New closed expression of the interaction kernel in the Bethe-Salpeter equation for quark-antiquark bound states

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The interaction kernel in the Bethe-Salpeter equation for quark-antiquark bound states is derived newly from QCD in the case where the quark and the antiquark are of different flavors. The technique of the derivation is the usage of the irreducible decomposition of the Green’s functions involved in the Bethe-Salpeter equation satisfied by the quark-antiquark four-point Green’s function. The interaction kernel derived is given a closed and explicit expression which shows a specific structure of the kernel since the kernel is represented in terms of the quark, antiquark and gluon propagators and some kinds of quark, antiquark and/or gluon three, four, five and six-point vertices. Therefore, the expression of the kernel is not only convenient for perturbative calculations, but also suitable for nonperturbative investigations.

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I. INTRODUCTION

The Bethe-Salpeter (B-S) equation which was proposed early in Refs. [1, 2], commonly, is recognized as a rigorous approach to the relativistic bound state problem [1-13]. The distinctive features of the equation are: (1) The equation is derived from the quantum field theory and hence set up on the firm dynamical basis; (2) The interaction kernel in the equation contains all the interactions taking place in the bound states and therefore the equation provides a possibility of exactly solving the problem of relativistic bound states; (3) The equation is elegantly formulated in a manifestly Lorentz-covariant form in the Minkowski space which allows us to discuss the equation in any coordinate frame. However, there are tremendous difficulties in practical applications of the equation. One of the difficulties arises from the fact that the kernel in the equation was not given a closed form in the past. The kernel usually is defined as a sum of B-S (two-particle) irreducible Feynman diagrams each of which can only be individually determined by a perturbative calculation. This definition is, certainly, not suitable to investigate the subjects such as the nuclear force in nuclear physics and the quark confinement in hadron physics which must necessarily be solved by a nonperturbative method. This is why as said in Ref. [12] that" The Bethe-Salpeter equation has not led to a real breakthrough in our understanding of the quark-quark force".

Opposite to the conventional concept as commented in Ref. [13] that ”The kernel $K$ can not be given in closed form expression”, we have derived a closed and compact expression of the B-S kernel for quark-antiquark ($q\bar{q}$) bound states in a recent publication [14]. The expression is derived with the aid of equations of motion satisfied by the $q\bar{q}$ four-point Green’s function and some other kinds of Green’s functions and contains only a few types of Green’s functions which not only are easily calculated by the perturbation method, but also suitable to be investigated by a certain nonperturbation approach. In order to exhibit a more specific structure of the B-S kernel, in this paper, we are devoted to deriving a new expression of the kernel by means of the technique of irreducible decomposition of the Green’s functions involved in the B-S equation satisfied by the $q\bar{q}$ four-point Green’s function. The technique was successfully applied to derive the interaction kernel in the Dirac-Schrödinger equation for $q\bar{q}$ bound states in the author’s latest publication [15]. The new expression derived is represented in terms of the quark, antiquark and gluon propagators as well as some kinds of quark, antiquark and/or gluon three, four, five and six-line vertices. For brevity of the derivation, we restrict ourself in this paper to discuss the B-S kernel for the bound system consisting of a quark and an antiquark which are of different flavors.

The rest of this paper is arranged as follows. In Sect. 2, we sketch the B-S equation satisfied by the $q\bar{q}$ four-point Green’s function. In Sect. 3, we describe the B-S reducibility of the Green’s functions involved in the B-S equation by means of the technique of irreducible decomposition of Green’s functions. Section 4 is used to derive the final expression of the B-S kernel given by the irreducible decomposition of the Green’s functions. The last section serves to make summary and some remarks. In Appendix, we show some details of the irreducible decomposition which follows from the QCD generating functional.
II. B-S EQUATION AND DEFINITION OF THE INTERACTION KERNEL

