Acoustic horizons for axially and spherically symmetric fluid flow

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We investigate the formation of acoustic horizons for an inviscid fluid moving in a pipe in the case of stationary and axi-symmetric flow. We show that, differently from what is generally believed, the acoustic horizon forms in correspondence of either a local minimum or maximum of the flux tube cross-section. Similarly, the external potential is required to have either a maximum or a minimum at the horizon, so that the external force has to vanish there. Choosing a power-law equation of state for the fluid, \( P \propto \rho^n \), we solve the equations of the fluid dynamics and show that the two possibilities are realized respectively for \( n > -1 \) and \( n < -1 \). These results are extended also to the case of spherically symmetric flow.

Sonic horizons can be generated by the motion of a classical fluid. The simplest case is represented by a fluid with space-dependent velocity flowing through a tube with variable section (Laval nozzle). If the fluid flows in the direction where the tube becomes narrower, the fluid velocity \( v \) will increase downstream. Eventually, a point will be reached where \( v \) is equal to the local speed of sound \( c \) and \( v > c \) beyond this point. An acoustic perturbation generated in the supersonic region cannot reach the upstream subsonic region. We have generated a sonic horizon, the acoustic analogue of a spacetime horizon. The downstream subsonic region behaves, as long as only acoustic disturbances are concerned, as the acoustic analogue of a black hole, a *dumb hole.*

Starting from the original proposal of Unruh, acoustic (and other condensed matter analogues) black holes have been used to describe various kinematical aspects of general relativity such as event and cosmological horizons, field theory in curved spacetime, Hawking radiation etc. Recently, the fluid/gravity analogy has been extended also at a dynamical level. It has been used to gain information about Einstein’s equations governing the dynamics of a spherically symmetric gravitational black hole. The lack of direct experimental tests has always been an obstacle for the research on black holes. Investigations on “artificial” black holes could represent a useful bridge to make black holes experimentally more accessible.

Although conceptually very simple, the realization of acoustic horizons requires fine-tuning of the external potential acting on the fluid, of the tube profile and of the initial fluid velocity. In the case of the Laval nozzle, it is widely believed that the horizon must form exactly at the narrowest part of the nozzle. However, it is not easy to find necessary and sufficient conditions for the formation of the horizon. In the general case, the equations governing the dynamics of the fluid are very difficult to solve.

In this paper we investigate the formation of acoustic horizons in the case of a inviscid fluid with stationary, axially or spherically symmetric flow. When considering axi-symmetric fluid motion we will discuss separately the case of a flux tube of varying section and the presence of external forces. We show that, differently from what is generally believed, the acoustic horizon forms in correspondence of either a local minimum or maximum of the flux tube cross-section. Similarly, the external potential is required to have either a maximum or a minimum at the horizon, so that the external force has to vanish there. Choosing a power-law equation of state for the fluid, \( P \propto \rho^n \), we solve the equations of the dynamics and show that the two possibilities are realized respectively for \( n > -1 \) and \( n < -1 \). Moreover, we will determine the range in the initial conditions for which the horizon effectively forms.

**Axi-symmetric flow**

Let us first consider axially symmetric, stationary, inviscid fluid motion, which is constrained on a pipe. Indicating with \( x \) the coordinate along the symmetry axis, the fluid is described by its velocity \( v(x) \), density \( \rho(x) \) and pressure \( P(x) \). The flux tube is characterized by its cross-section \( A(x) \) and we will assume that external forces, characterized by the potential \( \psi(x) \), act on the fluid. Because of the symmetries of the problem all the parameters are functions of the coordinate \( x \) only. The equations governing the dynamics of the fluid are the continuity and Euler equations

\[
\frac{d(\rho A v)}{dx} = 0, \quad \rho v \frac{dv}{dx} = -\frac{dP}{dx} - \rho \frac{d\psi}{dx}. \tag{1}
\]

We will restrict ourself to the case of isentropic fluid flow \( P = P(\rho) \), which gives for the local speed of sound \( c = \sqrt{\frac{dP}{d\rho}} \).
We will discuss separately the case of a flux tube of variable cross-section and homogeneous external potential and that of a flux tube of constant section and variable external potential.

