Phase Structure and Instability Problem in Color Superconductivity

Kenji Fukushima
RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA

Received: date / Revised version: date

Abstract. We address the phase structure of color superconducting quark matter at high quark density. Under the electric and color neutrality conditions there appear various phases as a result of the Fermi surface mismatch among different quark flavors induced by finite strange quark mass: the color-flavor locked (CFL) phase where quarks are all energy gapped, the $u$-quark superconducting (uSC) phase where $u$-quarks are paired with either $d$- or $s$-quarks, the $d$-quark superconducting (dSC) phase that is the $d$-quark analogue of the uSC phase, the two-flavor superconducting (2SC) phase where $u$- and $d$-quarks are paired, and the unpaired quark matter (UQM) that is normal quark matter without pairing. Besides these possibilities, when the Fermi surface mismatch is large enough to surpass the gap energy, the gapless superconducting phases are expected. We focus our discussion on the chromomagnetic instability problem related to the gapless CFL (gCFL) onset and explore the instability regions on the phase diagram as a function of the temperature and the quark chemical potential. We sketch how to reach stable physical states inside the instability regions.

PACS. 12.38.-t Quantum chromodynamics – 12.38.Aw General properties of QCD

1 Family of color superconducting phases

The phase structure of matter composed of quarks and gluons described by Quantum Chromodynamics (QCD) has been investigated for many years, and in the high temperature and low baryon (or quark if deconfined) density region which is accessible in heavy-ion collisions interesting discoveries have been reported both in theories and in experiments. In the high density and low temperature region, on the other hand, our knowledge is still poor as compared with the rich physics expected in this region. Heavy-ion collisions are not suitable for the purpose to probe dense and cold quark matter, and such a system could be realized, if any, only in the cores of compact stellar objects. The experimental data from the universe is, however, quite limited and there is no smoking-gun for color superconductivity so far. Nevertheless, the theoretical challenge to explore the QCD phase structure is of great interest on its own. Also it would be potentially important in studying the structure and evolution of neutron stars.

In order to look into a dense quark system, some of concepts known in condensed matter physics have been imported into QCD in hope of analogous phenomena taking place. In this sense the physics of dense quark matter is, so to speak, “condensed matter physics of QCD” as articulated clearly in the review [1]. Superconductivity is definitely one of them. In general the Cooper instability inevitably occurs wherever there are a sharp Fermi surface below which particles are degenerated and an attractive interaction between particles on the Fermi surface. Even QCD matter is not an exception and the condensation of quark Cooper pairs leads to color superconductivity.

A major difference between ordinary electric superconductivity in metals and color superconductivity in quark matter arises from the fact that quarks have three colors and three flavors in addition to spin, so that quark matter allows for many pairing patterns. The color and flavor degrees of freedom make the dense QCD phase structure so complicated that subtleties still remain veiled. In this article we shall argue what has been clarified by now and what should be solved in the future mainly following the discussions in my recent papers [2, 3].

1.1 Pairing patterns

Many theoretical works have revealed that the predominant pairing pattern is anti-symmetric in spin (spin zero), anti-symmetric in color (color triplet), and anti-symmetric in flavor (flavor triplet). Moreover, the color-flavor locking is known to be favored in energy, so that the color and flavor indices are locked together. Then there are three independent diquark condensates or gap parameters [4]:

$$\langle \psi_i^a C \gamma_5 \psi_j^b \rangle \sim \Delta_1 e^{ab} \epsilon_{ij1} + \Delta_2 e^{ab} \epsilon_{ij2} + \Delta_3 e^{ab} \epsilon_{ij3}, \quad (1)$$

where $(i, j)$ and $(a, b)$ represent the flavor indices $(u, d, s)$ and the color triplet indices (red, green, blue) respectively.
The charge conjugation $C$ and the Dirac matrix $\gamma_5$ are required to make (1) a Lorenz scalar. Of course you can consider other kind of pairing between quarks which are totally anti-symmetric under exchange, and even different types of condensates may coexist. Actually diquark condensates such as spin-zero pairing in the color symmetric (color sextet) channel and spin-one pairing between quarks of the same flavor have been analyzed quantitatively [5, 6, 7, 8] and known to be much smaller than the predominant condensate. In this article we shall simply neglect them.

