Entanglement dynamics for two interacting spins

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We study the dynamical generation of entanglement for a quantum system consisting in a pair of interacting spins, $s_1$ and $s_2$, in a constant magnetic field. Three different situations are considered: a) $s_1 = s_2 = 1/2$, b) $s_1 \to \infty, s_2 = 1/2$ and c) $s_1 = s_2 \to \infty$, corresponding respectively to a fully quantum system (two interacting qubits), a quantum degree of freedom coupled to a semiclassical one (a qubit in contact with an environment) and a fully semiclassical system, which displays chaotic behavior. The time evolution of entanglement for initially separable states is measured using the von-Neumann entropy and the linear entropy, and a comparison is made between the two. Compactness of the phase space and chaos in the classical limit are seen to have a strong influence on the recoherences of the system.

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Recent technological advances have made it possible to create and manipulate individual quantum states in the laboratory, thus allowing direct observations of entanglement and decoherence, concepts that are central to quantum computation and quantum information. Two subsystems 1 and 2 are said to be entangled if it is not possible to write the total system density matrix in the form

$$\rho = \sum_i p_i \rho_i^1 \otimes \rho_i^2,$$

and decoherence is the suppression of off-diagonal elements in a subsystem reduced density matrix. Quantifying entanglement, however, is a difficult task. Many different measures have been proposed, such as entropy of formation, entropy of distillation, and many different distances. All of them are very hard to calculate in practice, although Wootters has provided a closed form for the entropy of formation in the case of a system consisting of only two qubits. If the initial state is separable, the entropy of formation for a bipartite system reduces to the von-Neumann entropy (we reserve the letter $S$ for spins)

$$\delta_N = -\text{Tr} [\rho_2 \log_d \rho_2] ,$$

where $\rho_2$ is the reduced density matrix of the subsystem 2 and $d$ is the dimension of its Hilbert space. Another easy-to-handle measure is the linear entropy (or idempotency defect)

$$\delta = \frac{d}{d-1} (1 - \text{Tr} [(\rho_2)^2]).$$

When one considers only globally pure states (as we do here), both the linear and the von-Neumann entropy are 0 for separable states and 1 for maximally entangled states. If the initial state is separable, one is in fact considering a dynamical, deterministic, generation of entanglement. In this situation how fast entropy grows indicates how fast one subsystem suffers decoherence due to entanglement with the other.

Another interesting question is about the dependence of quantum entanglement upon the classical dynamics of the system, more specifically what is the role of chaos in the decoherence process. This has been the subject of many recent investigations. In a seminal work Furuya et al. have studied the Jaynes-Cummings model without the rotating-wave approximation, and found the entanglement rate to be greater for chaotic initial conditions. In later works they have shown that a regular initial condition can sometimes lead to faster entanglement than a chaotic one and that recoherences were related to the shape of the spin orbits. In the past few years much work has been done in this area, considering continuous variables, tops, and spin chains.

In the present work we consider two spins, $S_1$ and $S_2$, in a constant magnetic field $\hat{B} = B_0 \hat{z}$, interacting according to the Hamiltonian

$$H = \epsilon_1 B_0 S_1^z + \epsilon_2 B_0 S_2^z + \alpha S_1^x S_2^x,$$

which is somewhat reminiscent of the usual XXZ Hamiltonian widely used in statistical mechanics. Spin Hamiltonians have been considered in connection to quantum computation proposals using NMR (see also | ) and cold atoms in optical lattices. Entanglement in spin chains at finite temperature was considered for example. For simplicity we shall assume $\epsilon_1 B_0 = \epsilon_2 B_0 = 1$. This very simple case already shows a rich behavior depending on the parameter $\alpha$ and the spin magnitudes $s_1$ and $s_2$. 