**Flux tube with variable cross-section**

Using Eqs. (1) with \( \psi = \text{const.} \), one easily obtain the well-known nozzle equations:

\[
\frac{1}{A} \frac{dA}{dx} = \frac{1}{v} \frac{dv}{dx} (M^2 - 1),
\]

where \( M = v/c \) is the Mach number. The nozzle equation (2) can be used to discuss the conditions for the formation of sonic horizons, i.e. of a surface with \( M = 1 \) separating a subsonic \( (M < 1) \) from a supersonic \( (M > 1) \) region. Assuming that \( v, v'(x) = dv/dx, A \) are everywhere finite and nonvanishing, one can easily see from Eq. (2) that, the if a horizon forms, this necessarily happens in correspondence of a local extremum of the tube cross-section \( A \).

Conversely, if the cross-section \( A(x) \) has a local extremum at a point \( x_h \) necessarily a sonic horizon must form at this point. Depending on the sign of \( v'(x) \) we have two different situations.

(i) The fluid velocity grows monotonically passing from the subsonic to the supersonic region. In this case the cross-section \( A \) has a local minimum at the horizon. This corresponds to the usual situation. The flux tube has the shape of the Laval nozzle, a converging pipe where the fluid is accelerated, followed by a throat and by a diverging pipe where the fluid continues to accelerate. The corresponding flow situation and qualitative behavior of the parameters are shown in Fig. (1).

(ii) The fluid velocity decreases monotonically passing from the subsonic to the supersonic region. In this case the cross-section \( A \) has a local maximum at the sonic horizon. This flow situation is very unusual and until now has not been considered in the literature. The flux tube has the form of a diverging pipe where the fluid is decelerated followed

FIG. 1: Flow situation (on the left) and qualitative behavior of the parameters (on the right) for a converging-diverging nozzle. The black, grey and dashed lines represent the tube cross-section \( A \), the fluid velocity \( v \) and the speed of sound \( c \), respectively. The vertical axis indicates the position of the horizon. \( c \) and \( v \) are normalized to their horizon values.

FIG. 2: Flow situation (on the left) and qualitative behavior of the parameters (on the right) for a diverging-converging nozzle. The black, grey and dashed lines represent the tube cross-section \( A \), the fluid velocity \( v \) and the speed of sound \( c \), respectively. The vertical axis indicates the position of the horizon. \( c \) and \( v \) are normalized to their horizon values.
by a mouth and by a converging pipe where the fluid continues the deceleration. The corresponding flow situation and qualitative behavior of the parameters are shown in Fig. 2.

Notice that if the tube cross-section is given as a function of the fluid velocity, we can write the nozzle equation in the equivalent form

\[ \frac{dA}{dv} = \frac{A}{v}(M^2 - 1). \] (3)

This equation relates the appearance of a horizon directly with the existence of local extrema of \( A(v) \) without the need of further conditions on the behavior of the space derivatives of \( v \). We will make use this equation when discussing the formation of horizons in a fluid with spherically symmetric flow.

Until now our discussion has been focused on the nozzle equation (2). The previous results are the whole information we can extract from this equation. There are two main points on which Eqs. (2) give no information. First, they do not say to us which of the two possibilities (i), (ii) is actually realized. Second, our results are based on the rather strong assumption \( v'(x) \neq \infty, 0 \). In general its validity depends on both the dynamics and the initial conditions \[17\]. In order to answer to the previous questions one has to choose an equation of state for the fluid and to solve Eqs. (1). To be more concrete we will consider a fluid with a power-law equation of state,

\[ P = \frac{a^2}{n} \rho^n. \] (4)