Under the pairing ansatz (1), $\Delta_1$ is a gap parameter for the Cooper pairing between $d$ and $s$ flavors and green and blue colors. That is, $\Delta_1$ is for $bd$-$qs$ and $gd$-$bs$ quarks and $\Delta_2$ and $\Delta_3$ are to be understood likewise:

<table>
<thead>
<tr>
<th>$\Delta_1$</th>
<th>$\Delta_2$</th>
<th>$\Delta_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$bd$-$qs$</td>
<td>$rs$-$bu$</td>
<td>$gu$-$rd$</td>
</tr>
<tr>
<td>$gd$-$bs$</td>
<td>$bs$-$ru$</td>
<td>$ru$-$gd$</td>
</tr>
</tbody>
</table>

Each gap parameter is either zero or finite and there are $2^3 = 8$ combinations accordingly. Only five of eight phase possibilities as listed in Table 1 are of our interest relevant to the QCD phase diagram. When three gap parameters are all nonzero, this state is called the color-flavor locked (CFL) phase. When only $\Delta_1$ is zero, this is the $u$-quark superconducting (uSC) phase named after the fact that remaining $\Delta_2$ and $\Delta_3$ are gap parameters for pairing involving $u$-quarks. The $d$-quark superconducting (dSC) phase is understood in the same way. The two-flavor superconducting (2SC) phase has only $\Delta_3$ which is non-vanishing. The question is: where and how they show up on the phase diagram. The next section is devoted to this issue.

It is worth mentioning that these phases can be characterized by global symmetry breaking patterns. In particular the second-order phase transitions between the CFL phase and the uSC or dSC phase belong to the universality class same as an $O(2)$ vector model [9]. In QCD, neither the sSC ($s$-quark superconducting), 2SCsu (2SC of $s$- and $u$-quarks), nor 2SCds (2SC of $d$- and $s$-quarks) phase is realized actually because any pairing containing massive $s$-quarks is disfavored by the Fermi surface mismatch energy. (The solution branch of the gap equations belonging to the 2SCsu phase has been examined in Refs. [10, 11] and confirmed to cost a larger energy.)

### 1.2 Gapless superconductors

The classification of superconducting phases under the pairing ansatz (1) is not complete until we take account of the gapless superconducting states. They do not break any new symmetry and the phase transition to a gapless superconductor exists only at zero temperature [2]. In this sense the gapless CFL (gCFL) phase for instance which we will closely discuss later is not a totally new phase but can be regarded as a variant of the CFL phase augmented by the presence of gapless quarks. Before addressing the CFL problem, we shall see the simplest example that is actually enough for abstracting the essence.

Let us assume that there are two species of particles, 1 and 2, which have the Fermi momenta $\bar{\mu} - \delta\mu/2$ and $\bar{\mu} + \delta\mu/2$ respectively and they form a Cooper pair. The energy dispersion relations without pairing are shown by the solid and dotted lines in Fig. 1. In the presence of the 1-2 pair condensate $\Delta$, the level repulsion by the energy $\Delta$ results in the dispersion relations which are smoothly connected between the hole state of 2 (or 1) and the particle state of 1 (or 2). In this simple model of superconductivity, therefore, the quasi-particle energy is expressed as

$$\epsilon^\pm(p) = \sqrt{(p - \bar{\mu})^2 + \Delta^2 + \frac{1}{2}\delta\mu^2}. \quad (2)$$

As is obvious from Fig. 1 as well as from the expression (2), the dispersion relation comes to cross zeros when $|\frac{1}{2}\delta\mu| > \Delta$. The momentum region from one zero to another zero of the dispersion relation is called the block-
ing momentum region because the pairing within this region is hindered by degenerated particles. (In the case of Fig. 1, particle-2 is degenerated and particle-1 is absent in the blocking region.) Generally once a superconductor enters the gapless state, the gap parameter significantly decreases with increasing blocking momentum region.

In the language of physics, the condition for the gapless onset $|\frac{1}{2}\delta \mu| > \Delta$ means that a Cooper pair is not stable energetically. The pairing energy $2\Delta$ is needed for breaking a Cooper pair into two particles and at the same time the mismatch energy $\delta \mu$ is released by doing so. If the mismatch is more expensive than the pairing, two particles would no longer form a Cooper pair. Roughly speaking, the realization of gapless superconductivity can be seen as a weak instability disrupting the Cooper pair only in a limited blocking region but not ruining the whole superconductivity.

We will end this subsection with one more remark. In this article (and in some literatures) $\delta \mu$ is frequently called the Fermi surface mismatch. It should be kept in mind that this mismatch is for the energy dispersion relations without pairing. In the presence of pairing, as observed in Fig. 1, the (approximate) Fermi surface is provided by the average $\bar{\mu}$. In other words two energy dispersions $\epsilon^{\pm}(p)$ in (2) have a common $\bar{\mu}$.

