi)})^2 + (J_1^{(\psi)})^2 \right]. \) The asymptotic behavior of this solution is

\[ f \rightarrow \frac{Q_0}{r^2} + \frac{\beta J_2^2}{12r^6} \quad (\text{as } r \rightarrow \infty) \]

\[ \rightarrow \frac{\beta J_2^2}{4Q_H^{(2)} r^4} \quad (\text{as } r \rightarrow 0). \]

(4.52)

The ADM mass is given by

\[ M_{ADM} = \frac{\pi}{4G_5} (Q_0 + Q_H^{(2)}). \]

(4.53)

Although \( r = 0 \) is not a singularity, it is not a horizon because it is a timelike hypersurface. In fact, setting \( J_1^{(\phi)} = -J_1^{(\psi)} = J \) (a supersymmetric spacetime), we find the surface area of \( r=\text{constant} \) as

\[ A(r) = 4\pi r^3 \int d\theta \cos \theta \sin \theta \left[ H_2(1 + f) - \frac{J_2^2}{8r^6} \right]^{1/2} \]

\[ \sim \pi \int d\theta \cos \theta \sin \theta |J| \left( \beta - \frac{1}{2} \right)^{1/2} \quad (\text{as } r \rightarrow 0). \]

(4.54)
This value becomes imaginary because $\beta < 1/2$. When $\beta = 1/2$, i.e., $B^A_i = 0$, the surface area (the entropy) vanishes.

This solution is also similar to the Brinkmann wave solution if $\beta > 0$. For $\beta = 0$, we have already discussed in the previous subsection. In the case of $\beta < 0$, $(1 + f)$ vanishes at finite radius $r = r_S(> 0)$, and then there exists a naked singularity.

V. CONCLUDING REMARKS

In this paper, we have studied a stationary “black” brane solution in M/superstring theory. Assuming a BPS type relation between the first-order derivatives of metric functions, we have shown how to construct a stationary “black” brane solution with a traveling wave. We consider two types of intersecting brane: (1) charged branes and (2) neutral branes with a current. The solutions are given by harmonic functions $H_A$ and $A_i$ plus a wave metric $f$ which satisfies the Poisson equation for $\beta \neq 0$ (Cases (2a)-(2c) in Table II) or the Laplace equation for $\beta = 0$ (Case (1) and Cases (2a)-(2c) with specific values for $\lambda_A$ in Table II). Since those differential equations are linear and independent except for the Poisson equation for $f$, we can easily construct general solutions by superposition of harmonic functions.

Using the hyperspherical coordinate system, we present exact solutions in eleven-dimensional M theory for the case with $M2 \perp M5$ intersecting branes and a traveling wave. Compactifying these solutions into five dimensions, we show that these solutions include the BMPV black hole and the Brinkmann wave solution. We have also found new solutions which are similar to the Brinkmann wave.

We have proved that the solutions preserve the 1/8 supersymmetry if $F_{ij}$ is self-dual. All solutions found in the hyperspherical coordinates preserve the 1/8 supersymmetry if the angular momenta satisfy some relation (e.g., $J_\phi = -J_\psi$).

We also discuss non-spherical “black” brane solutions (e.g., a ring topology and an elliptical shape solution) by use of hyperelliptical and hyperpolorical coordinates in Appendix B. Unfortunately, we could not find regular new solution, but in stead, solutions with a naked singularity. In particular, even if we use hyperpolorical coordinates, we could not obtain a supersymmetric black ring. We may have to extend our approach to find such a black ring solution. We have two possibilities: One is the extension of our metric form, and the other is different configuration of intersecting branes. Bena and Kraus [36] describe a black ring solution in M-theory. If we rewrite the solution in our null coordinate system, we find that we have to include another “gravi-electromagnetic” field in addition to $A_j$. While Elvang et al. [32] use three charged M2 branes and three m5 monopoles to describe a supersymmetric black ring. Thus we may have to consider a different type of brane configuration. Using the U-duality [44], we may construct a supersymmetric black ring [32] and other solutions from our brane solutions.

The charges of branes of the BMPV black hole correspond to the numbers of D-brane tension. While SO(4) rotational symmetries, which describe angular momenta of the black hole, corresponds to endmorphisms in the graded algebra that rotate the fermionic generators $G_m^{\alpha} [33]$. By this correspondence (AdS/CFT correspondence), we can discuss the properties of our solutions in the SCFT side.

Although we assume the BPS type relations for the metric, we have to solve the elliptic type differential equations if we want to find most general solutions, especially non-BPS spacetimes. For this purpose, we need a completely different approach such as a soliton technique to generate new solutions [45, 46, 47].

We have found that the BPS and non-BPS rotating asymptotically flat stringy black holes, from which we may learn more about connections between microscopic and macroscopic states of gravitating objects. In our framework, we consider a toroidally compactified string theory, but one may embed the BMPV type geometry in M-theory compactified on generic Calabi-Yau spaces, which would be more interesting.

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APPENDIX A: SUPERSYMMETRY IN ELEVEN-DIMENSIONAL “BLACK” BRANES

Here we discuss supersymmetry in the solution obtained in this paper. The invariance for supersymmetry transformation of a gravitino gives a criterion for existence of unbroken supersymmetry. This condition is given by the Killing equation for the Killing spinor \( \epsilon \) [18], i.e.,

\[
\delta \psi_{\hat{A}} = \left[ \epsilon_{\hat{A}} \partial_{\mu} + \frac{1}{4} w^{BC} \hat{A} \gamma_{BC} + \frac{1}{288} (\gamma_{\hat{A}} \hat{B} \hat{C} \hat{D} \hat{E} - 8 \delta_{\hat{A}} B \gamma_{\hat{C} \hat{D} \hat{E}}) F_{\hat{B} \hat{C} \hat{D} \hat{E}} \right] \epsilon = 0 ,
\]

where \( \gamma_{\hat{A}} \)’s are the antisymmetrized products of eleven-dimensional gamma matrices with unit strength on vielbein \( e^{\hat{A}}_{\mu} \), and a spin connection is given by

\[
w^{BC} \hat{A} = e^\mu_{\hat{A}} \left( e^{B \mu} \partial_{\mu} e^{C} \nu - e^{C \mu} \partial_{\mu} e^{B} \nu - e^{B \mu} e^{C} \sigma e_{D \mu} \partial_{\mu} e^{D} \sigma \right) .
\]

