Instability of coherent states of a real scalar field

Vladimir A. Koutvitsky and Eugene M. Maslov
Pushkov Institute of Terrestrial Magnetism, Ionosphere and Radiowave Propagation of the Russian Academy of Sciences (IZMIRAN), Troitsk, Moscow Region, 142190, Russia

We investigate stability of both localized time-periodic coherent states (pulsons) and uniformly distributed coherent states (oscillating condensate) of a real scalar field satisfying the Klein-Gordon equation with a logarithmic nonlinearity. The linear analysis of time-dependent parts of perturbations leads to the Hill equation with a singular coefficient. To evaluate the characteristic exponent we extend the Lindemann-Stieltjes method, usually applied to the Mathieu and Lamé equations, to the case that the periodic coefficient in the general Hill equation is an unbounded function of time. As a result, we derive the formula for the characteristic exponent and calculate the stability-instability chart. Then we analyze the spatial structure of the perturbations. Using these results we show that the pulsons of any amplitudes, remaining well-localized objects, lose their coherence with time. This means that, strictly speaking, all pulsons of the model considered are unstable. Nevertheless, for the nodeless pulsons the rate of the coherence breaking in narrow ranges of amplitudes is found to be very small, so that such pulsons can be long-lived. Further, we use the obtained stability-instability chart to examine the Affleck-Dine type condensate. We conclude the oscillating condensate can decay into an ensemble of the nodeless pulsons.

PACS numbers: 03.65.Pm, 05.45.Yv, 11.10.Lm, 11.27.+d

I. INTRODUCTION

Nonlinear localized field configurations, solitons, are currently considered as models of various physical objects, from elementary particles and collective excitations in condensed matter to giant lumps of dark matter in the form of soliton stars and galactic halos \[1\text{,}2\text{,}3\]. Stability properties of solitons were investigated by many authors, and a number of important results has been obtained (see, e.g., \[4\] and references therein). In particular, Hobart \[3\] and Derrick \[6\] have proved that static multidimensional scalar solitons are energetically unstable, and, hence, these objects cannot last in a real world for a long time. One way to avoid this theorem is to invoke time dependence. Along this line main efforts were focused on the stability analysis of the stationary states, i.e., coherent states of a complex scalar field oscillating harmonically in time. It turned out, however, that for a wide class of relativistic models these states can be only conditionally stable, i.e., stable with respect to a certain type of perturbations (e.g., conserving the scalar charge) \[4\]. As to the time-periodic states of a more general form, both complex and real, there are presently no strong analytical results on their stability.

In this paper we examine stability of time-periodic configurations of the form

$$\phi = \phi_0(t, r) = a(t)u(r)$$

satisfying the nonlinear Klein-Gordon equation

$$\phi_{tt} - \Delta \phi + U'(\phi) = 0.$$  \hspace{5mm} (2)

These are coherent states in the sense the field oscillates synchronously at all spatial points. It is necessary to stress that we consider real solutions, so that the energy density oscillates as well (in contrast to the stationary states for which \(a(t) \propto e^{\pm it}\)). Solitons with oscillating energy density are usually called pulsons.

It turns out that for real \(\phi\) the ansatz \[1\] determines uniquely the potential \(U(\phi)\) in Eq. \[2\]. Namely, if neither \(a(t)\) nor \(u(r)\) are constants, the only potential admitting such solutions will have the form \[5\]

$$U(\phi) = \frac{1}{2} \phi^2 [m^2 + \lambda (1 - \ln \phi^2)],$$  \hspace{5mm} (3)

where \(m^2\) and \(\lambda\) are arbitrary constants.

Originally, the Klein-Gordon equation with a logarithmic potential of this type has been introduced in the quantum field theory by G. Rosen \[8\]. Later on Bialynicki-Birula and Mycielski \[9\] have rediscovered this equation and also considered its nonrelativistic version, the nonlinear Schrödinger equation \[10\].

In inflationary cosmology and in modern supersymmetric field theories the logarithmic nonlinearities appear naturally when quantum corrections to effective potentials are allowed for \[11\text{,}12\text{,}13\text{,}14\]. In this context the expression in the square brackets of Eq. \[3\] can be treated as a dynamic inflaton mass term \(m^2\) that is the bare inflaton mass term plus the logarithmic correction. It can be represented in the commonly considered form \[14\text{,}15\text{,}16\text{,}17\text{,}18\text{,}19\text{,}20\text{,}21\text{,}22\] by the substitution \(\ln \phi^2 = 1 + \ln(\Phi/M)^2\), \(\lambda/m^2 = -K\), where \(\Phi\) is an inflaton scalar field, \(M\) is a large mass scale, \(K\) is a constant (usually negative and small). Thus our consideration is also relevant to dynamics of the pulson excitations of a real inflaton field oscillating around a vacuum value.

Note that the multidimensional pulsons probably exist in other scalar models as well. Thus the long-living oscillating spherically symmetric localized states were numerically found in the sine-Gordon, \(\phi^4\), and \(\phi^3 - \phi^4\) models \[22\text{,}23\text{,}24\text{,}25\text{,}26\] (see \[27\] for a review). Unfortunately, the analytic form of these solutions is so far unknown.
The model (2) and (3) is unique in the sense it has a whole family of exact pulson solutions of the form (11), all existing in any number of spatial dimensions 7. This is also true for complex version of the model (22, 23). The real pulsons we are dealing with are the limiting states of the complex ones, when the scalar charge tends to zero. Other limiting states are Q-balls for which $a(t) \propto e^{\omega t}$ 15, 16. It is believed that Q-balls can arise due to fragmentation of the Affleck-Dine condensate 18, 19, 21, 22. We will see below that the parametric instability of the oscillating condensate leads to the resonant fragmentation that can give rise to the pulson formation at the nonlinear stage. Like Q-balls 17, pulsons interact elastically or inelastically in collisions depending on their relative velocities, phases, and rest masses 30, 31. Thus, in model (2) and (3) the light pulsons with given relative velocities interact always elastically, independently of their phases. In contrast, the collisions of heavy pulsons can result in formation of the so-called explosions, localized states with exponentially growing amplitude 30. For the intermediate masses the picture depends essentially on the phases of the colliding pulsons and impact velocity determining the duration of the interaction 31.

