Cosmological constraints on \( f(R) \) gravity theories within the Palatini approach

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Received date / Accepted date

Abstract. We investigate \( f(R) \) theories of gravity within the Palatini approach and show how one can determine the expansion history, \( H(a) \), for an arbitrary choice of \( f(R) \). As an example, we consider cosmological constraints on such theories arising from the type Ia supernova observations. We find that best fit to the data is a non-null leading order correction to the Einstein gravity, but the current data exhibits no significant preference over the concordance \( \Lambda CDM \) model. Our results show that the often considered \( 1/R \) models are not compatible with the data. The results demonstrate that the background expansion alone can act as a good discriminator between modified gravity models when multiple data sets are used.

1. Introduction

The combination of Einstein’s General Relativity (GR) and ordinary matter, as described by the standard model of particle physics (Eidelman et al. 2004), cannot explain the current cosmological observations. Key observations confronting the matter only universe are the luminosity-redshift relationship from observations of supernovae type Ia (SNIa) (Riess et al. 2004), the power spectrum of large scale structure as inferred from galaxy redshift surveys like the Sloan Digital Sky Survey (SDSS) (Tegmark et al. 2004) and the 2dF Galaxy Redshift Survey (2dFGRS) (Colless et al. 2001), and the anisotropies in the Cosmic Microwave Background Radiation (CMBR) (Spergel et al. 2003). In order to account for the results from all of these cosmological probes within GR, two exotic components are required in the matter-energy budget of the Universe. These two components are respectively dark matter, a collisionless and pressureless fluid, which contributes about 25% of the universe energy budget, and a negative pressure fluid called dark energy. Currently the dark energy component dominates the energy density of the universe, causing accelerating expansion.


In this article we investigate a family of alternative models to the dark energy paradigm based on a generalization of the Einstein-Hilbert Lagrangian. These models are called Nonlinear Theories of Gravity (Bronnikov & Sokolowski 1994, Allemandi et al. 2000) or \( f(R) \) theories (Bronnikov & Chernikova 2005, Cognola et al. 2005, Nunez & Solganik 2004, Ezawa et al. 2003, Barraco et al. 2000, Schmidt 1998, Rippl et al. 1996, Barrow & Ottewill 1983), since the scalar curvature \( R \) in the Einstein-Hilbert Lagrangian is replaced by a general function \( f(R) \). The main motivation for this generalization is the fact that higher order terms in curvature invariants (such as \( R^2, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \))
et al) have to be added to the effective Lagrangian of the gravitational field when quantum corrections are considered (Buchbinder et al 1992, Birrell & Davies 1982, Viklovsky 1992, Gasperini & Veneziano 1992). Furthermore, there is no a \textit{priori} reason to restrict oneself to the simple Einstein-Hilbert action when a more general formulation is allowed. Higher order terms in the gravitational action also have interesting consequences in cosmology, like natural inflationary behavior at early times (Starobinsky 1980, Barrow & Cotsakis 1988, 1991, La & Steinhardt 1989), and late time acceleration of the Universe (Meng & Wang 2003, 2004, Carroll et al 2004, Quandt & Schmidt 1991, Dominguez & Barraco 2003). Consequently, several authors have investigated whether such theories are indeed compatible with current cosmological observations, big bang nucleosynthesis, and solar system constraints in its weak field limit (Clifton & Barrow 2005, Barrow & Clifton 2006, Quandt & Schmidt 1991, Domínguez & Barraco 2004, Sotiriou 2005a, b, Olmo 2005a, b, Capozziello 2002, Chiba 2003, Mena et al. 2005, Barraco et al. 2002, Hwang & Noh 2001, Mena et al. 2005, Allemendi et al. 2004a, b, 2005). Most of these investigations have been model dependent and the conclusions are somewhat contradictory (Planasari 2004, Vollick 2004).

When dealing with \( f(R) \) theories of gravity, the choice of the independent fields to vary in the action is a fundamental issue. In the so-called Palatini approach one considers the metric and the connection to be independent of each other, and the resulting field equations are in general different from those one gets from varying only the metric, the so-called metric approach. The two approaches lead to the same equations only if the resulting field equations are in general different from those one gets from varying only the metric, the so-called metric approach. The two approaches lead to the same equations only if \( f(R) \) is linear in \( R \). The correct choice of approach to derive the field equations is still a hot topic of research. Initially, \( f(R) \) theories were investigated in the metric approach. However, since this method leads to fourth-order equations and the Palatini approach leads to second-order equations, the latter is appealing because of its simplicity. Moreover, the equations resulting from the metric approach seem to have instability problems in many interesting cases (Dolgov & Kawasaki 2003, Chiba 2003) from which the Palatini approach does not suffer. However, recent work has cast doubts over these instabilities (Cembranos 2003, Sotiriou 2005). In the present paper we will concentrate on the Palatini approach.