Now we consider the M2\( \perp \)M5 “black” brane solution related to the five-dimensional black hole, which we have discussed in section IV. The metric functions \( \xi, \eta, \zeta_0 \) for the space with M2\( \perp \)M5 ‘intersecting branes are given as

\[
\begin{align*}
\xi &= - \left( \frac{1}{3} \ln H_2 + \frac{1}{6} \ln H_5 \right) \\
\eta &= \frac{1}{6} \ln H_2 + \frac{1}{3} \ln H_5 \\
\zeta_2(\cdots 5) &= \frac{1}{6} (\ln H_2 - \ln H_5) \\
\zeta_6 &= \frac{1}{3} (\ln H_2 - \ln H_5).
\end{align*}
\]

The index 2(\( \cdots \)5) denotes that it is either 2, 3, 4, or 5. Hence the metric is given by

\[
ds^2 = H_2^{1/3} H_5^{2/3} \left[ 2(H_2 H_5)^{-1} du + f du + \frac{A}{\sqrt{2}} + \sum_{\alpha = 2}^{5} H_5^{-1} d\eta_{\alpha}^2 + H_2^{-1} d\eta_6^2 + \sum_{i = 1}^{4} dx_i^2 \right].
\]

The non-trivial components of field strength \( F_{\hat{A} \hat{B} \hat{C} \hat{D}} \) are given by

\[
\begin{align*}
F_{ij \bar{\alpha} \bar{\beta}} &= -H_2^{-1/6} H_5^{-1/3} \partial_i H_2 \\
F_{ij \bar{\alpha}} &= -\frac{1}{\sqrt{2}} H_2^{-2/3} H_5^{-5/6} (F_{ij}^{(2)} + * F_{ij}^{(5)}) \\
F_{kl \bar{m} \bar{n}} &= \epsilon^{klm} H_2^{-1/6} H_5^{-1/3} \partial_i H_5
\end{align*}
\]

where \( F_{ij}^{(A)} \) and those duals \( * F_{ij}^{(A)} \) are given by

\[
\begin{align*}
F_{ij}^{(A)} &\equiv -2 H_A \left( \partial_i B_{ij}^{(A)} - A_{[i} \partial_{j]} E_A \right) = -2 H_A \left( \partial_i B_{ij}^{(A)} + \frac{1}{H_A} A_{[i} \partial_{j]} H_A \right) \\
* F_{ij}^{(A)} &= \frac{1}{2} \epsilon_{ijkl} F_{kl}^{(A)} .
\end{align*}
\]

If we have two charged branes (Case (1) in §II: \( B_{ij}^{(A)} = -A_{ij}/H_A \)), each \( F_{ij}^{(A)} \) coincides with \( F_{ij} \). For neutral branes with currents (Case (2) in §II), we find \( F_{ij}^{(A)} = \lambda_A F_{ij} \).
Using Eq. (2.40) and the above explicit expression for $F_{ABCD}$, we obtain the Killing equations $A1$ as

$$
\delta \psi_\alpha = \frac{1}{6} H_2^{-1/6} H_5^{-1/3} \left[ \frac{\partial_i H_2}{H_2} \gamma^j (1 - \gamma^{i\hat{a}y_6}) + \frac{1}{2} \frac{\partial_i H_5}{H_5} \gamma^j (1 - \gamma^{i\hat{a}y_2\cdots y_5}) \right] \epsilon
$$

$$
- \frac{1}{4 \sqrt{2}} H_2^{-2/3} H_5^{-5/6} \left[ \frac{1}{2} F_{ij} \gamma^j \frac{1}{3} \left( F_{(ij)}^{(2)} + * F_{ij}^{(5)} \right) \gamma^{i\hat{a}y_6} - \frac{1}{6} \left( F_{ij}^{(2)} + * F_{ij}^{(5)} \right) \gamma^{i\hat{a}y_6} \right] \epsilon
$$

$$
- \frac{1}{2} H_2^{-1/6} H_5^{-1/3} \partial_i f \gamma^j \epsilon,
$$

(A10)

$$
\delta \psi_0 = \frac{1}{6} H_2^{-1/6} H_5^{-1/3} \left[ \frac{\partial_i H_2}{H_2} \gamma^{i\hat{a}} (1 - \gamma^{i\hat{a}y_6}) + \frac{1}{2} \frac{\partial_i H_5}{H_5} \gamma^{i\hat{a}} (1 - \gamma^{i\hat{a}y_2\cdots y_5}) \right] \epsilon,
$$

(A11)

$$
\delta \psi_i = H_2^{-1/6} H_5^{-1/3} \partial_i \gamma
$$

$$
+ \frac{1}{\sqrt{2}} H_2^{-2/3} H_5^{-5/6} \left[ \frac{1}{4} F_{ij} \gamma^{i\hat{a}} + \frac{1}{2} \left( F_{ij}^{(2)} + * F_{ij}^{(5)} \right) \gamma^{i\hat{a}y_6} - \frac{1}{24} \left( F_{ij}^{(2)} + * F_{ij}^{(5)} \right) \gamma^{i\hat{a}y_6} \right] \epsilon
$$

$$
+ \frac{1}{6} H_2^{-1/6} H_5^{-1/3} \left[ \frac{\partial_i H_2}{H_2} \gamma^{i\hat{a}} (1 - \gamma^{i\hat{a}y_6}) + \frac{1}{2} \frac{\partial_i H_5}{H_5} \gamma^{i\hat{a}} (1 - \gamma^{i\hat{a}y_2\cdots y_5}) \right] \epsilon
$$

$$
+ \frac{1}{6} H_2^{-1/6} H_5^{-1/3} \left[ \frac{\partial_i H_2}{H_2} \gamma^{i\hat{a}y_6} + \frac{1}{2} \frac{\partial_i H_5}{H_5} \gamma^{i\hat{a}y_2\cdots y_5} \right] \epsilon
$$

$$
= H_2^{-1/6} H_5^{-1/3} \left( \partial_i + \frac{1}{6} \frac{\partial_i H_2}{H_2} + \frac{1}{24} \frac{\partial_i H_5}{H_5} \right) \epsilon
$$