1.3 Strange quark mass effect and neutrality

We have so far illustrated what the gapless superconducting state is like. This gapless phase is often called the Sarma phase and was first demonstrated as a metastable state by Sarma [12]. The Sarma phase has recently at-

sequently, the realization of gapless superconductivity can be seen as a weak instability disrupting the Cooper pair only in a limited blocking region but not ruining the whole superconductivity.

Thus the strange quark mass effect can be incorporated by a chemical potential shift proportional to $M_s^2/\mu$. It is actually the case that the physics should change as a function of $M_s^2/\mu$ alone.

In the grand canonical ensemble, the number density of particles is specified by a chemical potential. We should introduce one chemical potential $\mu_a$ for the electromagnetic charge and eight chemical potentials $\mu_3$ for the color charges which are not gauge-invariant in non-abelian gauge theories. For the purpose of imposing neutrality it is enough to constrain only two eigenvalues of the color charge since zero charge remains to be zero charge if it is rotated by any gauge transformations. We thus only have to solve the color neutrality conditions with respect to two chemical potentials $\mu_3$ and $\mu_8$ [15].

Quark matter with three flavors would be automatically electric and color neutral if $M_{s}$ is zero, meaning that $\mu_c, \mu_3,$ and $\mu_8$ should be of order $M_{s}^2/\mu$ just like the direct $M_{s}$ effect.

In summary, the direct and induced $M_s$ effects are concisely expressed in a form of the effective chemical potentials for respective quarks with color $a$ and flavor $i$ as

$$\mu_{ai} = \mu - \mu_3(Q)_{ii} + \mu_3(T_3)_{aa} + \mu_8(\sqrt{2}T_8)_{aa} - (M^2)_{ii}/2\mu, \quad (4)$$

where $Q = \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $M = \text{diag}(0, 0, M_s)$ in flavor $(u,d,s)$ space and $T_3 = \text{diag}(\frac{1}{2}, -\frac{1}{2}, 0)$ and $\sqrt{2}T_8 = \text{diag}(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$ in color (red, green, blue) space. From (4) the Fermi surface mismatch for all the pairings is easily inferred; for the $bd$-$gs$ pairing for instance the mismatch is

$$\delta \mu_{bd-gs} = \mu_{bd} - \mu_{gs} = \frac{1}{2}\mu_3 - \mu_8 + \frac{M_s^2}{2\mu}, \quad (5)$$

In the CFL phase at zero temperature a model-independent argument [15] yields $\mu_c = \mu_3 = 0$ and $\mu_8 = -M_s^2/2\mu$, and together with the above expression, the gapless onset condition (where gapless quarks begin appearing) is

$$\delta \mu_{bd-gs} = M_s^2/\mu > 2\Delta_1. \quad (6)$$
Numerical calculations in a model study \cite{2,16} have confirmed this onset condition being a good estimate.

**2 Phase diagram of dense quark matter**

We will present the phase diagram first and then look closely at each phase in turn. Figure \ref{fig:phase_diagram} is the phase diagram obtained in the Nambu–Jona-Lasinio (NJL) model. The model parameters are chosen as to yield $\Delta = 40$ MeV at $\mu = 500$ MeV and $M_s = T = 0$. The strange quark mass is fixed at $M_s = 150$ MeV which is the lowest estimate for the $M_s$ effect in the intermediate density region. In Ref. \cite{21} the authors solved the gap equations for the chiral condensates as well as the gap parameters and the chemical potentials. In the case of weak coupling, $\Delta$ is smaller and thus the critical value of the strange quark mass (or chemical potential) on the gCFL onset is smaller (or larger), and the gCFL phase is not substantially affected by the chiral dynamics then.

Even if the chiral condensate is not dealt with dynamically, a weird structure including a first-order phase boundary emerges on the phase diagram for large $M_s^2/\mu$. As discussed in Ref. \cite{2}, however, the complicated structure seen at large $M_s^2/\mu$ is model and parameter dependent. Unfortunately there is no guiding principle to avoid unphysical artifacts in a certain parameter choice and one has to try and show all the cases for conclusive analyses \cite{21}. In this article we dare not to touch that subtle case \cite{2,16} (for details). As a result the $rs$-$bu$ dispersion relations are kept to be almost quadratic in the entire gCFL region. Therefore only $\Delta_1$ in the $bd$-$gs$ quark sector significantly drops in the gCFL region due to the spreading blocking region with increasing $M_s^2/\mu$.