The aim of this article is to use current cosmological data to consider possible deviations from GR by combining a number of different cosmological observations. The data used are the latest Supernovae Ia gold set (Riess et al 2004), the CMBR shift parameter (Bond et al. 1997), the baryon oscillation length scale (Eisenstein et al. 2005) and the linear growth factor at the 2dFGRS effective redshift (Hawkins et al. 2003). As far as we are aware, this is the first time one uses all of the main cosmological data sets in order to constrain these models.

The structure of the paper is as follows: In section 2 we analyze the evolution of linear perturbations and probe large scale structure formation in these models using the linear growth factor derived from the 2dFGRS data. Finally, section 3 contains a summary of our work and our conclusions.

2. General \( f(R) \) Gravity theories

The action that defines \( f(R) \) gravity theories in the Palatini formalism is the following:

\[
S[f; g, \Gamma, \psi_m] = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_m[g_{\mu\nu}, \psi_m] \tag{1}
\]

where \( \kappa = 8\pi G, S_m[g_{\mu\nu}, \psi_m] \) is the matter action which depends only on the metric \( g_{\mu\nu} \) and on the matter fields \( \psi_m \), \( R \equiv R(g, \Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma) \) is the generalized Ricci scalar and \( R_{\mu\nu}(\Gamma) \) is the Ricci tensor of the affine connection \( \Gamma \), which in the Palatini approach is independent of the metric. The generalized Riemann tensor is given by (Vollick 2003):

\[
R'_{\mu\nu\beta} = \Gamma^\alpha_{\mu\rho\beta} - \Gamma^\alpha_{\mu\beta\rho} + \Gamma^\lambda_{\mu\beta\lambda} \Gamma^\alpha_{\lambda\rho} - \Gamma^\lambda_{\mu\rho\lambda} \Gamma^\alpha_{\lambda\beta}. \tag{2}
\]

We define the Ricci tensor by contracting the first and the third indices of the Riemann tensor. The field equations will be obtained using the Palatini formalism i.e. we vary both with respect to the metric and to the connection.

Varying the above action with respect to the metric, we obtain the generalized Einstein equations

\[
f'(R)R_{\mu\nu}(\Gamma) - \frac{1}{2} f(R) g_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{3}
\]

where \( f'(R) \equiv df/dR \) and \( T_{\mu\nu} \) is the energy momentum tensor

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{4}
\]

Varying with respect to the connection \( \Gamma \) and contracting gives us the equation that determines the generalized connection (Vollick 2003):

\[
\hat{\nabla}_\mu [f'(R) \sqrt{-g} g^{\mu\nu}] = 0, \tag{5}
\]

where \( \hat{\nabla} \) is the covariant derivative with respect to the affine connection \( \hat{\Gamma} \). This equation implies that we can write the affine connection as \( \Gamma_{\mu\nu} = f'(R) g_{\mu\nu} \). The Levi-Civita connection of a new metric \( h_{\mu\nu} = f'(R) g_{\mu\nu} \). The Levi-Civita connections of the metrics \( g_{\mu\nu} \) and \( h_{\mu\nu} \) are then related by a conformal transformation. This allows us to write the affine connection as

\[
\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} + \frac{1}{2f'} [\delta^\sigma_{\nu\rho} f'' - g^{\sigma\tau} g_{\mu\tau} \partial_\rho f''], \tag{6}
\]

where \( \Gamma^\sigma_{\mu\nu} \) is the Levi-Civita connection of the metric \( g_{\mu\nu} \).

The generalized Ricci tensor can now be written as

\[
R_{\mu\nu} = R_{\mu\nu}(g) - \frac{3}{2} \frac{\nabla_{\mu} f' \nabla_{\nu} f'}{f^2} + \frac{\nabla_{\mu} f \nabla_{\nu} f'}{f} + \frac{1}{2g_{\mu\nu}} \frac{\nabla^\sigma \nabla_{\mu} f'}{f}. \tag{7}
\]

Here \( R_{\mu\nu}(g) \) is the Ricci tensor associated with \( g_{\mu\nu} \) and \( \nabla_\mu \) the covariant derivative associated with the Levi-Civita connection of the metric.
Since we are interested in cosmological solutions we consider the spatially flat FRW metric,
\[ ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j. \] (8)
and the perfect fluid energy-momentum tensor \( T_{\mu\nu}^\text{f} = \text{diag}(-p, p, p, p) \).