where \( a, n \neq 0 \) are real constants. This equation of state describes almost all physically interesting fluids: perfect fluid \((n = 1)\), Bose-Einstein condensate \((n = 2)\), Chaplygin gas \((n = -1)\). Even in the simple one-dimensional case under consideration explicit solution of Eqs. (1), (4) cannot be found. However, one can find solutions in implicit form, writing \( \rho, c, A \) as a function of the fluid velocity \( v \). The solution of Eqs. (1) for a fluid with equation of state given by Eq. (4) and \( n \neq 1 \) reads,

\[ \rho(v) = \left( \frac{n - 1}{2a^2} (\alpha - v^2) \right)^{-\frac{1}{n-1}}, \quad c^2(v) = \frac{n - 1}{2} (\alpha - v^2), \quad A(v) = \frac{\beta}{\rho v}. \] (5)

where \( \alpha, \beta \) are integration constants determined by the initial conditions. Formation of the sonic horizon requires \( c(v_h) = v_h \), which yields for \( n \neq -1 \)

\[ v_h = \sqrt{\frac{n - 1}{n + 1} \alpha}, \] (6)

where \( v_h \) is the fluid velocity at the horizon. Formation of the sonic horizon is possible in the following range of variation of the parameters. For \( n > 1, \alpha > 0 \) and \( v < \sqrt{\alpha} = \sqrt{(n - 1)/(n + 1)} v_h \). For \( -1 < n < 1, \alpha < 0 \). For \( n < -1, \alpha > 0, v > \sqrt{\alpha} \). The Chaplygin gas \((n = -1)\) represents a limiting case. Formation of the horizon requires \( \alpha = 0 \), which in turns implies \( A = \text{const} \) and \( v = c \), identically.

Using Eqs. (5) we can also determine which of the two possibilities (i), (ii) is actually realized. Differentiating two times \( A(v) \) we get \((dA/dv)|_{v_h} = 0 \) and \((d^2A/dv^2)|_{v_h} > 0 \) for \( n > -1 \) \((n < -1)\). Thus, the horizon forms in correspondence of a local minimum (maximum) of the function \( A(v) \) for \( n > -1 \) \((n < -1)\). Moreover \( d^2A/dv^2 \) is everywhere nonzero excluding the possibility of a flex point. To infer about the behavior of \( A(x) \), once the behavior of \( A(v) \) is known, we need just to use the trivial identities \( dA/dx = (dA/dv)(dv/dx) \), \( d^2A/dx^2 = (d^2A/dv^2)(dv/dx)^2 + (dA/dv)(d^2v/dx^2) \). When \( dv/dx \) is finite and nonvanishing, local maxima (minima) of \( A(v) \) correspond to local maxima (minima) of \( A(x) \). We have therefore shown that in the case of a fluid with a power-law equation of state (4), the acoustic horizon forms in correspondence of a minimum (maximum) of the cross-section when \( n > -1 \) \((n < -1) \). The Chaplygin gas \((n = -1)\) represents a limiting case for which the cross-section must be constant and \( v = c \) everywhere along the tube.

**Flux tube with constant cross-section and non-homogeneous external potential**

This case can treated similarly to the previous case. Setting \( A = \text{const} \) in Eqs. (4) one derives an equation similar to the nozzle equation (2), in which \( A \) is traded for \( \psi \)

\[ \frac{d\psi}{dx} = -\frac{v}{M^2} \frac{dv}{dx}(M^2 - 1). \] (7)
The main difference between Eq. 4 and 7 is the change of sign in the right hand side. This means that, differently to what happens for \( v'(x) \) and \( A'(x) \) in the nozzle Eq. 4, \( v'(x) \) and \( \psi'(x) \) have the same (opposite) sign in the subsonic (supersonic) region.