$$
+ \frac{1}{\sqrt{2}} H_2^{-2/3} H_5^{-5/6} \left[ \frac{1}{4} F_{ij} + \frac{1}{2} \left( F_{ij}^{(2)} + * F_{ij}^{(5)} \right) \right] \gamma^{i\hat{a}} \epsilon
$$

$$
+ \frac{1}{6} \left( F_{ij}^{(2)} + * F_{ij}^{(5)} \right) \gamma^{i\hat{a}} (1 - \gamma^{i\hat{a}y_6}) + \frac{1}{12} \left( F_{ij}^{(2)} + * F_{ij}^{(5)} \right) \gamma^{i\hat{a}} (1 - \gamma^{i\hat{a}y_2\cdots y_5}) \epsilon
$$

$$
+ \frac{1}{6} H_2^{-1/6} H_5^{-1/3} \left[ \frac{\partial_i H_2}{H_2} \gamma^{i\hat{a}y_6} + \frac{1}{2} \frac{\partial_i H_5}{H_5} \gamma^{i\hat{a}y_2\cdots y_5} \right] \epsilon
$$

$$
- \frac{1}{6} H_2^{-1/6} H_5^{-1/3} \left[ \frac{\partial_i H_2}{H_2} (1 - \gamma^{i\hat{a}y_6}) + \frac{1}{2} \frac{\partial_i H_5}{H_5} (1 - \gamma^{i\hat{a}y_2\cdots y_5}) \right] \epsilon,
$$

(A12)

$$
\delta \psi_{y_2(\cdots 5)} = - \frac{1}{12} H_2^{-1/6} H_5^{-1/3} \left[ \frac{\partial_i H_2}{H_2} \gamma^{i\hat{a}y_2(\cdots 5)} (1 - \gamma^{i\hat{a}y_6}) - \frac{1}{2} \frac{\partial_i H_5}{H_5} \gamma^{i\hat{a}y_2(\cdots 5)} (1 - \gamma^{i\hat{a}y_2\cdots y_5}) \right] \epsilon
$$

$$
+ \frac{1}{24 \sqrt{2}} H_2^{-2/3} H_5^{-5/6} \left( F_{ij}^{(2)} + * F_{ij}^{(5)} \right) \gamma^{i\hat{a}y_2(\cdots 5) y_6} \epsilon,
$$

(A13)

$$
\delta \psi_{y_6} = \frac{1}{6} H_2^{-1/6} H_5^{-1/3} \left[ \frac{\partial_i H_2}{H_2} \gamma^{i\hat{a}y_6} (1 - \gamma^{i\hat{a}y_6}) - \frac{1}{2} \frac{\partial_i H_5}{H_5} \gamma^{i\hat{a}y_6} (1 - \gamma^{i\hat{a}y_2\cdots y_5}) \right] \epsilon
$$

$$
- \frac{1}{12 \sqrt{2}} H_2^{-2/3} H_5^{-5/6} \left( F_{ij}^{(2)} + * F_{ij}^{(5)} \right) \gamma^{i\hat{a}y_6} \epsilon.
$$

(A14)
Most parts of the above equations vanish if we impose the following condition for the Killing spinor $\epsilon$:

$$ (1 - \gamma^{i_1_i_2_\ldots i_n})\epsilon = 0, \quad (1 - \gamma^{\tilde{i}_1_{\tilde{i}}_2_{\tilde{i}}_3_{\tilde{i}}_4})\epsilon = 0, \quad \gamma^{\tilde{i}}\epsilon = 0. $$

(A15)

These conditions can be rewritten as

$$ (1 + \gamma^{\tilde{i}_1_{\tilde{i}}_2_{\tilde{i}}_3_{\tilde{i}}_4})\epsilon = 0, \quad (1 + \gamma^{i_1_{i_2_{i_3_{i_4}}}})\epsilon = 0, \quad (1 + \gamma^{0_1_{0_2_{0_3}}})\epsilon = 0. $$

(A16)

However, two terms remain. One term is

$$ \left( \partial_i + \frac{\partial_i H_2}{6H_2} + \frac{\partial_i H_5}{12H_5} \right) \epsilon, $$

and the other term is

$$ \left\{ \frac{1}{2} \mathcal{F}_{ij} + \frac{1}{3} \left( * \mathcal{F}^{(2)}_{ij} + \mathcal{F}^{(5)}_{ij} \right) - \frac{1}{6} \left( \mathcal{F}^{(2)}_{ij} + * \mathcal{F}^{(5)}_{ij} \right) \right\} \gamma^{ij}\epsilon. $$

(A17)

(A18)

The former term vanishes if $\epsilon$ is described as

$$ \epsilon = H_2^{-1/6} H_5^{-1/12} \epsilon_0, $$

(A19)

where $\epsilon_0$ is a constant spinor. The latter term also vanishes if $\mathcal{F}^{(A)}_{ij} \propto \mathcal{F}_{ij}$ ($A = 2, 5$) and $\mathcal{F}_{ij}$ is self-dual ($* \mathcal{F}_{ij} = \mathcal{F}_{ij}$). In fact, this term is proportional to

$$ \mathcal{F}_{ij} \gamma^{ij}\epsilon, $$

(A20)

which vanishes for the self-dual field $\mathcal{F}_{ij}$ as shown below. From Eqs. (A16), we have

$$ \gamma^{0_1_{0_2_{0_3}}} = -\epsilon, \quad \gamma^{0_{0_2_{0_3}}} = \epsilon, \quad \gamma^{0_1_{0_2}} = \epsilon. $$

(A21)

We also assume that

$$ \gamma^{0_1 \ldots 0_{\tilde{i}_1_{\tilde{i}}_2_{\tilde{i}}_3_{\tilde{i}}_4}} = -\epsilon, $$

(A22)

which corresponds to the chiral state. Then, we find

$$ \gamma^{\tilde{i}_1_{\tilde{i}}_2_{\tilde{i}}_3_{\tilde{i}}_4} = \epsilon. $$

(A23)

This equation is rewritten as

$$ \gamma^{ij}\epsilon = -\frac{1}{2} \epsilon_{ijkl} \gamma^{kl}\epsilon. $$

(A24)

We then obtain

$$ \mathcal{F}_{ij} \gamma^{ij}\epsilon = -\frac{1}{2} \mathcal{F}_{ij} \epsilon_{ijkl} \gamma^{kl}\epsilon = -* \mathcal{F}_{kl} \gamma^{kl}\epsilon = -\mathcal{F}_{kl} \gamma^{kl}\epsilon. $$

(A25)

The last equality is found by the self-duality of $\mathcal{F}_{ij}$. This equation yields

$$ \mathcal{F}_{ij} \gamma^{ij}\epsilon = 0. $$

(A26)

In the case of the BMPV type solution discussed in §IV B, this self-dual condition gives the relation between $J_\phi$ and $J_\psi$, that is, $J_\phi = -J_\psi = J$.