The above results suggest that there is a domain of parameters where pulsons are stable, at least in short time interactions. But in what sense? How long a pulson conserves its characteristic features once interaction ends? If pulsons are long-lived objects they will be interesting candidates for the dark matter constituents having its perfect stability. No deviations from the exact solution (11) were found after about one thousand oscillations. However, our preliminary numerical experiments 32, 33 have shown that the pulsons of certain amplitudes, even perturbed by computer round-off errors only, gradually lose their coherency, remaining well-localized oscillating objects. This has motivated the closer examination.

In the present paper we clarify how long the pulsons can conserve the coherency depending on their parameters. For this purpose we investigate stability of the spherically symmetric pulson solutions (11) with respect to small initial perturbations of an arbitrary form.

The paper is organized as follows. In Sec. II the main properties of the real pulsons of the model considered are reviewed. Section III is wholly devoted to the linear stability analysis. We arrive at the singular Hill equation and generalize the Lindemann-Stieljes method to evaluate the characteristic exponent. On this basis we examine stability of the pulsons and discuss fragmentation of the oscillating Affleck-Dine type condensate. In Sec. IV we make some remarks concerning the complex pulsons and summarize the main results.

II. PULSONS AS COHERENT STATES

Assuming $\lambda$ positive, let us first eliminate the constants $m^2$ and $\lambda$ from consideration by the scaling $t \to \lambda^{-1/2} t$, $r \to \lambda^{-1/2} r$, $\phi \to \phi \exp \frac{m^2}{\lambda}$. In the new variables the field $\phi$ may be thought of as satisfying Eq. (2) with the potential

$$U(\phi) = \frac{1}{2} \phi^2 (1 - \ln \phi^2).$$

(4)

It is the potential we will deal with. It has local minimum at $\phi = 0$ and two maxima at $\phi = \pm 1$, at the minimum the potential having the singularity: its second derivative tends to infinity as $\phi \to 0$.

The substitution of the ansatz (11) into Eq. (2) leads then to two independent equations,

$$a_{tt} = -\frac{d}{da} \left[ \frac{1}{2} a^2 (1 - \ln a^2) \right],$$

(5)

$$\Delta u = -\frac{d}{du} \left[ \frac{1}{2} u^2 (\ln u^2 - 1) \right].$$

(6)

Note that the potentials in the square brackets of Eqs. (5) and (6) have the same form as the potential (4) taken with plus and minus signs, respectively. The existence of the oscillating localized solutions (11) is thus apparent from consideration of motion of a mechanical particle in these potentials.

Let us consider in more detail the oscillatory solutions of Eq. (5). Using the Hamiltonian and denoting $\xi = a/a_{\text{max}}$ ($0 < a_{\text{max}} < 1$, $-1 \leq \xi \leq 1$), we obtain

$$\dot{\xi}_t^2 = \omega_0^2 (1 - \xi^2) + \xi^2 \ln \xi^2,$$

(7)

where

$$\omega_0^2 = 1 - \ln a_{\text{max}}^2 > 1.$$

(8)

In the case of small amplitudes, $a_{\text{max}}^2 \ll 1$, $\omega_0^2 \gg 1$, Eq. (7) gives

$$\xi(t) \approx \cos \omega_0 t.$$

(9)

Thus, we have quasi-harmonic high-frequency oscillations which are however nonlinear since their period,

$$T \approx \frac{2\pi}{\ln a_{\text{max}}^2 / 2},$$

(10)

depends on the amplitude 33. In the next approximation from Eq. (7) we find

$$T = \frac{2\pi}{\omega_0} \left( 1 + 0.307 \frac{\omega_0^2}{\omega_0^2} + O \left( \frac{1}{\omega_0^2} \right) \right).$$

(11)

In the case of near-critical amplitudes, when $a_{\text{max}}^2 \to 1$, the oscillations become almost rectangular and have the period

$$T \approx 2\sqrt{2} \ln \frac{1}{1 - a_{\text{max}}^2}.$$

(12)
Examples of solutions of Eq. 15 are shown in Fig. 1. The spatial structure of a pulson is determined by Eq. 16. In the spherically symmetric case this equation has a discrete spectrum of localized N-nodal solutions $u_N(r)$ with the first derivatives vanishing at the origin [34] (see Fig. 2). The simplest of them, the nodeless solution, has a Gaussian-like shape,

$$u_0(r) = e^{-(3-r^2)/2},$$

and is usually called gausson [8, 9, 10]. It is agreed that its effective radius equals $\sqrt{2}$. In the multinodal solutions, as $r$ increases, the field undergoes spatial oscillations of the half-wavelength $L \lesssim 2\sqrt{2}$ and then decays as

$$u_N(r) \approx C_N e^{-(r-r_N)^2/2} \quad (r \gg r - r_N, \ N \gg 1),$$

where $C_N$ is the value of the last extremum of $u_N(r)$ attained at $r = r_N$, $C_N \rightarrow (-1)^N e^{1/2}$ ($N \rightarrow \infty$), $r_N \sim NL$. Thus, the pulsions of the model 12 and 13 are well-localized states of the inhomogeneity length $L$, at all points the field oscillating coherently with the period $T$. (To return to the physical units these scales should be multiplied by $\lambda^{-1/2}$.) In our dimensionless variables the pulsions are characterized by two parameters only: the amplitude $a_{\text{max}}$ and the number of the nodes $N$ (or $T$ and $u_N(0)$, respectively).

It should be stressed that, due to nonanalyticity of $U(\phi)$ at $\phi = 0$, the right-hand sides of Eqs. 15 and 16 are nonanalytic when $a$ and $u$ become zero. Hence, the solutions $a(t)$ and $u(r)$ themselves become nonanalytic at those points $t$ and $r$ where they pass through zero. Thus in the solution 16 we have dropped the terms which are small (of the order of $\omega_0^{-2}$) but nonanalytic when $\xi(t) = 0$. In general case from Eq. 17 it follows that $\xi(t)$ passes through zero at $t = t_m$ as

$$\xi(t) = \pm \omega_0 (t - t_m) \left[ 1 + \frac{1}{6} (t - t_m)^2 \ln (t - t_m)^2 \right] + O \left( (t - t_m)^2 \right).$$

As we will see below, the nonanalyticity of $U(\phi)$ gives rise to some specific features of the stability analysis.