Because this is an essential point in considering the instability problem later, we shall reiterate here; \cite{2,16}

- **bd-gs pairing with $\Delta_1$** — gapless quarks appear at $M_s^2/\mu = 2\Delta_1$; the blocking region increases for larger $M_s^2/\mu$.
- **rs-bu pairing with $\Delta_2$** — gapless quarks appear at $M_s^2/\mu = 2\Delta_2$; the dispersion relations are kept to be almost quadratic in the entire gapless phase.
- **gu-rd pairing with $\Delta_3$** — gapped quarks only.
- **ru-gd-bs** pairing with all $\Delta$'s — gapped quarks only.

**2.2 uSC phase**

The uSC phase results from the presence of the gCFL phase at zero temperature. In the gCFL phase, as explained in the last section, $\Delta_1$ is significantly reduced than $\Delta_2$ and $\Delta_3$. The pairing between $u$-$d$ quarks is not reduced but $\Delta_3$ is enhanced in the gCFL side as $\Delta_1$ and $\Delta_2$ decreases \cite{21}. (Note that $\Delta_3$ evaluated in the 2SC state is in general larger than $\Delta_3$ in the CFL state for the same diquark interaction.) In this way the ordering $\Delta_1 < \Delta_2 < \Delta_3$ is realized.

It is a well-known fact that the critical temperature is of order of the gap parameter at zero temperature. We can anticipate that $\Delta_1$ would melt first at finite temperature right above the gCFL region on the phase diagram. This means that the uSC phase where only $\Delta_1$ is zero is expected when gCFL matter is heated. Figure \ref{fig:phase_diagram} clearly shows that the phase structure is certainly as anticipated.

**2.3 dSC phase**

It is not the Fermi surface mismatch but the average Fermi momentum that is more relevant away from the gCFL region (see the expression \cite{2} or Fig. \ref{fig:gap_equations}). At zero temperature the average Fermi momenta are common in all the quark
sectors, that leads to equality in the number of nine (three colors and three flavors) quarks, and thus the electric and color neutrality is enforced \(^2\).

The enforced neutrality at zero temperature is broken at small temperatures of a few MeV \(^2\). At higher temperature, especially in the vicinity of the critical temperatures, it is a good approximation to estimate \(\mu\)'s in the normal phase, i.e., \(\mu_s = -M_s^2/4\mu\) and \(\mu_5 = \mu_8 = 0\). Then the ordering of the Fermi momenta, \(\mu_s < \mu_u < \mu_d\), is concluded, which can be understood in an intuitive way; the number of \(s\)-quarks is suppressed by \(M_s\) and so \(d\)-quarks should be more abundant than \(s\)-quarks to maintain electric neutrality. From this, the average Fermi momenta should obey the following ordering; \(\bar{\mu}_{su} < \bar{\mu}_{ds} < \bar{\mu}_{ud}\).

The gap equation to determine \(\Delta\)'s contains the momentum integration around the Fermi surface which effectively picks up the density of states at the Fermi momentum. The larger the density of states is, the greater the gap parameter becomes. In this way the ordering \(\Delta_2 < \Delta_1 < \Delta_3\) is realized from the average Fermi momenta ordering, which follows the presence of the dSC phase accordingly, as first discussed in Ref. \(^2\).

### 2.4 Doubly critical point

The phase boundary on which \(\Delta_1\) goes to zero crosses the phase boundary on which \(\Delta_2\) goes to zero at the “doubly critical point” where two phase transitions with respect to \(\Delta_1\) and \(\Delta_2\) take place simultaneously. Since the existence of the uSC and dSC phases are robust and model-independent, so is the doubly critical point.

We would shortly comment upon a puzzling question concerning the doubly critical point. The question is the following; what are the effects of gauge field fluctuations on the doubly critical point?

In weak coupling the gauge field fluctuations bring about an induced first-order phase transition and the critical temperature is shifted \(^2,\!^2\!^6\). Between the CFL phase and the uSC or dSC phase, no manifest effects would be expected because eight gluons are all massive in both phases. Therefore we can conclude that the phase transitions from the CFL phase toward either the uSC or dSC phase is surely of second order belonging to the same universality class as an O(2) vector model.

The gauge field fluctuations play an important role, on the other hand, between the 2SC phase and the uSC or dSC phase and the phase transition is forced to be of first order.