The condition of $\mathcal{F}^{(A)}_{ij} \propto \mathcal{F}_{ij}$ leads either to Case (1): two charged branes, or Case (2): neutral branes with currents discussed in §II. We conclude that the solutions discussed in this paper preserve 1/8 supersymmetry if $\mathcal{F}_{ij}$ is self-dual.

We also expect that the Kalza-Klein compactification into five-dimensional spacetime does not break any supersymmetry, because all coordinates to be compactified are cyclic. Thus the five dimensional solution obtained here is a BPS state if $\mathcal{F}_{ij}$ is self-dual.
APPENDIX B: “BLACK” BRANE SOLUTIONS IN M-THEORY BY USE OF DIFFERENT COORDINATE SYSTEMS

We may adopt several different curvilinear coordinates for the following reason. If we have a complete set of solutions, any solution can be described by some linear combination of these solutions. Therefore, we may not need to introduce another coordinate system. However, if we use some coordinate system, we may be able to describe some interesting solution by a few multipole moments. In order to describe it in the different coordinate system, one may need an infinite series of eigenfunctions. For example, a black ring solution may need an infinite sum to describe it in the Cartesian coordinate system. But if we use an appropriate coordinate system, the expression could be much simpler. Therefore, in this Appendix, we discuss two other curvilinear coordinates; hyperelliptical and hyperpolorical coordinate systems.

We start with the metric form of the four-dimensional Euclidian space written by some orthonormal curvilinear coordinates, i.e.,

$$ds^2 = h_{\xi\xi} d\xi^2 + h_{\eta\eta} d\eta^2 + h_{\phi\phi} d\phi^2 + h_{\psi\psi} d\psi^2.$$  \hfill (B1)

We assume that there are two rotation symmetries, as discussed in the text (§IV). Hence $h_{ij}$ depends only on two coordinates: $\xi$ and $\eta$.

Eqs. (4.2), (4.6), (4.7) and (4.4) are explicitly written as

$$\partial_\xi \left( \frac{h_{\eta\eta} h_{\psi\psi}}{h_{\xi\xi}} \partial_\xi H_A \right) + \partial_\eta \left( \frac{h_{\xi\xi} h_{\psi\psi}}{h_{\eta\eta}} \partial_\eta H_A \right) = 0 \quad (A = 2, 5),$$  \hfill (B2)

$$\partial_\xi \left( \frac{h_{\eta\eta} h_{\psi\psi}}{h_{\xi\xi}} \partial_\xi A_\phi \right) + \partial_\eta \left( \frac{h_{\xi\xi} h_{\psi\psi}}{h_{\eta\eta}} \partial_\eta A_\phi \right) = 0,$$  \hfill (B3)

$$\partial_\xi \left( \frac{h_{\eta\eta} h_{\psi\psi}}{h_{\xi\xi}} \partial_\xi A_\psi \right) + \partial_\eta \left( \frac{h_{\xi\xi} h_{\psi\psi}}{h_{\eta\eta}} \partial_\eta A_\psi \right) = 0,$$  \hfill (B4)

$$\partial_\xi \left( \frac{h_{\eta\eta} h_{\phi\phi}}{h_{\xi\xi}} \partial_\xi f \right) + \partial_\eta \left( \frac{h_{\xi\xi} h_{\phi\phi}}{h_{\eta\eta}} \partial_\eta f \right) = \frac{\beta}{4H_2 H_5} \left[ \frac{h_{\eta\eta} h_{\psi\psi}}{h_{\xi\xi} h_{\phi\phi}} (\partial_\xi A_\phi)^2 + \frac{h_{\xi\xi} h_{\psi\psi}}{h_{\eta\eta} h_{\phi\phi}} (\partial_\eta A_\phi)^2 + \frac{h_{\eta\eta} h_{\phi\phi}}{h_{\xi\xi} h_{\psi\psi}} (\partial_\xi A_\psi)^2 + \frac{h_{\xi\xi} h_{\phi\phi}}{h_{\eta\eta} h_{\psi\psi}} (\partial_\eta A_\psi)^2 \right].$$  \hfill (B5)

The ADM mass is determined from the asymptotic behaviors of $H_A$ and $f$ as

$$M_{ADM} = \frac{\pi}{4G_5} \left( Q_0 + Q_H^{(2)} + Q_H^{(5)} \right),$$  \hfill (B6)

where

$$H_A \rightarrow 1 + \frac{Q_H^{(A)}}{r^2},$$  \hfill (B7)

$$f \rightarrow \frac{Q_0}{r^2},$$

as

$$r \equiv \left( \sum_{i=1}^4 x_i^2 \right)^{1/2} \rightarrow \infty.$$  \hfill (B8)

If the horizon is given by $\xi = \xi_h$, the area $A_h$ is

$$A_h = 4\pi^2 \int_{\xi=\xi_h} d\eta \left( h_{\eta\eta} h_{\phi\phi} h_{\psi\psi} \right)^{1/2} \left[ H_2 H_5 (1 + f) - \frac{1}{8} \left( \frac{A_\phi^2}{h_{\phi\phi}} + \frac{A_\psi^2}{h_{\psi\psi}} \right) \right]^{1/2},$$  \hfill (B9)

which gives the entropy of a “black” object.
1. Hyperelliptical Coordinates

First we adopt the hyperelliptical coordinates \((\xi, \eta, \phi, \psi)\), which are defined by the following transformation:

\[
x_1 + ix_2 = R \cosh \xi \cos \eta e^{i\phi}, \quad x_3 + ix_4 = R \sinh \xi \sin \eta e^{i\psi},
\]

where \(R\) is a constant, \(\xi \geq 0, 0 \leq \eta \leq \pi\), and \(0 \leq \phi, \psi \leq 2\pi\).