## III. THE LINEAR STABILITY ANALYSIS

Consider a small fluctuation $\eta(t, r)$ around the spherically symmetric pulson 11, $\phi = \phi_0(t, r) + \eta(t, r)$. In the linear approximation the equation for $\eta$ reads

$$\eta_{tt} - \Delta \eta - (2 + \ln \phi_0^2) \eta = 0.$$  

(17)

Seeking a solution in the form $\eta(t, r) \propto X(t) \Psi(r)$ we arrive at the equations

$$X_{tt} + (E - 2 - \ln a^2) X = 0,$$  

$$\Delta \Psi + (E + \ln u^2) \Psi = 0,$$  

(18)  

(19)

where $E$ is some constant.

The expression in the brackets of Eq. 17 is $-U''(\phi_0)$. It becomes infinite, as well as the expressions in the brackets of Eqs. 18 and 19, at the points $t_m$ and $r_n$ where $a(t)$ and $u(r)$ become zero. Thus we need to analyze the second order differential equations with singular coefficients. We begin with Eq. 18 which has the periodic singular coefficient $\ln a^2(t)$ and hence belongs to the class of Hill equations.
A. Singular Hill equation and generalized Lindemann-Stieltjes method

It turns out to be very useful to look at the problem as a whole, considering first the Hill equation of a general form

\[ X_{tt} + h(z(t))X = 0. \]  
(20)

We will assume that \( h(z) \) is an integral function of \( z \), while \( z(t) \) is a real-valued periodic (of a period \( \tau \)) even function of \( t \), having, in general, singularities, but such, that \( h(z(t)) \) remains still integrable.

It is well known that the Hill equation describes the physical systems in which the parametric resonance can occur. In the context of our stability analysis we will be interested in real resonant solutions of Eq. (20). In accordance with the Floquet theory (see, e.g., [35]), any one of these solutions can be represented as a linear combination of the fundamental solutions

\[ X_+(t) = \varphi(t)e^{\mu t}, \quad X_-(t) = \varphi(-t)e^{-\mu t}, \]  
(21)

where \( \varphi(t) \) is a \( \tau \)-periodic or \( \tau \)-antiperiodic real function, \( \mu > 0 \) is the characteristic exponent. In the case that \( z(t) \) is unbounded it is impossible to obtain the solutions and evaluate \( \mu \) by expansions in Fourier series, following the standard Hill approach. Another way is to apply the Lindemann-Stieltjes method [36]. In some cases it allows one to obtain the results in a closed analytical form [37, 38, 39, 40]. We first outline this method in the context of the general Hill equation (20) with an extension to the case that the periodic function \( z(t) \) is unbounded. In doing so we follow the paper [39] where the method was used to construct the resonant solutions of the Lamé equation.

The main idea is as follows. Let us treat \( z \) as a new “time” variable instead of \( t \). In each interval of monotonicity of \( z(t) \) we define

\[ y(z) = X(t). \]  
(22)

Assume that the periodic function \( z(t) \) satisfies the equation

\[ \zeta_t^2 = g(z), \]  
(23)

where \( g(z) \) is an integral function of \( z \). Eq. (21) then becomes

\[ g(z)y'' + \frac{1}{2}g'(z)y' + h(z)y = 0 \]  
(24)

(hereinafter the prime denotes \( d/dz \)).

Let us first suppose \( z(t) \) is bounded. Equation (24) then shows that it is differentiable. Zeros of the function \( g(z) \) on the complex \( z \) plane, taken to be isolated, are singular points of Eq. (24). Since \( z(t) \) is periodic and real-valued, among singular points there are two, \( \zeta_1 \) and \( \zeta_2 \), lying on the real axis and being minimal and maximal values which \( z(t) \) acquires at the end points of the intervals of monotonicity. Also, it follows that \( g'/(\zeta_{1,2}) \neq 0 \). Physically, this is well understood, since \( \zeta_1 \) and \( \zeta_2 \) can be treated as turning points in periodic motion of a mechanical particle, e.g., of a nonlinear oscillator, under the action of the force \( g'/2 \). From Eq. (23) it is clear that the interval \([\zeta_1, \zeta_2]\) does not contain other singular points of Eq. (24).

For example, in the case of the Mathieu equation we have \( z(t) = \cos^2 t \), \( g(z) = 4z(1-z) \), so that Eq. (24) has the regular singular points \( z = \zeta_1 = 0, z = \zeta_2 = 1 \), both being the turning points. In addition, the equation has an irregular singularity at infinity. For the Lamé equation \( z(t) = \sin^2(t, \pi), g(z) = 4z(1-z)(1-x^2z) \). Equation (24) then has the regular singular points \( z = \zeta_1 = 0 \), \( z = \zeta_2 = 1 \), \( z = x^{-2} > 1 \), first two of them being the turning points, and a regular singularity at infinity.

In general, it is easy to verify that the turning points \( \zeta_{1,2} \) are regular singular points of Eq. (24), the exponents at each being 0 and 1/2. This implies that in the vicinity of each turning point \( \zeta \) there exist two independent solutions of Eq. (24), \( y^{(0)}(z; \zeta) \) and \( y^{(1/2)}(z; \zeta) \), having asymptotics \( 1 + O(z-\zeta) \) and \( (z-\zeta)^{1/2}[1 + O(z-\zeta)] \), correspondingly.