The doubly critical point is the point at which two phase transitions meet and three gluons become massless.
around it since it faces the 2SC phase. This suggests that the phase transition must be of first order and the critical temperature should be shifted at the doubly critical point. But how is it possible if the phase boundaries facing the CFL phase are not shifted at all? Or, are they shifted near the doubly critical point by nearly massless gluons? To answer this question, further clarification on the treatment of the gauge field fluctuations is waited.

3 Chromomagnetic instability

The gapless superconductors could be stabilized by the neutrality conditions, at least, in the variational space spanned by gap parameters and chemical potentials, but after all, they have turned out to be unstable in a wider space including the gauge fields. The chromomagnetic instability signifies that the Meissner screening mass squared becomes negative, that is, the Meissner mass is imaginary. Before addressing the chromomagnetic instability problem first pointed out by Huang and Shovkovy \[27\] in the case of the g2SC phase and analyzed in Refs. \[3,28,29\]. For the gCFL phase, we shall make a brief overview on the Debye and Meissner screening masses in the CFL phase, which would be useful to make a deeper insight into the instability problem. Our results are summarized in Fig. 3 and the goal of this section is to explain what is going on inside the instability regions presented by shadowed regions in this figure.

3.1 Debye and Meissner screening masses

The Debye and Meissner screening masses are the screening masses for the longitudinal and transverse gauge fields respectively, which are defined by

\[
m_{D,\alpha\beta}^2 = -\lim_{q \to 0} \Pi^{\alpha\beta}_{\mu\nu}(\omega = 0, q),
\]

\[
m_{M,\alpha\beta}^2 = \frac{1}{2} \lim_{q \to 0} (\delta_{ij} - q_i q_j) \Pi^{ij}_{\alpha\beta}(\omega = 0, q),
\]

where \(q_i = q_i/|q|\) and \(\Pi^{ij}_{\alpha\beta}\) is the polarization tensor for the gauge fields \(A_\mu^\alpha\) with the color and Lorenz indices denoted by \(\alpha\) and \(\mu\). At high enough density the polarization tensor is dominated by the quark-loop contributions alone and so we shall neglect the gluon and ghost loops that would not depend on \(\mu\) at the one-loop level.

Let us make one example to elaborate the quark-loop contributions for each gluon. The \(A_1\) and \(A_2\) gluons with the color indices labeled according to the Gell-Mann matrices in color space couple red quarks to green quarks and vice versa, while the flavor is not changed at any gluon vertices. For example, a diagram graphically shown as

3.2 Unstable channels and mixing with photon

If there is something funny in connection with gapless quarks in the quark-loop polarization like Example 1, it is expected to happen when both two quark propagators involve gapless energy dispersion relations. At the gapless onset, in fact, the quark-loop polarization consisting only of \(bd\), \(gs\), \(rs\), and \(bs\) quarks is divergent, leading to the divergent contributions to the Debye and Meissner masses for \(A_{1,2}\), \(A_3\), and \(A_8\) gluons. You can easily check that the polarization diagrams relevant to the \(A_{4,5}\) and \(A_{6,7}\) gluon channels must have at least one of \(gu\), \(rd\), \(ru\), \(bd\), and \(bs\) quarks which are gapped.

It should be noted here about mixing between the \(A_\gamma\) photon and the \(A_3\) and \(A_4\) gluons. In the symmetric CFL phase with \(M_8 = 0\), there is no mixing with respect to \(A_3\), but in the present case with \(M_8 \neq 0\) and thus \(\mu_8 \neq 0\) in the gCFL phase, there is \(A_3\)-mixing as well due to the isospin symmetry breaking. The mass squared matrix for \(A_1, A_3, A_8\) is a \(3 \times 3\) matrix with nonvanishing off-diagonal components generating mixing. We will denote the eigenmodes of the mass squared matrix by \(\tilde{A}_1\), \(\tilde{A}_3\), and \(\tilde{A}_8\). The rotated photon represented by the \(\tilde{A}_1\) field is massless all the way because the CFL and gCFL phases preserve a rotated electromagnetic \(U(1)\) symmetry. Once mixing occurs among \(A_1, A_3, A_8\), it is simply a matter of convention which eigenmode should be identified as \(\tilde{A}_3\) or \(\tilde{A}_8\). By this reason, though Figure 3 has the instability region with the label \(\tilde{A}_3\) and \(\tilde{A}_8\), it does not mean that both of two eigenmodes suffer from the instability, but the fact is actually that only one of them does.