The line element is given by

\[
ds_{A}^2 = R^2 \left[ (\sinh^2 \xi + \sin^2 \eta)(d\xi^2 + d\eta^2) + \cosh^2 \xi \cos^2 \eta d\phi^2 + \sinh^2 \xi \sin^2 \eta d\psi^2 \right].
\]

Eq. (B2) is

\[
\frac{1}{\sinh \xi \cosh \xi} \partial_{\xi} (\sinh \xi \cosh \xi \partial_{\xi} H_A) + \frac{1}{\sin \eta \cos \eta} \partial_{\eta} (\sin \eta \cos \eta \partial_{\eta} H_A) = 0 \quad (A = 2, 5).
\]

Setting \(H_A = h_A(\xi) j_A(\eta)\), we find two ordinary differential equations:

\[
\begin{align*}
    \frac{1}{\sinh \xi \cosh \xi} \frac{d}{d\xi} \left( \sinh \xi \cosh \xi \frac{dh_A}{d\xi} \right) - M h_A &= 0, \\
    \frac{1}{\sin \eta \cos \eta} \frac{d}{d\eta} \left( \sin \eta \cos \eta \frac{dj_A}{d\eta} \right) + M j_A &= 0,
\end{align*}
\]

where \(M\) is a separation constant. Using new variables \(\rho = \cos 2\xi\) and \(\mu = \cos 2\eta\), these equations are rewritten by the Legendre equation as

\[
\begin{align*}
    \frac{d}{d\rho} \left( (\rho^2 - 1) \frac{dh_A}{d\rho} \right) - \frac{M}{4} h_A &= 0, \\
    \frac{d}{d\mu} \left( (1 - \mu^2) \frac{dj_A}{d\mu} \right) + \frac{M}{4} j_A &= 0.
\end{align*}
\]

The regularity condition on the symmetric axis gives \(M = 4\ell(\ell + 1)\) \((\ell = 0, 1, 2, \cdots)\). A general solution for \(H_A\) is then

\[
H_A = \sum_{\ell=0}^{\infty} \left[ g^{(A)}_\ell P_\ell(\cosh 2\xi) + h^{(A)}_\ell Q_\ell(\cosh 2\xi) \right] P_\ell(\cosh 2\eta),
\]

where \(Q_\ell(z)\) is the second kind Legendre function, and \(g^{(A)}_\ell\) and \(h^{(A)}_\ell\) are arbitrary constants.

From the condition of asymptotically flatness \((H_A \to 1\) as \(r \to \infty\), \(H_A\) is given by

\[
H_A = 1 + \sum_{\ell=0}^{\infty} h^{(A)}_\ell Q_\ell(\cosh 2\xi) P_\ell(\cosh 2\eta) \quad (A = 2, 5),
\]

because \(Q_\ell(z)\) vanishes at \(z = \infty\). The explicit form for \(\ell = 0, 1, 2\) is as follows:

\[
\begin{align*}
    Q_0(z) &= -\frac{1}{2} \ln \left( \frac{z + 1}{z - 1} \right), \\
    Q_1(z) &= \frac{z}{2} \ln \left( \frac{z + 1}{z - 1} \right) - 1, \\
    Q_2(z) &= \frac{1}{4} \left( 3z^2 - 1 \right) \ln \left( \frac{z + 1}{z - 1} \right) - \frac{3z}{2}.
\end{align*}
\]

The solution of \(H_A\) with the lowest moment \((\ell = 0)\) is

\[
H_A = 1 + h^{(A)}_0 \ln (\tanh \xi).
\]

If we define a charge \(Q^{(A)}_H\) by the asymptotic behavior of \(H_A\) as \(H_A \to 1 + Q^{(A)}_H/r^2\), we find that

\[
h^{(A)}_0 = -\frac{2Q^{(A)}_H}{R^2},
\]

where \(R\) is the Schwarzschild radius.
because $\ln(\tanh \xi) \sim -2/e^{2 \xi} \approx -R^2/(2r^2)$.

Now we solve Eqs. (B23) and (B24), which are

$$
\coth \xi \partial_\xi (\tanh \xi \partial_\xi A_\phi) + \cot \eta \partial_\eta (\tan \eta \partial_\eta A_\psi) = 0, 
$$

(B22)

$$
\tanh \xi \partial_\xi (\coth \xi \partial_\xi A_\psi) + \tan \eta \partial_\eta (\cot \eta \partial_\eta A_\psi) = 0.
$$

(B23)

Setting $A_\phi = a_\phi(\xi) b_\phi(\eta)$ and $A_\psi = a_\psi(\xi) b_\psi(\eta)$, we obtain the following ordinary differential equations:

$$
\frac{d^2 a_\phi}{d\rho^2} + \frac{1}{\rho - 1} \frac{da_\phi}{d\rho} - \frac{K}{\rho^2 - 1} a_\phi = 0, 
$$

(B24)

$$
\frac{d^2 b_\phi}{d\mu^2} = \frac{1}{1 - \mu} \frac{db_\phi}{d\mu} + \frac{K}{1 - \mu^2} b_\phi = 0, 
$$

(B25)

$$
\frac{d^2 a_\psi}{d\rho^2} + \frac{1}{\rho + 1} \frac{da_\psi}{d\rho} - \frac{L}{\rho^2 - 1} a_\psi = 0, 
$$

(B26)

$$
\frac{d^2 b_\psi}{d\mu^2} = \frac{1}{1 + \mu} \frac{db_\psi}{d\mu} + \frac{L}{1 - \mu^2} b_\psi = 0, 
$$

(B27)

where $\rho = \cosh 2\xi$ and $\mu = \cos 2\eta$, and $K$ and $L$ are separation constants.