Now let us consider any one interval of monotonicity of the \( \tau \)-periodic even function \( z(t) \). Denote as \( y_1(z) \) and \( y_2(z) \) those two linearly independent solutions of Eq. (24) one of which coincides, by Eq. (22), with the increasing solution \( X_+(t) \), and another with the decreasing one, \( X_-(t) \), on the interval chosen. Since \( \varphi(t) \) is either \( \tau \)-periodic or \( \tau \)-antiperiodic, the product \( \varphi(t)\varphi(-t) = X_+X_- = y_1y_2 = w(z) \) is always \( \tau \)-periodic even function defined on the whole \( t \) axis. Hence, at the end points of the intervals of monotonicity of \( z(t) \), i.e., at \( t_m = m\pi/2 (m = 0, \pm 1, \ldots) \), the derivative \( [(X_+X_-)]_{t=t_m} = 0 \) or, what is the same,

\[ (w')^2 \sqrt{g} = 0. \]  
(25)

In the vicinity of a turning point \( \zeta \) the solutions \( y_1 \) and \( y_2 \) can be represented as linear combinations of the solutions \( y^{(0)} \) and \( y^{(1/2)} \). Consequently, the singularity \( (z-\zeta)^{1/2} \) is the only one which the function \( w(z) = y_1y_2 \) might have. But its existence is in contradiction with Eq. (24), because \( g'(\zeta) \neq 0 \) and, hence, \( g(z) \sim g'(\zeta)(z-\zeta) \). Therefore, the product \( y_1y_2 \) is analytic at \( z = \zeta_{1,2} \). Recall now that the interval \([\zeta_1, \zeta_2]\) does not contain other singular points of Eq. (24) and the singular points are assumed to be isolated. We thus conclude that on the complex \( z \) plane there exists a vicinity of the interval \([\zeta_1, \zeta_2]\), i.e., an open domain \( D > [\zeta_1, \zeta_2] \), in which \( y_1y_2 \) is an analytic function of \( z \). In addition, it follows that \( y_1^2 \) and \( y_2^2 \) of necessity have singularities of the type \( (z-\zeta)^{1/2} \) and, thus, cannot satisfy Eq. (24).

Now consider the case that one of the turning points or the both are at infinity. This implies that at the corresponding instants \( t_m \), the functions \( z(t) \) and \( z_1(t) \) become unbounded, the latter changing the sign. Nevertheless, we assume that in the vicinities of \( t_m \) the function \( h(z(t)) \)
in Eq. \(20\) is integrable and \(X\) is continuous, whence it follows that \(X_t\) and, therefore, \(w_t\) are also continuous. Hence, as before, \((w_t)_{t=t_o} = 0\) due to evenness and periodicity, so that we arrive at Eq. \(24\) again, where 
\[ \zeta_1 = -\infty \text{ and/or } \zeta_2 = +\infty. \]

It is easy to verify that the bilinear combinations \(y_1^2\), \(y_1 y_2\), and \(y_2^2\) constitute the fundamental system of solutions of the third-order differential equation
\[
g(z)w'''' + \frac{3}{2} y'(z) w'' + \left( \frac{1}{2} y''(z) + 4h(z) \right) w' + 2h'(z) w = 0.
\] \(26\)

Equation \(26\) is thus a common criterion for selection of the solution
\[
w = y_1 y_2
\] \(27\)

from the set of solutions of Eq. \(26\). In the case that \((z)\) is bounded, Eq. \(26\) is equivalent to the requirement that a solution of Eq. \(26\) be analytic in \(D\). If \((z)\) is unbounded, Eq. \(26\) will give the boundary conditions at infinity which must be satisfied in solving Eq. \(26\).

In case \((z)\) will be analytic in a vicinity \(D\) of one of the intervals \((-\infty, \zeta_2], \left[\zeta_1, \infty\right), (-\infty, \infty)\).

Thus, in a neighbourhood of any one point \(\zeta \in D\) we can write the expansions
\[
\begin{pmatrix} w(z) \\ g(z) \\ h(z) \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} w_n \\ g_n \\ h_n \end{pmatrix} (z - \zeta)^n,
\] \(28\)

Substitution of \(28\) into Eq. \(26\) leads to the following set of equations for the coefficients:
\[
m \sum_{n=0}^{m+2} a(m+n)g_m \omega_{n+1}w_n + 4 \sum_{n=0}^{m} (m+n)h_m \omega_n = 0
\] \(29\)

\((m = 1, 2, \ldots)\). Thus for \(m = 1\) we have
\[
6g_0 w_3 + 3g_1 w_2 + (g_2 + 4h_0) w_1 + 2h_1 w_0 = 0.
\] \(30\)

Assuming \(w(\zeta) \neq 0\), we normalize \(w(z)\) by \(w_0 = 1\). Then, at given \(w_1\) and \(w_2\) the remaining coefficients \(w_n\) are determined from Eqs. \(20\). The choice of \(w_1\) and \(w_2\) is not arbitrary but determined by Eq. \(26\). Thus, setting \(\zeta = \zeta_1\) and, hence, \(g_0 = 0\), we must choose \(w_1\) in such a way that the series \(26\) for \(w\) (or its continuation) converges at the second turning point \(\zeta_2\), or satisfies the boundary condition at infinity \(26\) if \(\zeta_2 = +\infty\). For the Mathieu and Lamé equations this leads to the function \(w(z)\) which is an integral one, for the latter it being a polinomial \(31\). In these cases the domain \(D\) is evidently the \(z\) plane with \(|z| < \infty\).

Let us suppose the function \(w(z)\) \(26\) is found. Return now to Eqs. \(20\) \(24\). Denote as \(W\) the Wronskian of the solutions \(21\),
\[
X_+ X_{-t} - X_{+t} X_- = W = \text{const.}
\] \(31\)

Setting
\[
y_1 = X_+, \quad y_2 = X_-\quad (z_t \geq 0),
\]
\[
y_1 = X_-, \quad y_2 = X_+\quad (z_t \leq 0),
\] \(32\)

we then obtain
\[
y_1 y_2 - y'_1 y'_2 = W/\sqrt{g}.
\] \(33\)

where \(\sqrt{g} \geq 0\) is assumed. The system of equations \(26\) and \(28\) can be easily solved, which gives
\[
y_{1,2}^2 = \exp \int \frac{f_+}{w_0 g} dz,
\] \(34\)

where
\[
f_\pm = w_0 \sqrt{g} \pm W.
\] \(35\)

