Kaluza-Klein Black Holes with Squashed Horizons

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Abstract

We study geometrical structures of charged static black holes in the five-dimensional Einstein-Maxwell theory. The black holes we study have horizons in the form of squashed $S^3$, and their asymptotic structure consists of a twisted $S^1$ bundle over the four-dimensional flat spacetime at the spatial infinity. The spacetime we consider is fully five-dimensional in the vicinity of the black hole and four-dimensional with a compact extra dimension at infinity.

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I. INTRODUCTION

Most attempts to construct theories in which gravity is unified with other forces are devoted to higher-dimensional spacetimes. Non-perturbative gravitational objects in higher-dimensional systems might provide insight for a quantum theory of gravity. In particular, there are many studies of higher-dimensional black holes, starting from pioneering works carried out several decades ago[1, 2, 3, 4].

Recently, the idea of large extra dimensions[5] has attracted much attention, because it suggests the interesting possibility that the unification of the electroweak and Planck scales takes place at the TeV scale. One of the most interesting phenomena predicted within this scenario is the possibility of the formation in accelerators of higher-dimensional black holes smaller than the size of extra dimensions[6]. Such higher-dimensional black holes would reside in a spacetime that is approximately isotropic in the vicinity of the black holes, but effectively four-dimensional far from the black holes[4]. We call higher-dimensional black holes with this property *Kaluza-Klein black holes*.

In this article, we study the geometry of a static charged black hole in the five-dimensional Einstein-Maxwell theory. The horizons of the black holes have the form of a squashed $S^3$, and the spacetime is asymptotically locally flat; i.e., it asymptotically approaches a twisted $S^1$ bundle over four-dimensional Minkowski spacetime. In other words, the spacetime is that of a five-dimensional Kaluza-Klein black hole.

In the context of the unification of interactions, the Einstein-Maxwell system with a Chern-Simons term is a theory of renewed interest as the bosonic part of supergravity[7]. There are many works treating black hole solutions in this system[8, 9]. Although the Chern-Simons term is absent in our action, this has no effect on the solutions discussed below. The solutions presented in this article are characterized by three parameters: the size of the $S^1$ fiber at infinity, the size of the inner horizon, and the size of the outer horizon. The singularity of the present static solution is hidden in the horizons, which are deformed owing to the non-trivial asymptotic structure. This contrasts with the fact that horizons are deformed by the rotation of rotating black holes. We also discuss some limiting cases of the solutions.
II. SOLUTIONS

We examine the five-dimensional Einstein-Maxwell theory described by the action

\[ S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left( R - F_{\mu\nu} F^{\mu\nu} \right), \]  

where \( R \) is the scalar curvature, \( F = dA \) is the Maxwell field strength 2-form corresponding to the gauge potential 1-form \( A \), and \( G \) denotes the five-dimensional Newton’s constant. The equations of motion derived from the action (1) are

\[ d^\ast F = 0 \quad \text{and} \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2 \left( F_{\mu\lambda} F^\lambda_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right). \]  

Equation (2) admits an electrically charged static black hole as a sourceless solution. The metric of the solution is

\[ ds^2 = -f dt^2 + \frac{k^2}{f} dr^2 + \frac{r^2}{4} \left[ k \left\{ (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right\} + (\sigma^1)_0^2 \right], \]  

where \( f \) and \( k \) are functions of \( r \) defined by

\[ f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4}, \quad k(r) = \frac{(r_0^2 - r_+^2)(r_0^2 - r_-^2)}{(r_0^2 - r^2)^2}, \]  

and the gauge potential is

\[ A = \pm \frac{\sqrt{3}}{2} \frac{r_+ r_-}{r^2} dt. \]  

Here, \( r_\pm \) and \( r_\infty \) are constants, and the quantities \( \sigma^i \) (\( i = 1, 2, 3 \)) satisfy the relation

\[ d\sigma^i = \frac{1}{2} C^i_{jk} \sigma^j \wedge \sigma^k, \]  

with \( C^1_{23} = C^3_{21} = C^3_{12} = 1 \) and \( C^i_{jk} = 0 \) in all other cases. (6)

The static spacetime of the metric (3) has the isometry group \( \text{SO}(3) \times \text{U}(1) \). Because the metric is apparently singular at \( r = r_\infty \), the radial coordinate \( r \) should be assumed to move the range \( 0 < r < r_\infty \). If we choose the parameters to satisfy \( 0 < r_- \leq r_+ < r_\infty \), the metric has horizons at \( r = r_+ \), the outer horizon, and at \( r = r_- \), the inner horizon.