3.3 Divergences at the gCFL onset

The divergences in the Debye and Meissner screening masses at the gapless onset derive from the density of states which is divergent when the energy dispersion relations take a quadratic form; \(\epsilon(p) \sim (p - \bar{\mu})^2/2\Delta\). That is, the density of states \(n(p)\) is given by

\[
n(p) = 4\pi p^2 \left[ \frac{d\epsilon(p)}{dp} \right]^{-1}
\]

and obviously \(n(p) \to \infty\) when the slope of \(\epsilon(p)\) is zero at \(p = \bar{\mu}\).

The next question is whether the divergence is positive or negative in the Debye and Meissner masses. A
naive intuition would be that both are positive, for in the CFL phase it is well established that $m^2_M = \frac{1}{3} m^2_D$ should hold \cite{9, 30, 31} and one might well consider that they are correlated in a similar way even in the presence of $M_s \neq 0$.

This expectation is partly true but in an unexpected manner as we will shortly see below.

To find the relations between the Debye and Meissner masses, it is convenient to split the quark propagator into four distinct parts. If the quark mass effect is approximated as (3) near the Fermi surface, the energy projection operator divides the propagator into the particle part that is a function of $p - \mu$ and the antiparticle part that is a function of $p + \mu$. Moreover, in the Nambu-Gor'kov formalism, the quark propagator is a 2 $\times$ 2 matrix and its diagonal component is a normal propagation of particles, and its off-diagonal component is an abnormal propagation mediated by diquark condensates. There are thus four distinct combinations; diagonal–particle, diagonal–antiparticle, off-diagonal–particle, and off-diagonal–antiparticle.

The diagram shown in Example 1 is one example consisting of only the diagonal propagators. We can construct another example in the same gluon channel composed of only the off-diagonal propagators as follows:

\[
\begin{array}{c}
\text{ru} \\
gd \\
gu \\
rd \\
\text{T}_1 \\
\text{T}_2 \\
\end{array}
\]

(Example 2)

Fig. 3. A magnified drawing of the phase diagram around the gCFL and uSC phases with the instability regions overlaid.

where the upper propagator is the off-diagonal component proportional to $\Delta$’s connecting ru and gd quarks, and the lower one is the off-diagonal component proportional to $\Delta_3$ connecting gu and rd quarks.

It is important to note that the possible singular behavior originating from gapless quarks can reside only in the particle parts because antiparticles would never be gapless. The Debye mass squared receives the contributions only from particles which can be separated into the diagonal part, $[m^2_D]_{\text{diag}}$, and the off-diagonal part, $[m^2_D]_{\text{off}}$.

The Meissner mass, on the other hand, has all of the particle-particle, particle-antiparticle, and antiparticle-antiparticle contributions. In Ref. \cite{8} interesting relations have been found:

\[
[m^2_M]_{\text{diag}(pp)} = -\frac{1}{3} [m^2_D]_{\text{diag}},
\]

\[
[m^2_M]_{\text{off}(pp)} = \frac{1}{3} [m^2_D]_{\text{off}},
\]

where $[m^2_M]_{\text{diag}(pp)}$ is a part of the Meissner mass squared coming from the diagrams with two diagonal particle propagators (like Example 1) and $[m^2_M]_{\text{off}(pp)}$ with two off-diagonal particle propagators (like Example 2). The relation \cite{10} might be strange at a glance, for we already know that $m^2_M = \frac{1}{3} m^2_D$ in the CFL phase at $M_s = 0$. Then, how can we retrieve the relation $m^2_M = \frac{1}{3} m^2_D$? Actually, as long as the system stays in the CFL side, the diagonal particle-antiparticle contribution $[m^2_M]_{\text{diag}(pa)}$ is twice larger than $[m^2_M]_{\text{diag}(pp)}$ and changes the overall sign in the relation between the Debye and Meissner masses. The other con-
tions, \( [m_M^2]_{\text{diag}(aa)} \), \( [m_M^2]_{\text{off}(pa)} \), and \( [m_M^2]_{\text{off}(aa)} \) are all negligibly small.

It might be surprising that these relations \( \text{(11)} \) and \( \text{(11)} \) are satisfied as they are even in the gCFL phase! The diagonal and off-diagonal parts both can generate divergent contributions at the gCFL onset. As far as the QCD problem is concerned, the diagonal contributions are always larger than the off-diagonal ones, leading to the Debye mass squared diverging positively and Meissner mass squared diverging negatively at the gCFL onset. The negative sign in \( \text{(11)} \) cannot be compensated in the gCFL phase since, unlike in the CFL phase, \( [m^2]_{\text{diag}(pp)} \) is no longer a match for \( [m^2]_{\text{diag}(pp)} \).