Again, assuming that \( v \) and \( v'(x) \) are everywhere finite and nonvanishing, one can easily show using Eq. 7 that the horizon must form in correspondence of local extrema of the external potential \( \psi \). This gives as null (external) force condition at the horizon location. Depending on the sign of \( v'(x) \) we have two different situations. (I) The fluid velocity grows monotonically passing from the subsonic to the supersonic region. In this case the external potential \( \psi \) has a local maximum at the horizon. (II) The fluid velocity decreases monotonically passing from the subsonic to the supersonic region. Now the external potential has a local minimum at the sonic horizon. The qualitative behavior of the parameters for both cases (I) and (II) is depicted in Fig. 3.

Also in the case under consideration we can solve the fluid-dynamical equations for a fluid with equation of state given by Eq. 4. We have for \( n \neq 1 \)

\[
\rho(v) = \frac{\beta}{v} \quad \text{and} \quad \psi(v) = \frac{a^2}{1-n} \left( \frac{\beta}{v} \right)^{n-1} - \frac{v^2}{2} + \alpha, \tag{8}
\]

where \( \alpha, \beta \) are integration constants. Solving the equation \( c(v) = v \) one easily finds the fluid velocity on the horizon, \( v_h = (a^2 \beta^{n-1})^{\frac{1}{n(n+1)}} \). Differently from the case of a flux tube with nonconstant section, in this case the formation of the horizon does not imply limitations on the range of variation of \( v \). Using the previous equations one can now easily show that \( (d\psi/dv)|_{v_h} = 0 \) and \( (d^2\psi/dv^2)|_{v_h} = -(n+1) \) so that \( (d^2\psi/dv^2)|_{v_h} > 0 \) \((< 0)\) for \( n > -1 \)(\( n < -1 \)). Using the same arguments used for the flux tube with non-constant section, one concludes that for \( n > -1 \)(\( n < -1 \)) the horizon forms in correspondence of a maximum (minimum) of the external potential. The value \( n = -1 \) is also here a limiting case with the same behavior as that previously discussed.

In solving Eqs. 4 in the two cases of constant and non constant cross-section, we have not considered \( n = 1 \) in the equation of state 4. Solutions 6 and 8 become singular for \( n = 1 \) and this case has to be considered separately. For \( n = 1 \) the solutions for a flux tube with variable cross-section read

\[
\rho(v) = a e^{-\frac{\beta^2}{2v}}, \quad A(v) = \frac{\beta}{\alpha v} e^{\frac{\beta^2}{2v}}, \quad c = a. \tag{9}
\]

For a flux tube of constant section we have instead,

\[
\rho(v) = \frac{\beta}{v}, \quad \psi(v) = -a^2 \ln \frac{\alpha}{v} - \frac{v^2}{2}, \quad c = a. \tag{10}
\]

In both cases we have for the fluid velocity at the horizon \( v_h = a \). A straightforward calculation shows that, as expected, the horizon forms for a minimum of the tube cross-section and for a maximum of the external potential.

As we have already noted one main physical requirement for the formation of an acoustic horizon is that \( v'(x) \) remain finite and non vanishing throughout the flux tube. In general to keep \( v'(x) \neq 0, \infty \) one needs a fine-tuning of the initial conditions. For instance, if \( v \) is too large at the entrance of the Laval nozzle the flow will become supersonic at a point upstream from the throat so that \( v'(x) \to -\infty \) \( \frac{4}{3} \). This will result in the generation of a shock wave, which

![FIG. 3: Qualitative behavior of the parameters for a flux tube of constant section and varying external potential. The figure on the left (right) corresponds to case I, (II), respectively. The black, grey and dashed lines represent the external potential \( \psi \), the fluid velocity \( v \) and the speed of sound \( c \), respectively. The vertical axis indicates the position of the horizon. \( c \) and \( v \) are normalized to their horizon values.](image-url)
will destroy the stationary flow of the fluid. We will not discuss here how this fine-tuning can be realized. We will just point out that the condition \( v'(x) \neq 0, \infty \) can be obtained as necessary consequence of imposing a dynamical analogy between the fluid flow and a gravitational, spherically symmetric black hole \(^{20,21}\).