III. SHAPE OF HORIZONS

A time-slice \( t = \text{const.} \) of the spacetime, which is orthogonal to the time-like Killing vector, is foliated by three-dimensional surfaces, which are specified by \( r = \text{const.} \), say \( \Sigma_r \). Each surface \( \Sigma_r \) is regarded as a Hopf bundle, an \( S^1 \) fiber over the \( S^2 \) base space, with the metric

\[ ds^2_{\Sigma_r} = \frac{r^2}{4} \left[ k(r) \left\{ (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right\} + \chi^2 \right], \]  

(7)
where

\[ d\Omega^2_{S^2} = d\theta^2 + \sin^2 \theta d\phi^2, \quad \chi = \sigma^3 = d\psi + \cos \theta d\phi. \]

\[ (0 \leq \theta < \pi, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \psi < 4\pi) \] (8)

The surface \( \Sigma_r \) takes the form of a deformed \( S^3 \) on which \( \text{SO}(3)\times\text{U}(1) \) acts as an isometry group. The aspect ratio of the \( S^2 \) base space to the \( S^1 \) fiber, which characterizes the squashing of \( S^3 \), is denoted by the function \( k(r) \). The degree of squashing increases monotonically as \( r \) increases towards \( r_\infty \). The divergence of \( k \) at \( r_\infty \) indicates the collapse of \( S^3 \) to \( S^2 \) there.

The shapes of \( \Sigma_\pm \), time-slices of the outer and inner horizons, are described by the three-dimensional metric (7) with \( r = r_\pm \), respectively. The squashing function \( k(r) \) on \( \Sigma_\pm \) is

\[ k(r_\pm) = \frac{r_\pm^2 - r_0^2}{r_\infty^2 - r_0^2}. \] (9)

Because \( k(r_+) \geq 1 \geq k(r_-) \), the outer horizon is ‘oblate’, with \( S^2 \) larger than \( S^1 \), while the inner one is ‘prolate’, with \( S^2 \) smaller than \( S^1 \). In the degenerate case, \( r_+ = r_- \), the shape of the horizon is the round \( S^3 \).

IV. ASYMPTOTIC STRUCTURE

The apparent singularity at \( r = r_\infty \) of the metric (3) is the spatial infinity. To observe this, we introduce a new radial coordinate \( \rho \) as

\[ \rho = \rho_0 \frac{r^2}{r_\infty^2 - r^2}, \] (10)

where

\[ \rho_0^2 = k_0 \frac{r_\infty^2}{4}, \quad k_0 = k(0) = f_\infty = f(r_\infty) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{r_\infty^4}. \] (11)

The new coordinate \( \rho \) varies from 0 to \( \infty \) when \( r \) varies from 0 to \( r_\infty \). The metric (3) can be rewritten in terms of \( \rho \) and \( T = \sqrt{f_\infty} t \) as

\[ ds^2 = -VdT^2 + \frac{K^2}{V} dp^2 + R^2 d\Omega^2_{S^2} + W^2 \chi^2, \] (12)

where \( V, K, R \) and \( W \) are functions of \( \rho \) in the form

\[ V = \frac{(\rho - \rho_+)(\rho - \rho_-)}{\rho^2}, \quad K^2 = \frac{\rho + \rho_0}{\rho}, \]

\[ R^2 = \rho^2 K^2, \quad W^2 = \frac{\rho_0^2}{4} K^{-2} = (\rho_0 + \rho_+)(\rho_0 + \rho_-) K^{-2}. \] (13)

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In (13) we have used new parameters defined by

$$\rho_{\pm} = \rho_0 \frac{r^2}{r_\infty^2 - r^2}. \quad (14)$$

As \(\rho \to \infty\), i.e., \(r \to r_\infty\), the metric (12) with (13) approaches

$$ds^2 = -dT^2 + d\rho^2 + \rho^2 d\Omega^2_{S^2} + \frac{r_\infty^2}{4} \chi^2. \quad (15)$$

Actually, in the limit of large \(\rho\), the leading-order term of the Kretschmann scalar is

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \sim \frac{12 [(\rho_+ + \rho_-)^2 - \rho_+ \rho_- + 2(\rho_0 + \rho_+)(\rho_0 + \rho_-)]}{\rho^5}. \quad (16)$$

Therefore, the limit \(\rho \to \infty\), and equivalently \(r \to r_\infty\), corresponds to the spatial infinity, and the spacetime is locally asymptotically flat, i.e., topologically not a direct product but a twisted \(S^1\) fiber bundle over four-dimensional Minkowski spacetime.