To put it in another way, the key relation \( \text{(11)} \) can be stated as follows: In the diagonal part, we can say that the particle-particle loops tend to induce paramagnetism, while diamagnetism stems from the particle-antiparticle loops. Usually in the superconducting phase, the diamagnetic tendency is greater enough to exhibit the Meissner effect. In gapless superconductors, however, antiparticles are never gapless and only the particle-particle loops are abnormally enhanced due to the presence of gapless quarks. As a result of that, the diamagnetism gives way to the chromomagnetic instability.

### 3.4 Away from the gCFL onset

Figure 3 shows the instability regions for respective gluons. The difference between the \( A_{1,2} \) behavior and the \( A_3-A_8 \) behavior can be understood from the difference between the \( bd-gs \) and \( rs-bu \) quark dispersion relations. It is only the diagram shown in Example 1 that causes singular behavior in \( A_{1,2} \) around the gCFL onset. This diagram has two quark propagators; one of gapless \( gs \)-quarks and the other of quadratic \( rs \)-quarks. Since \( rs \) quarks are kept to be almost quadratic in the entire gCFL region, as we have explained before, the \( A_{1,2} \)-instability extends over the whole gCFL region at small temperatures. The instability would not persist into regions at higher temperatures because the quadratic dispersion relations are easily affected by thermal excitations.

In contrast, the same species of quarks constitute the \( A_3-A_8 \) instability region. The instability caused by two quadratic quark propagations lies in the entire gCFL region like the \( A_{1,2} \)-instability but only at tiny temperatures of order eV. Instability boundaries at such low temperatures are not visible actually on the phase diagram. The instability caused by two gapless quark propagations is located near the gCFL onset and spreads towards high temperatures than the \( A_{1,2} \)-instability because it has nothing to do with quadratic quarks.

Although we would not go further into details, the Meissner screening masses evaluated in the g2SC phase \( \text{[27]} \) indicate that the \( A_8 \)-instability occurs starting at the g2SC onset and there also arises the instability for \( A_{4,5,6,7} \) gluons whose boundary is found not at the g2SC onset but inside the 2SC region. In our results the \( A_{4,5} \) and \( A_{6,7} \) instability regions are located at large Fermi surface mismatches and enter the uSC phase at higher temperature and then the 2SC phase farther. We conjecture that these instability regions for \( A_{4,5} \) and \( A_{6,7} \) would be linked to the instability found in the 2SC phase. The nature of the instability with respect to \( A_{4,5} \) and \( A_{6,7} \) is presumably different from the instability near the gCFL onset that we have seen in great details. In the aim of disclosing the QCD phase diagram in the intermediate density region, in particular, the \( A_{4,5} \) and \( A_{6,7} \) instability deserves further investigation.

### 4 Speculations

This final section is devoted to sketching some speculations on how to reach the stable states inside the instability regions on the phase diagram, which has been barely succeeded so far.

There are already some attempts to interpret and resolve the chromomagnetic instability problem \( \text{[32,33,34,35]} \). The most important among them is, in my opinion, the observation pointed out by Giannakis and Ren in Ref. \( \text{[32]} \) that the instability with respect to gluons can be interpreted as the instability toward a plane-wave crystalline superconducting phase.

The most straightforward interpretation of the chromomagnetic instability is, of course, spontaneous generation of the expectation value for the transverse gluons. The gauge field itself is, however, not a physical quantity depending on the gauge choice. Actually, what Giannakis and Ren realized in Ref. \( \text{[32]} \) is that the spontaneously generated gauge fields can be absorbed in the phase of the gap parameters by means of the gauge transformation. It should be noted that whether the gauge invariance is maintained or not does not matter. The gauge transformation in this manipulation simply means the change of variables.

The crystalline phase had been already studied \( \text{[36]} \) before the chromomagnetic instability was discovered. It is called the “crystalline” phase because the gap parameter takes a form of

\[
|\Delta(x)| = |\Delta|e^{i\mathbf{q} \cdot \mathbf{x}},
\]

which breaks the translational and rotational invariance. The essential point is that there is another quark basis where the phase factor of \( \text{(12)} \) vanishes at the price of the vector potential arising in the effective action. Therefore, taking care of a vector potential turns out to be equivalent with dealing with the gap parameter with a phase factor corresponding to the given vector potential.