The Einstein equations for a spherically symmetric black hole can be put in correspondence with the fluid motion described by Eqs. \(^{1}\) and constrained by

\[
2 \frac{dY}{dr} - X \frac{dF}{dr} + 2e^{-r} \frac{d\psi}{dr} = \lambda^2 V. \quad (11)
\]

where \( X = \frac{2}{4} (c^2 - v^2), Y = \rho c, F = \ln \frac{\rho}{\rho_0}, r \) is the radial coordinate for the black hole spacetime, \( \lambda^2 \) is the inverse of Newton constant and \( V \) is a function of the coordinate \( r \), which depends on the particular gravitational model under consideration \(^{21}\). The constraint \(^{11}\) represent a constraint on the geometrical form of the flux tube, which forces the cross-section to have a local extremum exactly at the position of the horizon. The constraint \(^{11}\) also implies that \( v'(x) \) is finite and nonvanishing at the horizon. This can be demonstrated using the trivial identity \( dv/dx = (dv/dY)(dY/dX)(dX/dr)(dr/dx) \) and showing that each factor in the product is individually finite and nonvanishing at the horizon. The third factor is proportional to the Hawking temperature of the horizon, the fourth is the fluid density \(^{20,21}\); therefore both must be \( \neq 0, \infty \). The first and the second can be shown to be finite and non-vanishing at the horizon by using the explicit solution of the constrained fluid dynamics derived in Ref. \(^{21}\).

Spherically symmetric flow

Let us now consider the inviscid, isentropic, spherically symmetric flow of a fluid with equation of state given by \(^{3}\). For simplicity we will set to zero the external forces acting on the fluid. Using spherical coordinates \( (r, \theta, \phi) \), the continuity and Euler equations yield,

\[
\frac{d(r^2 v \rho)}{dr} = 0, \quad \frac{v}{r} \frac{dv}{dr} + a^2 \rho^{n-2} \frac{d\rho}{dr} = 0. \quad (12)
\]

These equations can be easily generalized to the case when a pointlike source is located at the origin of the spherical coordinate system. Whereas the Euler equation remains unchanged the continuity equation gets a contribution proportional to \( \delta(r) \). In this more general case for \( r > 0 \) the dynamics of the system is still described by Eqs \((12)\).

The formation of sonic horizons for the spherically symmetric flow can be easily discussed noticing that Eqs. \((12)\) are formally identical to those describing an axi-symmetric flow along a tube with cross-section \( A(r) = 4\pi r^2 \). Because the section increases monotonically, in view of the previous derived results we should conclude that an horizon cannot form. However, these results rely on the assumption that \( v'(r) = dv/dr \) is everywhere finite. Relaxing this assumption could allow for the formation of an acoustic horizon even though \( A(r) \) does not have local extrema. Divergence of \( v'(r) \) implies that the fluid has infinite acceleration. The discussion has therefore no direct physical relevance. However, considering this more general situation sheds light on the physical mechanism on which horizon formation is based.

Setting \( \beta = r^2 v \rho \), Eqs. \((12)\) are solved by Eqs. \(^{5}\) with \( A(r) = 4\pi r^2 \). The condition for horizon formation is also given by Eq. \(^{4}\). Using Eqs. \(^{5}\) and the continuity equation in \((12)\) one gets \( A(v) = 4\pi r^2 n(a - v^2)^{1/(n-1)} \). One can easily show that for \( n > -1 \) (\( n < -1 \)) \( A(v) \) has a minimum (maximum) in correspondence of \( v = v_h \) with \( v_h \) given by Eq. \(^{4}\). It is immediately evident from the equation \( dA/dv = (dA/dr)(dr/dv) \) that in this case local extrema of the function \( A(v) \) are not related with local extrema of \( A(r) \) but with the divergence of \( dv/dr \). The function \( A(v) \) contains more information about the formation of acoustic horizons than the function \( A(x) \) (or \( A(r) \)). The vanishing of \( dA/dv \) is a necessary and sufficient condition for the existence of a point where \( v = c, \).

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