**V. PHYSICAL PROPERTIES**

First, let us consider the physical properties near a black hole. We use the metric form (3). In the near region, where \(r \ll r_\infty\), the function \(k(r)\) is nearly constant, and therefore the hyper-area of \(\Sigma_r\), i.e., squashed \(S^3\), is proportional to \(r^3\). Thus, the coordinate \(r\) plays the role of the hyper-area radius of \(\Sigma_r\). The temporal component \(f\) of the metric in coordinates in which \(r\) plays such a role has the same form as that of the five-dimensional Reissner-Nordström black hole[1], and hence in the region \(r \ll r_\infty\), the spacetime we consider is similar to that of a Reissner-Nordström black hole. For the case \(r_+ \ll r_\infty\), there exists an outer region satisfying \(r_+ < r \ll r_\infty\), and the spacetime would behave as that of a five-dimensional black hole for observers in this region.

Inside the horizon, \(\Sigma_r\) shrinks to a point with the constant \(k_0\) as \(r \to 0\). It is obvious that a time-like singularity exists at \(r = 0\), as in the case of the Reissner-Nordström spacetime. The Kretschmann scalar diverges as

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \sim \frac{508r_+^4r_-^4}{k_0^8r^12} \quad (17)$$

in the limit \(r \to 0\). The value of \(k_0\) denotes the elongation of the central singularity.

Next, we consider the region far from the black hole by using the metric form (12). In the region \(\rho_0 \ll \rho\) (equivalently \(r_\infty - r \ll r_\infty\) in the original coordinates), the function \(K^2\) is almost constant. Hence, the spacetime is effectively four-dimensional for phenomena with energies that
are lower than the scale of the inverse size of the extra dimension. If \( \rho_0 \ll \rho_+ \), \( V \) depends on \( \rho \) more strongly than \( K^2 \). In this case, the spacetime would behave as that of a four-dimensional Reissner-Nordström black hole for distant observers.

The size of the outer horizon for a distant observer can be regarded as the circumference radius of the \( S^2 \) base space on the horizon:

\[
R_+ = \frac{r_+}{2} \sqrt{\frac{r^2_+ - r^2_-}{r^2_+ - r^2_+}}.
\]  

(18)

When the size of the \( S^1 \) fiber on the outer horizon, \( r_+ \), approaches the size of the fifth dimension at infinity, \( r_\infty \), we could effectively obtain a very large four-dimensional black hole.

The surface gravity of the outer horizon, \( \kappa_+ \), is given by

\[
\kappa_+ = \frac{r^2_+ r^2_+ - r^2_-}{r^2_+ r^2_+ - r^2_-} \sqrt{\frac{r^2_+ - r^2_-}{r^2_+ - r^2_+}} = \frac{\rho_+ - \rho_-}{2r^2_+} \sqrt{\rho_+ + \rho_0}.
\]  

(19)

For the degenerate case, \( r_+ = r_- \) (equivalently \( \rho_+ = \rho_- \)), the surface gravity of the black hole is vanishing, as in the usual case.

Using the time-like Killing vector \( \xi = \partial_t/\sqrt{f_\infty} = \partial_T \), which is normalized at the spatial infinity, we can define the mass of the black hole as

\[
M = -\frac{3}{32\pi G} \int_\infty dS_{\mu \nu} \nabla^\mu \xi^\nu,
\]  

(20)

where the integral is taken over the three-dimensional topological sphere at the spatial infinity. For the metric form (3) with (4), or (12) with (13), we find

\[
M = \frac{3\pi}{8G\sqrt{f_\infty}} \left( r^2_+ + r^2_- - \frac{2r^2_+ r^2_+}{r^2_\infty} \right) = \frac{3\pi r_\infty}{4G} (\rho_+ + \rho_-).
\]  

(21)

The electric charge is defined as

\[
Q = \frac{1}{8\pi G} \int_S dS_{\mu \nu} F^{\mu \nu},
\]  

(22)

where the integral is taken over the three-dimensional topological sphere surrounding the black hole. For the gauge potential (5), we have

\[
Q = \frac{\sqrt{3}\pi}{2G} r_+ r_- = \frac{\sqrt{3}\pi r_\infty}{G} \sqrt{\rho_+ \rho_-}.
\]  

(23)
VI. LIMITS

We consider four kinds of limits of the metric parameterized by \( r_-, r_+ \) and \( r_\infty \).