Now let us insert \(y_{1,2} \) \(26\) back into Eq. \(24\). We obtain
\[
2gww'' + g'ww' - gw'^2 + 4hw^2 + W^2 = 0.
\] \(36\)

By this formula one can find the constant \(W^2 = 0\) from a knowledge of \(w(z)\) in a vicinity of any point \(z\). Thus, calculating \(36\) at any one finite turning point \(\zeta\) we obtain (in terms of expansions \(28\) with normalization \(w_0 = 1\))
\[
W^2 = -4h_0 - g_1 w_1.
\] \(37\)

Alternatively, one can take zeros \(z_i\) of \(w(z)\). (The functions \(g(z)\) and \(w(z)\) do not have common zeros because otherwise the Wronskian \(31\) would be zero.) Then we find
\[
W^2 = g(z_i)w^2(z_i).
\] \(38\)

The requirement for positivity of \(W^2\) determines the values of parameters of Eq. \(20\) (resonance zones) for which the resonant solutions exist.

Let us construct these solutions. Consider the intervals of monotonicity \(t_1 \leq t \leq t_2 (z_t \geq 0), t_2 \leq t \leq t_3 (z_t \leq 0), \) etc. According to \(22\) and \(23\) we can write
\[
X^2(t) = X^2(t_1) \exp \int_{t_1}^{t} \frac{f_+}{w_0 g} dz,
\]
\[
X^2(t) = X^2(t_2) \exp \int_{t_2}^{t} \frac{f_+}{w_0 g} dz, \text{ etc.}
\] \(39\)

To find the characteristic exponent \(\mu\) consider, e.g., the growing solution \(X_1(t)\). Setting \(t = t_2, z(t_2) = \zeta_2\) in the first equation of \(39\) and \(t = t_3, z(t_3) = \zeta_1\) in the second one, we can express \(X^2(t_3)\) through \(X^2(t_1)\). Using Eq. \(21\) and taking into account that \(t_3 = t_1 + \tau, \varphi(t + \tau) = \pm \varphi(t)\), we thus obtain
\[
\mu = -\frac{W}{2} \int_{\zeta_1}^{\zeta_2} \frac{dz}{w_0 g(z)}.
\] \(40\)

Recall that \(\tau\) is the period of \(z(t)\), the constant \(W\) is determined from Eq. \(37\) or Eq. \(38\), its sign being taken opposite to that of the integral in \(40\) to provide for positivity of \(\mu\). Since \(w(z)\) has zeros, the integrals in Eqs. \(37\) and \(38\) are understood as their principal values. Formula \(40\) is a simple generalization of the ones used previously in Refs. \(37, 38, 39, 40\).
B. Evaluation of the characteristic exponent

Let us return to Eq. (18). It can be written in the form of Eq. (20) if we set

\[ z(t) = -\ln(a/a_{\text{max}})^2, \quad z(0) = 0, \quad (41) \]

\[ h(z) = E - 3 + \omega_0^2 + z. \quad (42) \]

Equation (41) then immediately gives

\[ g(z) = 4[\omega_0^2(e^z - 1) - z]. \quad (43) \]

Zeros of this function are shown in Fig. 3. Since \( \xi(t) = a/a_{\text{max}} \) oscillates with the period \( T \) in the interval \(-1 \leq \xi \leq 1 \) [see Eqs. (7) and (12)], the function \( z(t) \) oscillates with the period \( \tau = T/2 \) between the turning points \( \zeta_1 = 0 \) and \( \zeta_2 = +\infty \). To calculate the characteristic exponent by the formula (40), we need to know the function \( w(z) \) which is the solution of Eq. (20) with boundary conditions (26). Unfortunately, for given \( h(z) \) (12) and \( g(z) \) (13), equation (20) cannot be solved analytically. We solve it numerically for various values of the parameters \( E \) and \( \omega_0^2 = 1 - \ln a_{\text{max}}^2 \). Doing so, we use the conditions (25) in the following way. As discussed above, the fulfillment of (25) at a finite turning point means analyticity of \( w(z) \) in some vicinity of this point. Therefore, we can use the expansions (28) setting there \( \zeta = \zeta_1 = 0, \quad g_0 = 0, \quad g_1 = 4(\omega_0^2 - 1), \quad g_n = 4\omega_0^2/n! \) \( (n = 2, 3, \ldots) \), \( h_0 = E + \omega_0^2 - 3, \quad h_1 = 1 \). Equation (30) then gives

\[ w_2 = -\frac{1 + (2E + 3\omega_0^2 - 6)w_1}{6(\omega_0^2 - 1)}. \quad (44) \]

We thus solve Eq. (20) with the following conditions at \( z = 0 \): \( w(0) = 1, \quad w'(0) = w_1, \quad w''(0) = 2w_2 \). Given values of \( E \) and \( \omega_0^2 \), we choose \( w_1 \) so as to satisfy the condition (25) at infinity,

\[ (w'\sqrt{g})_{z\rightarrow\infty} \rightarrow 0. \quad (45) \]

At the same time, since \( \mu \) assumed to be real, the values of \( E, \omega_0^2 \), and \( w_1 \) must provide for positivity of \( W^2 \),

\[ W^2 = 4[3 - E - \omega_0^2 - (\omega_0^2 - 1)w_1] > 0 \quad (46) \]

[see Eqs. (37) and (20)]. Conditions (15) and (16) determine the resonance zones in the space of parameters \( E \) and \( \omega_0^2 \) (or \( a_{\text{max}}^2 \)). Hereinafter the zones will be referred to as \( Z_j \) and numbered sequentially as \( E \) grows (with \( a_{\text{max}}^2 \) fixed) starting with \( j = -1 \) in the region \( E < 0 \). Figure 4 shows the solutions \( w(z) \) for zones \( Z_1, Z_2, \) and \( Z_3 \) lying in the region \( E > 2 \). Now, knowing \( w(z) \), we can calculate the integral in (40). Because \( w(z) \) has zeros, we first transform the integrand with the help of Eq. (39) extracting the total derivative. Owing to the condition (16), the latter does not contribute to the principal value of the integral, while the remaining terms give

\[ \int_0^\infty \frac{dz}{w\sqrt{g}} = -\frac{1}{2W^2} \int_0^\infty \sqrt{g}(w'\sqrt{g})' \ln w^2 + 8hw \] \[ \frac{dz}{\sqrt{g}}. \quad (47) \]

The integrand on the right-hand side of Eq. (47) is more convenient for numerical integration because its singularities are all integrable. We perform the integration in (47), calculate \( W^2 \) by the formula (40), and find the period \( T = 2\tau \) by integration of Eq. (7). These procedures are carried out numerically for a set of grid points in every resonance zone. In this way from (40) we obtain the characteristic exponent \( \mu \) as a function of \( E \) and \( a_{\text{max}}^2 \).