Eqs. (B25) and (B27) are the same as Eqs (B24) and (B26). Then we obtain the angular solutions by hypergeometric functions as $b_\phi = F(-m, m, 1, (1 - \mu)/2)$ and $b_\psi = F(-n, n, 1, (1 + \mu)/2)$. We have set $K = m^2$ and $L = n^2$ $(m, n = 1, 2, \ldots)$ from regularity conditions on the symmetric axis. The solutions for Eqs. (B24) and (B26) are also given by the hypergeometric functions. Imposing the asymptotically flatness condition at infinity ($\xi \to \infty$), we find the following solutions:

$$
A_\phi = \sum_{m=1}^{\infty} b_m^{(\phi)} \frac{1}{\sinh 2m \xi} F\left(m, m, 1 + 2m, -1 / \sinh^2 \xi\right) F(-m, m, 1, \sin^2 \eta),
$$

(B28)

$$
A_\psi = \sum_{n=1}^{\infty} b_n^{(\psi)} \frac{1}{\cosh 2n \xi} F\left(n, n, 1 + 2n, 1 / \cosh^2 \xi\right) F(-n, n, 1, \cos^2 \eta).
$$

(B29)

Here we show some hypergeometric functions, which we use later, explicitly:

$$
F(1, 1, 3, z) = \frac{2}{z^2} \left[z + (1 - z) \ln(1 - z)\right]
$$

(B30)

$$
F(1, 2, 3, z) = -\frac{2}{z^2} \left[z + \ln(1 - z)\right].
$$

If $\beta = 0$ (two charged branes or neutral branes with appropriately chosen current strength $\lambda_{A^\mu}$), the solution of $f$ is given by a harmonic function, which is

$$
f = \sum_{t=0}^{\infty} c_t \zeta(t \cosh 2\xi) P_t(\cos 2\eta),
$$

(B31)

where

$$
c_t = -\frac{2 \zeta}{R^2}.
$$

(B32)

The lowest moment solution $(t = 0, m = n = 1)$ in this case is

$$
H_A = 1 - \frac{2 Q_0}{R^2} \ln(\tanh \xi) \quad (A = 2, 5)
$$

(B33)

$$
f = -\frac{2 Q_0}{R^2} \ln(\tanh \xi)
$$

(B34)

$$
A_\phi = -2 J_1^{(\phi)} \left[1 + 2 \cosh^2 \xi \ln(\tanh \xi)\right] \cos^2 \eta
$$

(B35)

$$
A_\psi = 2 J_1^{(\psi)} \left[1 + 2 \sinh^2 \xi \ln(\tanh \xi)\right] \sin^2 \eta,
$$

(B36)
where $Q_H^{(2)}, Q_H^{(5)}, Q_0$ are charges and $J_1^{(\phi)}, J_1^{(\psi)}$ are angular momentum. We can show that this spacetime is supersymmetric if $J_1^{(\phi)} = -J_1^{(\psi)}$, which is the same condition as that for the BMPV black hole solution.

The ADM mass of this object is

$$M_{\text{ADM}} = \frac{\pi}{4G_5} \left( Q_0 + Q_H^{(2)} + Q_H^{(5)} \right).$$  \hfill (B37)

$H_A$ and $(1 + f)$ diverge at $\xi = 0$, which may correspond to the horizon. Calculating the Kretschmann curvature invariant, we show that it is a naked singularity. Therefore, this solution does not provide a black hole spacetime, but instead, describes the spacetime of a rotating singular disk.

In the case of $\beta \neq 0$, in order to obtain the solution for $f$, we have to expand $f$ and the source term $\tilde{S}$ by the Legendre function $P_1(\cos 2\eta)$, just as in the case of the previous hyperspherical coordinates (Eqs. \ref{4.49} and \ref{4.50}). $\tilde{S}$ is defined by

$$\tilde{S}(\xi, \eta) = R^2(\sin^2 \xi + \sin^2 \eta) S(\xi, \eta) = \frac{3R^2(\sin^2 \xi + \sin^2 \eta) F_1 F^1}{8H_2H_5}$$

$$= \frac{\beta}{4R^2H_2H_5} \left[ \sinh^2 \xi \cos^2 \eta \left( (\partial_\xi A_\phi)^2 + (\partial_\eta A_\phi)^2 \right) + \frac{1}{\sinh^2 \xi \sin^2 \eta} \left( (\partial_\xi A_\psi)^2 + (\partial_\eta A_\psi)^2 \right) \right].$$  \hfill (B38)

We then have

$$\frac{d}{d\rho} \left( (\rho^2 - 1) \frac{df}{d\rho} \right) - \ell(\ell + 1)f = \frac{\tilde{S}_\ell}{4},$$  \hfill (B39)

where $\rho = \cosh 2\xi$.

Let us show one concrete example, which is the lowest moment solution ($m = n = 1$). Setting $H_2 = H_5 = 1$ (Case (2c) in Table II) and

$$A_\phi = -2J_1^{(\phi)} \left[ 1 + 2\cosh^2 \xi \ln(\tanh \xi) \right] \cos^2 \eta$$

$$A_\psi = 2J_1^{(\psi)} \left[ 1 + 2\sinh^2 \xi \ln(\tanh \xi) \right] \sin^2 \eta,$$  \hfill (B40, B41)

we find

$$\tilde{S}_0(\rho) = \frac{8\beta J_1^2}{R^2} \left[ \frac{2\rho}{\rho^2 - 1} + \frac{\rho - 1}{\rho + 1} + \frac{2}{\rho} \left( \ln \left( \frac{\rho - 1}{\rho + 1} \right) \right)^2 \right],$$

$$\tilde{S}_1(\rho) = \frac{8\beta J_1^2}{R^2} \left[ \frac{2}{\rho^2 - 1} - \frac{1}{2} \left( \ln \left( \frac{\rho - 1}{\rho + 1} \right) \right)^2 \right],$$  \hfill (B42, B43)

where $2J_1^2 = (J_1^{(\phi)})^2 + (J_1^{(\psi)})^2$. Integrating Eq. \ref{B39}, we find the exact solution as

$$f(\xi, \eta) = f_0(\xi) + f_1(\xi)P_1(\cos 2\eta),$$  \hfill (B44)

with

$$f_0(\xi) = -\frac{2Q_0}{R^2} \ln(\tanh \xi) + \frac{2\beta J_1^2}{R^2} \cosh 2\xi \ln(\tanh \xi) \right)^2,$$  \hfill (B45)

$$f_1(\xi) = \frac{2Q_1}{R^2} \left[ 1 + \cosh 2\xi \ln(\tanh \xi) \right] - \frac{\beta J_1^2}{2R^2} \ln(\tanh \xi)^2.$$  \hfill (B46)

The ADM mass is

$$M_{\text{ADM}} = \frac{\pi}{4G_5} Q_0.$$  \hfill (B47)

Although this is an exact solution, it is very complicated. Unless $\beta = 0$, the horizon, even if it exists, is not described by a surface of $\xi = \text{constant}$.