First, we take the limit \( r_\infty \to \infty \). The squashing function behaves as \( k \to 1 \) in this limit, and therefore the metric (3) reduces to that of the five-dimensional Reissner-Nordström black hole with the symmetry of round \( S^3 \), i.e., the SO(4) isometry group[3]. In this case, the spacetime asymptotically becomes five-dimensional Minkowski spacetime. The surface gravity (19) reduces to the well-known form.

Second, in the limit \( r_- \to 0 \), the metric describes a neutral black hole with a gravitational radius \( r_+ \):

\[
 ds^2 = -\left(1 - \frac{r_+^2}{r_-^2}\right)dt^2 + \left(1 - \frac{r_+^2}{r_-^2}\right)^{-1}k^2dr^2 + \frac{r_+^2}{4}\left[kd\Omega^2_S + \chi^2\right].
\]  

When \( r_\infty \) is finite, the metric of the black hole is deformed by the squashing function \( k(r) \) with \( r_- = 0 \). The metric in the neutral case has been studied in the context of Kaluza-Klein theory[2]. If we take the limit \( r_\infty \to \infty \) in (24), the metric reduces to the five-dimensional Schwarzschild black hole with SO(4) isometry, while in the limit \( r_+ \to 0 \), the metric reduces to the Gross-Perry-Sorkin monopole[10].

Third, when we set \( r_- = r_+ \), the metric describes a deformed version of the five-dimensional extremal Reissner-Nordström black hole with a degenerate horizon. In this case, using the new coordinate \( \tilde{r}^2 = r^2 - r_+^2 \) and the parameter \( \tilde{r}_\infty^2 = r_\infty^2 - r_+^2 \), we can rewrite the metric in the harmonic form

\[
 ds^2 = -H^{-2}dt^2 + H\,d\tilde{s}_{T-NUT}^2,
\]

where

\[
 d\tilde{s}_{T-NUT}^2 = \frac{\tilde{r}_\infty^8}{(\tilde{r}_\infty^2 - \tilde{r}^2)^4}dr^2 + \frac{\tilde{r}^2}{4}\left[\frac{\tilde{r}_\infty^4}{(\tilde{r}_\infty^2 - \tilde{r}^2)^2}d\Omega^2_S + \chi^2\right]
\]

is the metric of the Euclidean self-dual Taub-NUT space in four-dimensions, and we have

\[
 H = 1 + \frac{r_+^2}{\tilde{r}^2}.
\]

In the case \( r_+ = r_- \) (equivalently \( M = \frac{\sqrt{2}}{2}Q \)), the metric is a special case of the supersymmetric solution to five-dimensional supergravity[9, 11].

Finally, we consider the limit \( r_+, r_- \to r_\infty \), with \( \rho_\pm \) finite. It is convenient to use the metric (12). Because \( \rho_0 \to 0, \ K^2 \to 1 \) in this limit, the metric becomes

\[
 ds^2 = -\frac{(\rho - \rho_+)(\rho - \rho_-)}{\rho^2}dT^2 + \frac{\rho^2}{(\rho - \rho_+)(\rho - \rho_-)}d\rho^2 + \rho^2d\Omega^2_S + \rho_+\rho_-\chi^2.
\]
This metric describes a four-dimensional Reissner-Nordström black hole with a twisted $S^1$ bundle, where the size of the $S^1$ fiber takes the constant value $\sqrt{\rho_+ \rho_-} = r_\infty/2$. The surface gravity (19) reduces to that of a four-dimensional black hole in this limit. Furthermore, if we take the limit $\rho_+ \to 0$ in the metric (28), after redefining the angular coordinate $\psi$ in (8) as $\psi' = \sqrt{\rho_+ \rho_-} \psi$, the metric (28) approaches the direct product metric of the four-dimensional Schwarzschild black hole and $S^1$,

$$ds^2 = - \left(1 - \frac{\rho_+}{\rho}\right) dT^2 + \left(1 - \frac{\rho_+}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2_{S^2} + d\psi'^2,$$

whose universal covering space describes a black string.

It would be interesting to clarify the stability and phase structure of five-dimensional gravitational objects[12] by using the family of solutions presented in this article. It is also important to investigate the thermodynamics of black holes including parameters representing the sizes of the extra dimensions.

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