To check this result we derive \( \mu(E, a_{\text{max}}^2) \) directly from analysis of numerical solutions of Eq. (18). Examples of these solutions for resonance zones \( Z_1, Z_2, \) and \( Z_3 \) are shown in Fig. 5. The growth of the amplitude with time is clearly seen. The function \( \mu(E, a_{\text{max}}^2) \) so derived is found to be fully coincident with the one obtained by the formula (40).
FIG. 5: Resonant solutions of Eq. (18), \( \varphi(t) = \varphi(t) e^{it} \), (solid lines) and the function \( z(t) = -\ln(a/a_{\text{max}})^2 \) (dashed lines): (a) zone \( Z_1 \), (b) zone \( Z_2 \), (c) zone \( Z_3 \). The initial conditions are: \( a(0) = a_{\text{max}}, \ a_t(0) = 0, \ X_t(0) = 0 \). \( X(t) \) is normalized in a proper way. The values of \( E \) and \( a_{\text{max}}^2 \) in each zone are the same as in Fig. 4. It is seen that \( \varphi(t) \) is \( \tau \)-periodic in \( Z_1 \), \( \tau \)-antiperiodic in \( Z_2 \), \( \tau \)-periodic in \( Z_3 \), \( Z_2 \), \( Z_3 \), and so on, in accordance with the solutions [see Eqs. (35) and (38) and Fig. 4].

The resulting stability-instability chart is presented in Fig. 6. Figure 6(a) shows the region \( E > 2 \). There is an infinite series of narrow resonance zones \( Z_1, Z_2, Z_3, \ldots \), the first one having the highest magnitude \((\approx 0.08 \text{ at the maximum})\). All these zones originate from the point \( E = 2, a_{\text{max}}^2 = 1 \) at which \( \mu = 0 \) [see Eqs. (40) and (41)]. In the region \( E < 2 \) we have two zones, \( Z_0 \) and \( Z_{-1} \), lying in the ranges \( 0 < E < 2 \) and \( E < 0 \), correspondingly. Since in these zones the values of \( \mu \) proved to be much greater than in \( Z_1, Z_2, \ldots \), we depict the surface \( \mu(E, a_{\text{max}}^2) \) for this region separately, in Fig. 6(b).

FIG. 6: The stability-instability chart: (a) \( E > 2 \), first ten zones are shown, (b) \( E \leq 2 \).

C. Spatial structure of the perturbation

Consider now Eq. (19). It has the form of the Schrödinger equation for a quantum particle of the energy \( E \) moving in the potential \(-\ln u^2\). Since the potential tends to \(+\infty\) with growing \( r \) [as \( r^2 \), see Eqs. (13) and (14)], the energy spectrum is discrete, \( E = E_n \), and the corresponding eigenfunctions \( \Psi_n(r) \) are all localized. In the case of the nodeless pulsion [13] we have the isotropic harmonic oscillator. Its eigenfunctions are well known (see, e.g., [11]). We write them as follows:

\[
\Psi_n(r) = \sum_{l=0}^{n} [1 + (-1)^{n-l}] R_{nl}(r) Y_l(\theta, \varphi),
\]  

\[
R_{nl}(r) = r^l e^{-r^2/2} \Phi \left( -\frac{n-l}{2}, l + \frac{3}{2}, r^2 \right),
\]  

\[
Y_l(\theta, \varphi) = \sum_{m=-l}^{l} c_{l,m} P_l^{(m)}(\cos \theta) e^{im\varphi}.
\]

Here \( \Phi(\alpha, \gamma, x) \) is the Kummer function, \( P_l^{(m)}(x) \) are the associated Legendre functions, \( c_{l,m} \) are constants, \( c_{l,-m} = c_{l,m}^{*} \). The energy spectrum is given by

\[
E = E_n = 2n \quad (n = 0, 1, 2, \ldots).
\]

(Our energy levels are shifted with respect to the conventional ones since the minimum of the potential \(-\ln u_0^2\) is
In the case of the nodal pulsons the picture becomes more complicated due to the loss of the orbital degeneracy. The corresponding eigenfunctions and eigenvalues can be calculated only numerically. As an example, in Fig. 7 is shown the energy spectrum for perturbations of the one-nodal pulson.

Note that there always exist the eigenvalues \( E = 0, l = 0 \) and the corresponding eigenfunction \( \Psi_0(r) \propto u(r) \). This fact immediately follows from the comparison of Eqs. (14) and (19). The corresponding \( X_0(t) \) in \( \eta(t, r) \) is an oscillating function with the amplitude growing linearly with time. It is easy to see, however, that this mode is physically meaningless. Indeed, it will formally appear if we perturb the pulson by a small variation of its amplitude \( a_{\text{max}} \) but not the form \( u(r) \). Due to nonlinearity, this results in a pulson with slightly shifted frequency. Then the difference of the perturbed and unperturbed pulsons, i.e., \( \eta(t, r) \), will have the form of beats generated by two oscillations with close frequencies and the same profile \( u(r) \). The function \( X_0(t) \) approximates the initial, linearly growing part of a beat. We exclude this mode from the subsequent consideration, since it belongs to the class of perturbations that conserve a pulson as a whole. Next, for the nodal pulsons only, there is a mode with \( E = 0, l = 1 \) (see Fig. 7). Since this mode cannot grow faster than linearly in time, we also do not take it into account. Further, we should exclude the mode resulting from a small translation of the pulson. The corresponding eigenfunction is proportional to \( \mathbf{n} \mathbf{v} u \), where \( \mathbf{n} \) is a displacement vector. Using Eqs. (6) and (14) one can easily show that this mode corresponds to \( E = 2, l = 1 \). Thus the resulting perturbation is written as

\[
\eta(t, r) = \sum_n X_n(t) \Psi_n(r),
\]

where \( X_n \) is a solution of Eq. (18) with \( E = E_n, E_n \neq 0,2 \). If \( E_n \) and \( a_{\text{max}}^2 \) are in a resonance zone, \( X_n(t) \) will be represented as a linear combination of the solutions \( 24 \) and, hence, will grow with time as \( e^{\mu(E_n, a_{\text{max}}^2)t} \).