The article is organized as follows: in section I we consider \( s_1 = s_2 = 1/2 \), which corresponds to a pair of qubits. In section II we let \( s_1 \to \infty \) to simulate an environment and study its ability to induce decoherence upon spin \( s_2 \). Section 3 deals with the relation between chaos and decoherence and we conclude in section IV.

## I. TWO QUBITS \((s_1 = s_2 = 1/2)\)

Let us first consider the case \( s_1 = s_2 = 1/2 \). The Hamiltonian in this case is a \( 4 \times 4 \) matrix that can easily be diagonalized. Its eigenvectors, written in the standard \( st := \{ |11\rangle, |10\rangle, |01\rangle, |00\rangle \} \) basis are

\[
\begin{align*}
v_1 &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 4(1 + \beta)/\alpha \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\
v_4 &= \begin{pmatrix} 4(1 - \beta)/\alpha \\ 0 \\ 0 \\ 1 \end{pmatrix},
\end{align*}
\]

with respective eigenvalues

\[
e_1 = \frac{\alpha}{4}, e_2 = -\frac{\alpha}{4}, e_3 = \beta, e_4 = -\beta,
\]

where

\[
\beta = \frac{\sqrt{\alpha^2 + 16}}{4}.
\]

In order to study dynamical entanglement generation, we use several different initially separable configurations. At \( t = 0 \), the spins \( s_1 \) and \( s_2 \) are in the states

\[
|z_i\rangle = \frac{|0\rangle + z_i|1\rangle}{\sqrt{1 + |z_i|^2}}, \quad i = 1, 2
\]

and, after some symmetry considerations, we have decided to investigate the following values of the parameters:

\[
a) z_1 = z_2 = 0, \quad b) z_1 = 1, z_2 = 0 \\
c) z_1 \to \infty, z_2 = 0, \quad d) z_1 = 0, z_2 = 1 \\
e) z_1 = 0, z_2 = i, \quad f) z_1 = z_2 = 1 \\
g) z_1 = 1, z_2 = i, \quad h) z_1 = z_2 = i
\]

Note that \( z_1 = \pm 1 \) and \( z_1 = \pm i \) correspond to eigenstates of \( S_1^z \) and \( S_1^y \), respectively, and the same holds for spin \( s_2 \).

A measure of entanglement exists for two spins: the entropy of formation, which measures “the amount of entanglement necessary to create the entangled state” \([3]\). If the system is initially separable, which is the case we consider here, the entropy of formation is equal to the von-Neumann entropy. For each one of the chosen initial conditions we followed the time evolution of this quantity and of linear entropy. Both measures have approximately the same form, with maxima and minima at the same points, even tough they are not proportional to each other. We have also observed that \( \delta \leq \delta_N \) for all chosen initial conditions, but this is not a general property, since the opposite occurs in section III. In Fig. 1 we see the case \( a \), which is very similar to cases \( b, c, \) and \( e \), all of which at least one of the spins is in the \( |0\rangle \) state. Conditions \( f, g, h \) are also quite similar, all in which \( |z_1| = |z_2| = 1 \). Condition \( f \) can be seen in Fig. 1b. All time scales present in the system can be related to the eigenvalues \([3]\), and are therefore determined by \( \alpha \).

## II. A QUBIT AND AN ENVIRONMENT \((s_1 \to \infty, s_2 = 1/2)\)

We now consider \( s_2 = 1/2 \) and \( s_1 = 200 \). Such a huge value of spin is not intended to be realistic, but only to simulate an environment for dissipation, when infinitely many degrees of freedom are required (see \([3]\) and references therein). We consider spin \( s_1 \) to be, at \( t = 0 \), in three different conditions: in a coherent state \([3]\)

\[
|z_1\rangle = \sum_{m=-s_1}^{s_1} \sqrt{\frac{2s_1}{s_1 + m}} \frac{z^{s_1+m}}{(1+|z|^2)^{s_1/2}} |m\rangle,
\]
in a non-localized state

$$|\psi\rangle = (2s_1 + 1)^{-\frac{1}{2}} \sum_{m_1 = -s_1}^{s_1} |m_1\rangle \quad (11)$$

and in a thermal mixture

$$\rho_1 = \frac{1}{N} \sum_{m_1 = -s_1}^{s_1} e^{-m_1/T} |m_1\rangle\langle m_1| \quad (12)$$

where $N$ is a normalization factor (the partition function).