2.1. The leading correction to Einstein gravity

In order to investigate to what extent observations allow deviations from general relativity (where \( f(R) = R \)), consider the following gravity Lagrangian:
\[ f(R) = R \left( 1 + \alpha \left( -\frac{R}{H_0^2} \right)^{\beta-1} \right), \] (16)
where \( \alpha, \beta \) are dimensionless parameters (note that in out notation \( R \) is negative). Specializing to a matter dominated universe \((w = 0)\), we get from Eq. (9) the relation between the curvature scalar and matter density:
\[ \kappa \rho_m = -R \left( 1 + \alpha(2 - \beta) \left( -\frac{R}{H_0^2} \right)^{\beta-1} \right). \] (17)
We wish to recover standard behaviour at early times. This implies that the correction term in the Lagrangian must vanish for large \(|R|\), and hence we must demand that \( \beta < 1 \). Furthermore, we must demand that the right hand sides of equations (16) and (17) always remain positive, which restricts the parameter space further.

Defining \( \Omega_m \equiv \kappa \rho_m^0/(3H_0^2) \), and choosing units so that \( H_0 = 1 \), we can solve for \( R_0 \) and \( \Omega_m \), and hence that substituting the obtained value of \( R_0 \) into Eq. (12) must give \( H_0 = 1 \). Hence, given \( \alpha, \beta, \Omega_m \) is fixed.

As an example, consider the case \( \beta = 0 \), which corresponds to the \( \Lambda \)CDM model. From Eq. (12), we have \( H_0^2 = (3\alpha H_0^2 - 2R_0)/6 \) and from Eq. (17), \( 3\Omega_m H_0^2 = 2\alpha H_0^2 - R_0 \), and hence \( \Omega_m = 1 + \alpha/6 \) is fixed.

3. Observational constraints from the Background Evolution

Armed with the modified Friedmann equation, we can now consider the constraints arising from cosmological observations. In this section we will consider quantities related to the background expansion of the Universe: the SNIa luminosity distance-redshift relationship, the CMBR shift parameter and the baryon oscillation length scale.

3.1. CMBR Shift Parameter

The CMBR shift parameter (Bond et al. 1997; Melchiorri et al. 2003; Odman et al. 2003) in a spatially flat universe is given by
\[ \mathcal{R} = \sqrt{\Omega_m H_0^2} \int_0^{z_{\text{dec}}} \frac{dz}{H(z)}. \] (18)
where \( z_{\text{dec}} \) is the redshift at decoupling. The WMAP team (Spergel et al. 2003) quotes \( z_{\text{dec}} = 1088_{-2}^{+1} \) and \( \mathcal{R} = 1.716 \pm 0.062 \). In writing the shift parameter in this form we have implicitly assumed that photons follow geodesics determined by the Levi-Civita connection. In (Koivisto 2005) this is shown to be the case if there is no torsion present. Furthermore, in order to use the shift parameter, the evolution of the universe needs to be standard up to very late times so that at decoupling we recover standard matter dominated behaviour. Hence, we restrict our analysis to \( \beta < 1 \).
In order to incorporate measurements from SNIa, it is useful to rewrite the expression for the luminosity distance as
\[
D_L (z) = (1 + z) \int^z_0 \frac{dz}{H(z)} = \frac{1}{3} \sqrt{\Omega_m H_0^2 \frac{R}{H(R)}} \int^{R_f}_0 \frac{R f'' - f'}{R (R f' - 2 f)^{2/3}} \frac{dR}{H(R)}.
\]

For the supernova data, we use the “Gold data set” from Riess et al. (2004). The contour plots showing the constraints on the parameters from the supernovae can be seen in figure 2. To get these plots we marginalized over the Hubble parameter $h$.

With the added information from the CMBR in the form of the shift parameter, the situation improves as one can see from figure 3. Still quite a large degeneracy persists on the 99% level, but on the 68% level, the model is quite well constrained and centered around the concordance $\Lambda$CDM model.
recombination, which can be estimated from the CMBR data (Spergel et al. 2003). Measuring the apparent size of the oscillations in a galaxy survey allows one to measure the angular diameter distance at the survey redshift. Together with the angular size of the CMB sound horizon, the baryon oscillation size is a powerful probe of the properties and evolution of the universe.

The imprint of the primordial baryon-photon acoustic oscillations in the matter power spectrum provides us therefore with a ‘standard ruler’ via the dimensionless quantity $A$ (Linder 2003, 2005; Hu & Sugiyama 1996; Eisenstein & Hu 1998; Eisenstein & White 2004):

$$A = \sqrt{\Omega_m} E(z_1)^{-1/3} \left[ \frac{1}{z_1} \int_0^{z_1} \frac{dz}{E(z)} \right]^{2/3}$$

where $E(z) = H(z)/H_0$.