Note that although this solution is very complicated, it is still supersymmetric if $J_1^{(\phi)} = -J_1^{(\psi)}$.
2. hyperpolorical coordinates

Our next example is the hyperpolorical coordinates \((\xi, \eta, \phi, \psi)\), which are defined by the transformation

\[
x_1 + ix_2 = \frac{R \sinh \xi}{\cosh \xi - \cos \eta} e^{i\psi}, \quad x_3 + ix_4 = \frac{R \sin \eta}{\cosh \xi - \cos \eta} e^{i\phi},
\]

where \(\xi \geq 0, 0 \leq \eta \leq \pi, \) and \(0 \leq \phi, \psi \leq 2\pi\). This coordinates could be used to describe a ring topology. In this case, the infinity corresponds to \(\xi = 0\), which also describes one of the symmetric axis.

The line element is given by

\[
ds_{\xi,\eta}^2 = \frac{R^2}{(\cosh \xi - \cos \eta)^2} (d\xi^2 + \sinh^2 \xi d\eta^2 + d\eta^2 + \sin^2 \eta d\phi^2).
\]

With this coordinate system, Eq. (B2) is written as

\[
\sum_{\ell=0}^{\infty} A_\ell (\cos \xi) P_\ell (\cosh \xi) = 0, \quad (\ell = 0) \quad (B54)
\]

Using new variable \(\tilde{H}_A\), which is defined by \(H_A(\xi, \eta) = 1 + (\cosh \xi - \cos \eta)\tilde{H}_A(\xi, \eta)\), Eq. (B50) is rewritten as

\[
\partial_\xi^2 \tilde{H}_A + \coth \xi \partial_\xi \tilde{H}_A + \partial_\eta^2 \tilde{H}_A + \cot \eta \partial_\eta \tilde{H}_A = 0. \quad (B51)
\]

Setting \(\tilde{H}_A = \tilde{h}_A(\xi, \eta)\), we can separate the variables and find the following two ordinary differential equations:

\[
(\rho^2 - 1) \frac{d^2 \tilde{h}_A}{d\rho^2} + 2\rho \frac{d\tilde{h}_A}{d\rho} - M \tilde{h}_A = 0, \quad (B52)
\]

\[
(1 - \mu^2) \frac{d^2 \tilde{j}_A}{d\mu^2} - 2\mu \frac{d\tilde{j}_A}{d\mu} + M \tilde{j}_A = 0, \quad (B53)
\]

where \(\rho = \cosh \xi\) and \(\mu = \cos \eta\), and \(M\) is a separation constant.

We find that the general solution is described by the Legendre functions as

\[
H_A = 1 + (\cosh \xi - \cos \eta) \sum_{\ell=0}^{\infty} h_\ell^{(A)} P_\ell (\cosh \xi) + g^{(A)} Q_\ell (\cosh \xi) \right] P_\ell (\cos \eta), \quad (B54)
\]

where the separation constant \(M\) is given by an integer \(\ell\) as \(M = \ell (\ell + 1)\) because of the regularity on the symmetric axis. \(g^{(A)}\) and \(h_\ell^{(A)}\) are arbitrary constants.

The asymptotically flatness condition yields

\[
H_A = 1 + (\cosh \xi - \cos \eta) \sum_{\ell=0}^{\infty} h_\ell^{(A)} P_\ell (\cosh \xi) P_\ell (\cos \eta). \quad (B55)
\]

Since \(r^2 = R^2(\cosh \xi + \cos \eta)/(\cosh \xi - \cos \eta)\), looking at the asymptotic behavior at infinity, we find \((\cosh \xi - \cos \eta) \sim 2R^2/r^2\) as \(r \to \infty(\xi, \eta \to 0)\). This gives a relation between the coefficient \(h_0^{(A)}\) and charge \(Q_H\) as

\[
h_0^{(A)} = \frac{Q_H^{(A)}}{2R^2}. \quad (B56)
\]

Now we discuss Eqs. (B3) and (B4), which are

\[
\frac{1}{\sinh \xi} \partial_\xi (\sinh \xi \partial_\xi A_\phi) + \sin \eta \partial_\eta \left(\frac{1}{\sin \eta} \partial_\eta A_\phi\right) = 0 \quad (B57)
\]

\[
\sinh \xi \partial_\xi \left(\frac{1}{\sinh \xi} \partial_\xi A_\psi\right) + \frac{1}{\sin \eta} \partial_\eta \sin \eta \partial_\eta A_\psi = 0. \quad (B58)
\]
Setting $A_\phi = a_\phi(\xi) b_\phi(\eta)$ and $A_\psi = a_\psi(\xi) b_\psi(\eta)$, we obtain the following ordinary differential equations:

\begin{align}
\frac{d^2 a_\phi}{d\rho^2} + \frac{2\rho}{\rho^2 - 1} \frac{da_\phi}{d\rho} - \frac{K}{\rho^2 - 1} a_\phi &= 0, \\
\frac{d^2 b_\phi}{d\mu^2} + \frac{K}{1 - \mu^2} b_\phi &= 0, \\
\frac{d^2 a_\psi}{d\rho^2} - \frac{L}{\rho^2 - 1} a_\psi &= 0, \\
\frac{d^2 b_\psi}{d\mu^2} - \frac{2\mu}{1 - \mu^2} \frac{db_\psi}{d\mu} + \frac{L}{1 - \mu^2} b_\psi &= 0 ,
\end{align}

where $\rho = \cosh \xi$ and $\mu = \cos \eta$, and $K$ and $L$ are separation constants.

The solutions for Eqs. (B60) and (B62) are given by the Legendre functions. We set $K = n(m+1)$ and $L = n(n+1)$ ($m, n = 1, 2, \cdots$) because of regularity conditions on the symmetric axis.