Let us first consider both spins to be initially in coherent states with $z_1 = z_2 = 0$. The linear entropy is shown in Fig. 2 (the von-Neumann entropy is indistinguishable from it in this scale). We see that the entanglement reaches its maximum value very fast, and remains saturated for some time, but a strong and localized recoherence takes place in a nearly periodic way. As already observed in [? ], such recoherences are related to the mean value of the $z$-component of the spin, $\langle S_z^2 \rangle$, or equivalently to how close is the spin to the north pole of the Bloch sphere [? ]. In other words, recoherence is an effect of the compactness of the phase space.

In Fig. 3a we see the results for $z_1 = 0, z_2 = 1$. Even though the value of $z_1$ is the same as in the previous case, the entanglement is completely different. The linear entropy is now less than the von-Neumann entropy, but they have essentially the same information. Entanglement increases with time, but much slower than the previous case. Instead of raising immediately to its maximum value, it increases by jumps, alternating fast increments with plateaus.

We also consider $z_1 = z_2 = 1$, shown in Fig. 3b. The behavior of the entropies is quite similar to the previous case, but in a very different scale: entanglement is nearly a hundred times smaller in this case, i.e., the system is practically entanglement-free. It is clear that the entanglement rate strongly depends on the initial state of the environment, even if this is a coherent state.

Finally, we have considered the environment to be initially in the non-localized state (11) and in the thermal mixture (12). In all cases we have used $z_2 = 1$, and in Fig. 4 we show the behavior of the linear entropy. The qualitative behavior is the same for all initial states (in the thermal mixture we have used $T = s_1/10$), but the ability to induce decoherence is considerably greater for the coherent state.

### III. ENTANGLEMENT AND CHAOS ($s_1 = s_2 \to \infty$)

We now consider both spins to be large (semiclassical), namely we use $s_1 = s_2 = 15$. For any spin Hamiltonian it is possible to obtain a semiclassical dynamics using $su(2)$ coherent states [10], in which the quantum dynamics is then approximated by Hamilton equations of motion [? ]

$$\dot{z} = -i \frac{(1 + |z|^2)^2}{2\hbar} \frac{\partial \mathcal{H}}{\partial z^*}, \quad (13)$$

where $\mathcal{H} = \langle z|H|z \rangle$ plays the part of a classical Hamiltonian. In our case this is given by [? ]

$$\mathcal{H} = \frac{1}{2}(A_1 + A_2 - 31) + \frac{\alpha}{4} q_1 q_2 \sqrt{(4s_1 - A_1)(4s_2 - A_2)}, \quad (14)$$

where

$$A_i = q_i^2 + p_i^2 \quad (15)$$

and the canonical coordinates are defined by

$$\frac{q_j + ip_j}{\sqrt{4s_j}} = \frac{z_j}{\sqrt{1 + |z_j|^2}}, \quad j = 1, 2. \quad (16)$$

In Fig. 5 we see a Poincaré section showing the $(q_1, p_1)$ coordinates at $p_2 = 0$. We use the marked points as initial conditions for our coherent states to be evolved. The trajectory marked with a triangle is chaotic, the one marked with a square is regular and the circle denotes a periodic orbit. Note that we do not evolve the state semiclassically: a semiclassical evolution of the kind [13] would preserve, by construction, the coherence of the initial state, and thus produce no entanglement.
In Fig. 6 we see the linear entropy as a function of time. The rate of entanglement for short times is very similar for all initial conditions, contrary to what is observed for example in the Jaynes-Cummings system \[\text{[?]}.\] We also note that the maximal value of \(\delta\) is higher for the periodic orbit than for the more generic regular trajectory. Numerical analysis indicate that both the short-time entanglement rate and the maximum entanglement value are related not to the classical dynamics but to strictly quantum properties of the wave packet, such as energy uncertainty.