Recently the acoustic signature associated with the baryonic oscillations has been identified at low redshifts in the distributions of Luminous Red Galaxies in the Sloan Digital Sky Survey (Eisenstein et al. 2005), with a value of

$$A = D_A(z = 0.35) \frac{\sqrt{\Omega_m H_0^2}}{0.35c} = 0.469 \pm 0.017,$$

where

$$D_A(z) = \left[ D_M(z)^2 \frac{cz}{H(z)} \right]^{1/3}$$

and $D_M(z)$ is the comoving angular diameter distance. For instance, in the case of $\Lambda$CDM with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ and $h = 0.7$ we have $D_A(z = 0.35) = 1334$ Mpc.

Using the observed baryon oscillation length scale, one can hence constrain the cosmological model. The confidence contours for the modified gravity model we are concerned with in this paper, are shown in figure 4.

Once again there is a large degeneracy in the $\alpha$ vs $\beta$ plane, similar to the case when we fit SNIa (figure 2) and the CMBR shift parameter (figure 1). Such degeneracies are strongly restricted, however, when one combines all of the data sets in one single plot (see figure 5).

Using the baryon oscillations length scales and combining it with the SNIa and the CMBR shift parameter data hence imposes strong constraints on possible deviations from Einstein’s General Relativity. This also demonstrates how combining the current data, one can efficiently study and constrain cosmological models from the background expansion only.

The best fit model to the three data sets is $\alpha = -3.6$ and $\beta = 0.09$. This is slightly different from the $\Lambda$CDM model, but the $\Lambda$CDM model is well within the 68% confidence contour.

4. Large Scale Structure: Formalism and Constraints

So far we have only considered observables related to the background evolution. In order to gain further information, it is useful to go beyond these ‘zeroth order’ tests and consider perturbations within $f(R)$ models. Knowledge on the evolution of perturbations allows one to confront models with large scale structure observations from galaxy surveys.

Several authors have investigated cosmological perturbations in generalized gravity theories in the metric approach (Hwang 1991d). However within the Palatini formalism, only very recently the first steps were made (Koivisto & Kurki-Suonio 2005). Here we will follow a spherical collapse formalism (Lue et al. 2004a; Multamaki et al. 2005) where by requiring that a general gravity theory respects
the Jebsen-Birkhoff theorem\(^2\) one derives the modified gravitational force law necessary to describe the evolution of the density perturbations at astrophysical scales.

### 4.1. Spherical collapse in \(f(R)\) theories

Although the spherical collapse model has been set and used for a long time (Peebles 1993; Padmanabhan 1993) and in many different contexts (Lue et al. 2004a; Mota & Barrow 2004b; Mota & van de Bruck 2004; Clifton et al. 2005), it has not been applied previously to Nonlinear Gravity Theories. In [Lue et al. (2004a)], one starts by assuming a generalization of the Jebsen-Birkhoff theorem: for any test particle outside a spherically symmetric matter source, the metric observed by that test particle is equivalent to that of a point source of the same mass located at the center of the sphere. With this one assumption, one can deduce the Schwarzschild-like metric of the new hypothetical gravity theory. Armed with the Schwarzschild-like metric one can then investigate the evolution of spherical matter overdensities and compare it to the latest large scale structure data. The idea is the following: Consider a uniform sphere of dust. Imagine that the evolution inside the sphere is exactly cosmological, while outside the sphere is empty space, whose metric (given the Jebsen-Birkhoff theorem) is Schwarzschild-like (as defined by the metric equation (2.4) in [Lue et al. (2004a)]). The mass of the matter source (as determined by the form of the metric at short distances) is unchanged throughout its time-evolution. The surface of the spherical mass therefore charts out the metric through all of space as the sphere expands with time, as long as we define the cosmological metric inside the sphere without the need to consider what is happening outside the spherical mass itself. This is only possible if spherically symmetric configurations respect the metric equations (25) [Lue et al. 2004a]. Differentiating equation (26) twice with respect to time, and using equation (27), one obtains a new equation for \(\delta(t)\)

\[
\dot{\delta} + 2H\delta - \frac{4}{3} \frac{1}{1 + \delta} \dot{\delta}^2 = 3(1 + \delta)H_0^2 \left[ \frac{3}{2} x(1 + \delta)g'(x(1 + \delta)) - g(x(1 + \delta)) \right] - 3(1 + \delta)H_0^2 \left[ \frac{3}{2} xg'(x) - g(x) \right].
\]
solved will then be of the form

\[ H = \frac{d}{dt}\left(\alpha + \beta R\right). \]