The asymptotically flatness condition yields

\begin{align}
A_\phi &= \sum_{m=1}^{\infty} \frac{b_{m}^{(\phi)}}{m+1} P_m(\cosh \xi) [\cos \eta P_m(\cos \eta) - P_{m-1}(\cos \eta)] \\
A_\psi &= \sum_{n=1}^{\infty} \frac{b_{n}^{(\psi)}}{n+1} [\cosh \xi P_n(\cosh \xi) - P_{n-1}(\cosh \xi)] P_n(\cos \eta),
\end{align}

where $b_{m}^{(\phi)}$ and $b_{n}^{(\psi)}$ are arbitrary constants.

When $\beta = 0$, $f$ is given by the Legendre functions just as $H_A$, i.e.,

\begin{equation}
f = (\cosh \xi - \cos \eta) \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cosh \xi) P_\ell(\cos \eta),
\end{equation}

where $c_\ell$’s are arbitrary constants. For the lowest moment solution, we find

\begin{align}
H_A(\xi, \eta) &= 1 + \frac{Q_0^{(A)}}{2 R^2} (\cosh \xi - \cos \eta), \\
f(\xi, \eta) &= \frac{Q_0}{2 R^2} (\cosh \xi - \cos \eta), \\
A_\phi &= J_1^{(\phi)} \cosh \xi \sin^2 \eta \\
A_\psi &= J_1^{(\psi)} \sin^2 \xi \cos \eta .
\end{align}

The self-dual condition for supersymmetry implies $J_1^{(\phi)} = -J_1^{(\psi)}$.

In this spacetime, there is no horizon, but rather, a singularity at $\xi = \infty$, which locates at a ring with a radius $R$ in the flat 4D Euclidian space. Then this describes the geometry of a ring singularity.

To solve the equation for $f$ in the case of $\beta \neq 0$, we again expand $f$ and the source term $\tilde{S}$ by the Legendre functions as

\begin{align}
f(\xi, \eta) &= (\rho - \mu) \sum_{\ell=0}^{\infty} \tilde{f}_\ell(\rho) P_\ell(\mu) \\
\tilde{S}(\xi, \eta) &= \frac{R^2}{(\cosh \xi - \cos \eta)^3} S(\xi, \eta) = \frac{\beta R^2}{8 H_2 H_5 (\cosh \xi - \cos \eta)^3} \mathcal{F}_{ij} \mathcal{F}^{ij} \\
&= \frac{\beta (\cosh \xi - \cos \eta)}{4 R^2 H_2 H_5} \left[ \frac{1}{\sin^2 \eta} ((\partial_\xi A_\phi)^2 + (\partial_\eta A_\phi)^2) + \frac{1}{\sinh^2 \xi} ((\partial_\xi A_\psi)^2 + (\partial_\eta A_\psi)^2) \right] \\
&= \sum_{\ell=0}^{\infty} \tilde{S}_\ell(\rho) P_\ell(\mu),
\end{align}

where $\rho = \cosh \xi$ and $\mu = \cos \eta$. 
where $\rho = \cosh \xi$.

Setting $H_2 = H_5 = 1$ (Case (2c) in Table II) and

$$A_\phi = J_1^{(\phi)} \cosh \xi \sin^2 \eta$$

$$A_\psi = J_1^{(\psi)} \sinh^2 \xi \cos \eta,$$

we show the lowest moment solution here. We have now

$$\hat{S}_0 = \frac{\beta J^2}{3R^2} \rho (3\rho^2 - 1)$$

$$\hat{S}_1 = -\frac{\beta J^2}{5R^2} (7\rho^2 - 1)$$

$$\hat{S}_2 = \frac{\beta J^2}{3R^2} \rho (3\rho^2 + 1)$$

$$\hat{S}_3 = -\frac{\beta J^2}{5R^2} (3\rho^2 + 1),$$

where $2J^2 \equiv [(J_1^{(\phi)})^2 + (J_1^{(\psi)})^2]$, and then find general solutions as

$$\hat{f}_0(\rho) = c_0 P_0(\rho) + d_0 Q_0(\rho) + \frac{\beta J^2}{24R^2} \left[ 2\rho(\rho^2 + 1) + \ln \left( \frac{\rho - 1}{\rho + 1} \right) \right]$$

$$\hat{f}_1(\rho) = c_1 P_1(\rho) + d_1 Q_1(\rho) - \frac{\beta J^2}{40R^2} \rho \left[ 14\rho + 5 \ln \left( \frac{\rho - 1}{\rho + 1} \right) \right]$$

$$\hat{f}_2(\rho) = c_2 P_2(\rho) + d_2 Q_2(\rho) - \frac{\beta J^2}{40R^2} \rho \left[ -2\rho(2\rho^2 - 1) - (3\rho^2 - 1) \ln \left( \frac{\rho - 1}{\rho + 1} \right) \right]$$

$$\hat{f}_3(\rho) = c_3 P_3(\rho) + d_3 Q_3(\rho) + \frac{\beta J^2}{10R^2} \rho^2.$$  \hfill (B76)

Imposing the regularity ($\hat{f}_0$ finite, $\hat{f}_\ell = 0$ for $\ell \geq 1$) at infinity and on the axis ($\rho = 1$), we can fix the coefficients $c_\ell$ and $d_\ell$ ($\ell = 0, 1, 2, 3$) except for $c_0$. We obtain an exact solution as

$$f(\xi, \eta) = (\cosh \xi - \cos \eta) \sum_{\ell=0}^{3} \hat{f}_\ell(\cosh \xi) P_\ell(\cos \eta),$$

with

$$\hat{f}_0(\rho) = \frac{\beta J^2}{12R^2} (\rho - 1)(\rho^2 + \rho + 2) + \frac{Q_0}{2R^2}$$

$$\hat{f}_1(\rho) = -\frac{\beta J^2}{20R^2} (\rho - 1)(7\rho + 5)$$

$$\hat{f}_2(\rho) = +\frac{\beta J^2}{12R^2} (\rho - 1)(2\rho^2 + 5\rho + 1)$$

$$\hat{f}_3(\rho) = -\frac{\beta J^2}{20R^2} (\rho - 1)(5\rho + 3),$$

where $Q_0$ is an arbitrary charge. The ADM mass is given by

$$M_{\text{ADM}} = \frac{\pi}{4G_5} Q_0.$$  \hfill (B79)

This exact solution is also very complicated, but supersymmetric if $J_1^{(\phi)} = -J_1^{(\psi)}$. The horizon, even if it exists, is not described by a surface of $\xi=\text{constant}$. 

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