There is however a strong difference between the regular and chaotic initial conditions: the former present strong recoherences that are not present in the latter. As already observed in the previous section these recoherences are related to the mean value of the \(z\)-component of the spin, but in this case the relevant spin is \(s_1\), as we see in Fig. 7. Note that coherent states have localized Husimi distributions in phase space and thus some more general, less localized state, would be less likely to display recoherences.

Again the entanglement measures are qualitatively very similar, even tough they are not proportional, but in this case the von-Neumann entropy is less than the linear entropy at all times and for all initial conditions. We show this only for the periodic orbit in Fig. 8.

**IV. SUMMARY**

We have studied dynamical entanglement generation for a pair of interacting spins in a magnetic field. The case of two qubits is rather simple, since it allows an analytic treatment up to a point and is useful as a background against which we can compare the other cases. We noted for example that for all initial conditions \(\delta \leq \delta_N\). The case of a qubit interacting with an environment had a much more rich variety. We have seen that the decohering power of the environment strongly depends on its initial state. A comparison was made between different coherent states and between a coherent state, a non-localized state and a thermal mixture.

In the case of two semiclassical spins we have analyzed the influence of chaos upon the entanglement process. We saw that chaotic initial conditions presented no recoherences, while regular ones do so. We also noted that in this case \(\delta \geq \delta_N\) for all initial conditions, contrary to the low dimensional case of two qubits. In all cases the recoherences present in time evolution were related to the compactness of spin phase space and localization of coherent states. It would be interesting to investigate other measures of entanglement, specially for the case of initially mixed systems. Such work is underway.
FIG. 1: \( s_1 = s_2 = 1/2 \) Linear (dashed line) and von-Neumann (solid line) entropies for \( z_1 = z_2 = 0 \) (upper panel) and for \( z_1 = z_2 = 1 \) (lower panel). We have observed that \( \delta_N \geq \delta \) for all times and all initial conditions.
FIG. 2: \((s_1 = 200, \ s_2 = 1/2)\) Linear entropy (solid line) and angular momentum (dashed line) as functions of time for \(z_1 = z_2 = 0\). We see that \(\delta\) saturates very fast, but strong recoherences appear, related to \(\langle S_z^2 \rangle\).
FIG. 3: \((s_1 = 200, s_2 = 1/2)\) Linear (dashed line) and von-Neumann (solid line) entropies for \(z_1 = 0, z_2 = 1\) (upper panel) and for \(z_1 = z_2 = 1\) (lower panel). In both cases we see plateaus followed by jumps, but there is a large difference in magnitude.
FIG. 4: \((s_1 = 200, s_2 = 1/2)\) Linear entropy evolution, for the environment initially in a coherent state (dashed line), a thermal mixture (dotted line) and a non-localized state (solid line).

FIG. 5: Poincaré section of the classical Hamiltonian. We use the marked points as initial conditions for our coherent states to be evolved.
FIG. 6: \((s_1 = s_2 = 15)\) Linear entropy \(\delta(t)\) for three different initial conditions. The regular cases present strong recoherences, while these are suppressed in the chaotic case.

FIG. 7: \((s_1 = s_2 = 15)\) Linear entropy and angular momentum \(\sigma_1 = \frac{(S_z^1 + s_1)}{2s_1}\) for the periodic initial condition. Recoherences are related to the compactness of phase space.
FIG. 8: ($s_1 = s_2 = 15$) Linear (dashed line) and von-Neumann (solid line) entropies for the periodic initial condition. In contrast to section I, now $\delta \geq \delta_N$ for all times and all initial conditions.