At linear order in perturbation theory, equation (30) gives:

\[ \delta + 2H\delta = 4\pi G\rho \left[ g'(x) + 3x g''(x) \right]. \quad (31) \]

In our case both \( H \) and \( \rho \) are expressed as functions of the scalar curvature \( R \) and not the time explicitly. We therefore need to rewrite (31) in terms of derivatives of \( R \). The equation to be solved will then be of the form

\[ \frac{d^2\delta}{dR^2} + A(R) \frac{d\delta}{dR} = B(R)\delta. \quad (32) \]

where \( A(R) \) and \( B(R) \) are two rather unattractive functions of the scalar curvature. Using Eq. (32), one can solve for the evolution of the linear growth factor and compare to observations.

4.1.2. Constraints

The large scale structure information we choose to use here is the linear growth rate \( F(\alpha, \beta) = 0.51 \pm 0.11 \) measured by the 2dFGRS (Verde et al. 2002; Knop et al. 2003; Hawkins et al. 2003), where \( F = d \ln D/dlna \). We compare the theoretical value we get for the linear growth rate for our model with the value measured by the 2dFGRS at its effective redshift, \( z_{\text{eff}} = 0.15 \). The constraints arising from the linear growth data sets are plotted in figure 6. Combining this with all the other constraints leads to the confidence contours shown in Fig. 7.

Fig. 6. The 68, 95 and 99% confidence contours arising from fitting the linear growth factor using the spherical collapse approach. The parameter values corresponding to the concordance ΛCDM model (\( \Omega_m = 0.27, \Omega_\Lambda = 0.73 \) or \( \alpha = -4.38, \beta = 0 \)) are marked with a cross. The gray area represents a section of the parameter space that is not allowed.

In particular we have investigated the possible form of the leading correction to standard GR, parameterizing the Lagrangian for gravity as \( \mathcal{L}_G = R - \alpha R/H^2_0 \), and have used a combination of data sets to determine the allowed ranges of \( \alpha \) and \( \beta \). This is by no means an exhaustive study. Other interesting forms of \( f(R) \) that are definitely worth studying include \( f(R) = \ln(R) \) and perhaps especially \( f(R) = R - c_1/R + c_2 R^3 - c_3 R^4 \) (Sotiriou 2005a), but our main purpose here has been to set up the formalism and demonstrate the effectiveness of combining the current data sets.

Using a combination of data sets that probe the background evolution, we have found that the current data efficiently constrains the allowed parameter space of the leading correction to GR in the Palatini approach. The best fit models to the individual data sets are \( \alpha, \beta = (-10.0, -0.51) \) for the SNIa, \( \alpha, \beta = (-8.4, -0.27) \) for the CMBR shift parameter and \( \alpha, \beta = (-1.1, 0.57) \) for the baryon oscillations. The best fit to the combination of these data sets is \( \alpha, \beta = (-3.6, 0.09) \), but the ΛCDM concordance model is well within the 1σ contour. Note, however, that the commonly considered 1/R model is strongly disfavoured by the data.

In order to bring in additional information from the current galaxy surveys, we have also considered the growth of structures in these models of modified gravity. By assuming that the new gravitational physics obeys a limited version of the Jebsen-Birkhoff theorem, we can describe the evolution of overdensities in \( f(R) \) gravity theories at sub-horizon scales. We find the best fit model to the linear growth factor alone to be \( \alpha, \beta = (-4.25, 0.05) \) but the allowed parameter range is degenerate and does not improve constraints derived from the background evolution. In order to fully utilize the information available from the galaxy survey in form of the large scale mat-

5. Conclusions

We have investigated observational constraints on \( f(R) \) theories within the Palatini formalism. In order to relate these theories to observations, we have shown how one can determine the expansion history for a given \( f(R) \). In particular, in a matter dominated universe, determining \( H(a) \) is straightforward as expressed by Eqs. (14) and (15).
ter power spectrum, a more detailed analysis is needed along the lines presented in (Koivisto & Kurki-Suonio 2003).

In summary, modified gravities provide us with an interesting alternative to the cosmological concordance model with a dominant dark energy component. Modern cosmological data can efficiently constrain such models. These data indicate that currently there is no compelling evidence for non-standard gravity.

Acknowledgements. We would like to thank T. Koivisto for useful discussion. MA, ØE and DFM acknowledge support from the Research Council of Norway through project numbers 159637/V30 and 162830/V00. TM is supported by the Academy of Finland through project number 108658.

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