D. Instability of the pulsons

The arrangement of the resonance zones on the \( (E, a_{\text{max}}^2) \) plane indicates that for any spectrum \( E_n \) there always exist the ranges of \( a_{\text{max}}^2 \) where pulsons are unstable. But do the values of \( a_{\text{max}}^2 \) exist for which the pulsons are stable? To answer this question let us return to the surface \( \mu(E, a_{\text{max}}^2) \) depicted in Fig. 6. Take, at first, the spectrum for the nodeless pulson. We choose the sections \( \mu_n(a_{\text{max}}^2) \) of the surface \( \mu(E, a_{\text{max}}^2) \) by \( E = 2n \) and project them on the \( (\mu, a_{\text{max}}^2) \) plane. As a result, the pattern shown in Fig. 8(a) emerges. It is clearly seen the tendency to the total filling of the interval \( 0 < a_{\text{max}}^2 < 1 \) by the resonant peaks as the successively higher energy levels are accounted for. This implies that for any given \( a_{\text{max}}^2 \), there always exists an unstable mode with \( \mu = \mu_n(a_{\text{max}}^2) \), i.e., strictly speaking, all nodeless pulsons are unstable. On the other hand, the figure suggests that there are domains of \( a_{\text{max}}^2 \) where the peaks are very small. These domains are the gaps between the main peaks originated from the low-energy cross sections of the surface \( \mu(E, a_{\text{max}}^2) \) over a few first zones. In the gaps the exponent \( \mu \) is small, so that the corresponding pulsons are long-lived. For example, in Ref. 23 we observed numerically the nodeless pulson with \( a_{\text{max}}^2 = 0.49 \) that conserved its coherency against the radially symmetric perturbations over the course of several hundreds of periods.

Further, the above projective procedure is performed using the spectrum of the one-nodal pulson (Fig. 7). The main contribution here is made by the sections with the energies \( E_n = -0.7142 \) (\( l = 0 \)), 0.4833 (\( l = 2 \)), 1.1222 (\( l = 3 \)), and 1.8996 (\( l = 4 \)) falling into zones \( Z_{-1} \) and \( Z_0 \), the projections of the first and the third sections over-
lapping the other ones. The result is presented in Fig. 8(b). We see that $a^2_{\text{max}}$ axis is totally full. Thus, the one-nodal pulson has neither stability nor even quasi-stability domains. It seems likely that things will get worse, not better, if one goes to the multinodal pulsons. We thus conclude that, strictly speaking, all pulsons of the model considered are unstable. But nodeless pulsons can be quasi-stable in narrow ranges of amplitudes. It is the long-lived pulsons that can be of astrophysical and cosmological interest. If the dark matter consists of scalar long-lived pulsons that can be quasistable in narrow ranges of amplitudes. It is the model considered are unstable. But nodeless pulsons can thus conclude that, strictly speaking, all pulsons of the stability domains. It seems likely that things will get worse, if one goes to the multinodal pulsons. We see that $k\approx 0$ this state can be formally obtained from Eq. 11 if we set there $u(r)\equiv 1$. We thus assume that $\phi_0(t)$ obeys Eq. 15. Taking the perturbed state $\phi = \phi_0(t) + \eta(t,r)$, in the linear approximation from Eqs. 2 and 12 we readily obtain

$$A_{tt} + (k^2 - 2 - 2 \ln \phi_0^2)A = 0,$$

where $A(t,k)$ is the Fourier amplitude of the perturbation, and $k = |k|$. It is seen that the real and imaginary parts of this equation have the form of Eq. (20) with $h(z)$ given by Eq. (12), $z = -\ln(\phi_0/\phi_{0,\text{max}})^2$, $\omega^2_0 = 1 - \ln \phi_{0,\text{max}}$, and $E = k^2$. Returning to the stability-instability chart (Fig. 6) we note that in the region $E \geq 0$ maximal values of $\mu$ are attained in the zone $Z_0$ for which $0 < E < 2$. Interestingly, this band exactly coincides with the one obtained in Ref. 20 for the power-law potential approximating (39) when $\lambda \ll 1$. In the interior of $Z_0$ the exponent $\mu$ depends almost not at all on the amplitude of the condensate oscillations and is a sufficiently smooth function of $k^2$ with a maximum at $k^2 = k^2_0 \approx 1$ where $\mu \approx 0.5$. Therefore, if the initial power spectrum $|A(0,k)|^2$ lies in the region $0 \lesssim |k| \lesssim \sqrt{2}$ and, in addition, its characteristic width along $k$ is small, $\Delta k \ll \sqrt{2}$, then the growth of the perturbation amplitude will not be accompanied by significant changes in the structure of the perturbation. The limiting case of such perturbations is a harmonic wave. Otherwise, if $\Delta k \gtrsim \sqrt{2}$, the shape of the power spectrum will vary with time so that a maximum will appear at $k_0 \approx 1$. As a result, the effective width of the spectrum will become smaller, $\Delta k \lesssim 1$. In this case, if the initial spectrum is sufficiently isotropic in $k$ space, the parametric amplification of the perturbations will result in the emergence of the localized field configurations of the characteristic size $\Delta k \sim 1/\Delta k \gtrsim 1$ that agrees with the radius of the gausson (see Sec. II). At this scale the field practically does not undergo spatial oscillations since the corresponding wavelength $2\pi/k_0 \gtrsim 1$. We thus expect that at the nonlinear stage these configurations will turn into the nodeless pulsons. Their period will be equal to the period of the condensate oscillations since in the zone $Z_0$ the parametric amplification proceeds at the basic frequency. Gradually, the energy of the oscillating condensate will go to ensemble of the arising pulsons, this process resulting in the damping of the background oscillations. As to the pulsons themselves, they can be long-lived or short-lived depending on their amplitudes, in accordance with the results of the previous Subsection.