Let us rephrase the above mentioned idea in a slightly different way. Supposing we performed the derivative expansion of the thermodynamic potential \( \Omega_A[\Delta(x)] \) which is now gauged with gluons, then we have in general

\[
\Omega_A[\Delta(x)] \simeq \Omega_0[\Delta] - \kappa^{ab}[|\Delta| \text{Tr}[(\partial_i + iA_i^a)\Delta^*(x)a][\partial^i - iA^i] \Delta(x)]^b, \tag{13}
\]

where \( a \) and \( b \) represent the color triplet indices and \( A^i = A^i_\alpha T^\alpha \) with the color adjoint index \( \alpha \). The average over \( x \) is
symbolically implied in \( \text{Tr} \). Then, an alternative definition of the Meissner screening mass is immediately available from this thermodynamic potential as

\[
m_{M,\alpha\beta}^2 = \frac{1}{3} \sum_{i=1}^{3} \frac{\partial^2 \Omega}{\partial A_i^\alpha \partial A_j^\beta} \bigg|_{A=0} \sim 2\kappa^{ab}(T^\alpha)_{ca}(T^\beta)_{bd}\Delta^c\Delta^d,
\]

where \( \Delta(x) \) is assumed to be spatial constant. It might be instructive to see how this expression works actually. In the 2SC phase, \( \Delta^a \propto \delta^{a3} \) and \( (T^3)^{a3} = (T^2)_{a3} = (T^3)_{a3} = 0 \), and hence one can instantly conclude that the \( A_1 \), \( A_2 \), and \( A_3 \) gluons are not Meissner screened. In the g2SC case the \( A_8 \) gluon is unstable, which can be stated by the condition \( \kappa^{33} < 0 \) because \( m_{M,88}^2 \sim \kappa^{33}(\Delta^3)^2 \).

Now we shall assume the crystalline superconducting phase with (12). The curvature of the thermodynamic potential with respect to \( q \) is

\[
\frac{1}{3} \sum_{i=1}^{3} \frac{\partial^2 \Omega}{\partial q_i^a \partial q_i^b} \sim \kappa^{ab}\Delta^a\Delta^b. \tag{15}
\]

In the 2SC phase the instability condition for \( q \) to grow is given by \( \kappa^{33} < 0 \) again. This argument works well only in the (g)2SC case in which the \( A_8 \)-instability can be identified as the instability toward a crystalline superconducting phase equivalently. Essentially the above is what has been articulated in Refs. [32, 33].

The generalization to the (g)CFL problem is possible with a simple extension of the crystalline ansatz (12). As suggested by Giannakis and Ren, we can consider a colored crystalline superconducting phase with the gap parameter taking a form of

\[
\Delta(x) = |\Delta|e^{iT^\alpha q^\alpha x}, \tag{16}
\]

then it is almost obvious that the curvature with respect to \( q^\alpha \) is identical with the Meissner mass squared. In this sense, we can identify the instability regions in Fig. 3 with the colored crystalline phase with nonvanishing \( q^\alpha \).

What we should do next is now apparent; \( q^\alpha \) are the new variational parameters to be determined so as to minimize the thermodynamic potential. This is a quite tough task, however. The number of the new variables is five corresponding to \( A_{1,2}, A_{3,5}, A_{6,7}, A_3 \), and \( A_8 \). To be worse, the rotational symmetry is broken by the direction of \( q^\alpha \), and the momentum angle integration cannot simplify in evaluating the thermodynamic potential. So, the numerical calculations are too time-consuming to be done. Moreover, it is difficult to achieve an enough accuracy to get reliable outputs in the multi-dimensional numerical integration. Maybe we have to abandon this apparent strategy and instead need to invent a wiser simplification that would not lose the important physics.

One possibility would be to go to the higher orders in the derivative expansion (13). The calculation would be feasible because the expansion coefficients can be evaluated at vanishing \( q^\alpha \) and the rotational symmetry is not broken then. Although we certainly have a chance to find the energy minimum with some \( q^\alpha \), there is no guarantee that the next higher order terms can be adequate to stabilize the potential. In any case this kind of calculation has yet to be performed.

Finally, if I am excused to speak of my own opinion, all these theoretical efforts will be tested in the lattice QCD simulation someday when the sign problem at finite density will be solved and the lattice spacing can be small enough to describe high density matter. The phase diagram like Fig. 3 is to be confirmed then.

This article is based on the talk given for the new talent sessions at International School of Subnuclear Physics 43rd Course held at Erice in Italy from Aug. 29 to Sep. 7 in 2005. I thank all the organizers for stimulating lectures and discussions at school.

References

8. For a review on spin-one color superconductivity, see, A. Schmitt, arXiv nucl-th/0405076.

Kenji Fukushima: Phase Structure and Instability Problem in Color Superconductivity