Note, that numerical simulations performed for the complex version of the model 2 and 39 have shown the fragmentation of both the rotating 12 10 and oscillating 20 Affleck-Dine condensate. The localized configurations arising in the condensate have been identified with Q-balls. We belive, however, that the configurations observed in the oscillating condensate are in fact the complex pulsos (see Sec. IV), rather than the usual Q-balls. This possibility was early discussed in Ref. 12 where an attempt to simulate the complex pulson has been made.

The resonant exitation of the pulsos was also observed in the two-vacuum $\phi^4 - \phi^6$ model within a regularly oscilating background 39 and in the $\phi^4$ model within an initially thermalized background 43. Note that in two-vacuum models the pulsos can play the role of nuclei of a new phase. In Ref. 39 the general suggestion has been made that the parametric resonance can underlie the mechanism responsible for the first-order phase transitions in nonlinear non-dissipative systems. This conjecture turns out to be in agreement with recent results of Ref. 42 where the resonant nucleation within the thermalized background have been numerically observed in the $\phi^3 - \phi^4$ model. Note, in addition, that the dynamical nucleation can also take place in the nonlinear Schrödinger equation 45.

### IV. CONCLUDING REMARKS

In this paper we have examined only the linear stage of instability at which small deformations of the pulson’s shape result in loss of the coherence. There is numerical evidence that in time the growth of the perturbations becomes saturated due to nonlinear effects 22. We thus suggest that in the model considered the pulsos, while unstable, remain well localized objects with no tendency for spreading or collapsing.

Further, we dealt with a real scalar field. It would be interesting to perform the similar analysis for a complex scalar field too. It is believed that the existence of the scalar charge can stabilize a field lump. For Q-balls this fact is well established (so-called Q-theorem 2, 4, 27). In contrast, for the complex pulsos this is an open question. As it was shown in Refs. 25 29, the field equation 2 with $U' = -\phi \ln(\phi \phi^*)$ admits the exact pulson solutions

---

[The rest of the text would follow here, including any necessary notes or references.]
of the form \( \phi_0(t,r) = a(t)u(r)e^{i\theta(t)} \), where \( a(t) \), \( u(r) \), and \( \theta(t) \) are real. The function \( u(r) \) satisfies Eq. (4) as before, while \( a(t) \) oscillates with a period \( T \) in accordance with the equation

\[
a_{tt} = -\frac{d}{dt} \left[ \frac{1}{2}a^2(1 - \ln a^2) + \frac{q^2}{2a^2} \right], \tag{54}
\]

where \( q \) is a real constant, \( q^2 < (2e)^{-1} \), and \( \theta_0 = qa^{-2} \). The constant \( q \) is proportional to the charge of the scalar field. In contrast to Eq. (4), the potential in the square brackets of Eq. (54) prevents \( a(t) \) from being zero. Without loss of generality one may assume \( a(t) \) positive, so that the oscillations occur around the minimum of the potential at \( a = a_0 \), where \( a_0 \) is the least positive root of the equation \( a^4 \ln a^2 = -q^2 \). If \( a \) is at rest in this minimum, then \( \theta(t) = qa_0^{-2}t + \theta(0) \), and we have the standard Q-ball. Physically, Eq. (54) describes the motion of a mechanical particle with an angular momentum \( q \) in the potential \( (a^2/2)(1 - \ln a^2) \). The condition for its trajectory to be closed is \( \theta(T) - \theta(0) = 2\pi m/n \), where \( m \) and \( n \) are arbitrary integers. In fact, it relates the energy of the particle and its angular momentum whereby such trajectories exist. In our case this means periodicity of the solution \( \phi_0(t,r) \) with the period \( nT \). Obviously, there is an infinity of such solutions. Taking \( \phi_0(t,r) \) and considering the partial perturbation \( \eta \propto X(t)\Psi(r) \) one can find that the function \( \Psi(r) \), assumed to be real, satisfies Eq. (1) as before, while \( X(t) \) obeys the equation

\[
X_{tt} + (E - 1 - \ln a^2)X = e^{2i\theta}X^*, \tag{55}
\]

where \( E \) is a real constant. This equation can be represented as a system of four real first-order equations with periodic coefficients of the periods \( T \) and \( nT \). It is significant that, since \( a(t) \neq 0 \), these coefficients are bounded in time, so that one can attempt to estimate the characteristic exponent of the system using the standard methods [17].

Also, it would be interesting to examine stability of a self-gravitating pulson. Hopefully, gravitation can expand the domains of (quasi)stability, as it is the case for Q-balls [4]. These are possible subjects of our future work.

In the present paper we have investigated stability of both the coherent localized states (pulsons) and nonlocalized states (uniformly oscillating scalar condensate) of the real scalar field. Our main analytical result is the generalization of the Lindemann-Stieltjes method to the case that the periodic coefficient in the Hill equation is unbounded in time. Our main numerical result is the stability-instability chart with the values of characteristic exponent calculated in the resonance zones. Using this chart we have found the gaps in the set of the pulson amplitude values in which the real nodeless pulsions conserve the coherency for an extremely long time. Also, considering the oscillating scalar condensate, we have determined the wavelength of the most unstable mode. This wavelength turned out to be equal to the characteristic size of the nodeless pulson. We thus suggest the pulsions can be formed due to resonant fragmentation of the scalar condensate. These are our main physical results.

Acknowledgments

The authors thank I. Bogolubsky, Yu.P. Rybakov, and A. Shagalov for useful discussions. This work was partly supported by the RAS Presidium Program “Nonstationary phenomena in astronomy”.