In Ref. [14], it was shown that to derive the B-S interaction kernel for $q\bar{q}$ bound states, it is necessary to derive a B-S equation satisfied by the $q\bar{q}$ four-point Green’s function. In the case of quark and antiquark with different flavors, the $q\bar{q}$ four-point Green’s function is, in the Heisenberg picture, defined as [16]

$$G(x_1,x_2;y_1,y_2)_{\alpha\beta\rho\sigma} = \langle 0^+ | T \{ \psi_\alpha(x_1) \psi_\beta(x_2) \bar{\psi}_\rho(y_1) \bar{\psi}_\sigma(y_2) \} | 0^- \rangle$$

(1)

where $T$ stands for the time-ordering product, $\psi_\alpha(x)$ and $\psi_\beta(x)$ are the quark and antiquark field operators, respectively, $\bar{\psi}_\rho(x)$ and $\bar{\psi}_\sigma(x)$ are their corresponding Dirac conjugates,

$$\psi^c(x) = C \bar{\psi}^T(x), \bar{\psi}^c(x) = -\psi^T(x)C^{-1}$$

(2)

here $C$ is the charge conjugation operator. The advantage of using $\psi^c(x)$ other than $\bar{\psi}(x)$ to represent the antiquark field is that the antiquark would behave as a quark in the B-S equation so that the quark-antiquark equation formally is the same as the corresponding two-quark equation.

For deriving the B-S equation satisfied by the $q\bar{q}$ four-point Green’s function, it is necessary to use various equations of motion satisfied by the quark and antiquark propagators, the $q\bar{q}$ four-point Green’s functions and some other kinds of Green’s functions. The latter equations of motion are easily derived from the QCD generating functional [16]

$$Z[J,\bar{\eta},\eta,\bar{\xi},\xi] = \frac{1}{N} \int \mathcal{D}(A,\bar{\psi},\psi,\mathcal{C},C)e^{iI}$$

(3)

where

$$I = \int d^4x[\mathcal{L} + J^{a\mu}A^a_\mu + \bar{\eta}\psi + \bar{\psi}\eta + \bar{\xi}C + \mathcal{C}\xi]$$

(4)

in which $\mathcal{L}$ is the effective Lagrangian of QCD

$$\mathcal{L} = \bar{\psi}(i\partial - m + gA)\psi - \frac{1}{4}F^{a\mu\nu}F_{a\mu\nu} - \frac{1}{2\lambda^2}(\partial^\mu A^a_\mu)^2 + \bar{\mathcal{C}}C^\alpha \partial^\alpha (D^{a\mu}C^b)$$

(5)

here $\partial_x = \gamma^\mu \partial_\mu$, $A = \gamma^\mu T^{a}A^a_\mu$ with $A^a_\mu$ being the vector potentials of gluon fields and $T^{a} = \lambda^a/2$ the quark color matrix,

$$F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$

(6)

are the strength tensors of the gluon field,

$$D_{\mu}^{ab} = \delta^{ab}\partial_\mu + gf^{abc}A^b_\mu C^c$$

(7)

are the covariant derivatives, $C^a, C^b$ are the ghost fields, and $J^{a\mu}, \bar{\eta}, \eta, \bar{\xi}$ and $\xi$ denote the external sources coupled to the gluon, quark and ghost fields respectively. By the charge conjugation transformations shown in Eq. (2) for the quark fields and in the following for the external sources

$$\eta^c = C\bar{\eta}^T, \bar{\eta}^c = -\eta^T C^{-1}$$

(8)

it is easy to prove the following relation

$$\bar{\psi}(i\partial - m + gA)\psi + \bar{\eta}\psi + \bar{\psi}\eta = \bar{\psi}^c(i\partial - m + gA)\psi^c + \bar{\eta}^c\psi^c + \bar{\psi}^c\eta^c$$

(9)

where $A = \gamma^\mu T^{a}A^a_\mu$ with $T^{a} = -\lambda^{a*}/2$ being the antiquark color matrix.

Now let us sketch the derivation of the B-S equation satisfied by the $q\bar{q}$ four-point Green’s function. The equation may be set up by acting on the Green’s function with the operator $(i\partial_\mu - m_1 + \Sigma)(i\partial_\nu - m_2 + \Sigma^c)$ where, $m_1$ and $m_2$ are the quark and antiquark masses, and $\Sigma$ and $\Sigma^c$ stand for the quark and antiquark proper self-energies. For this purpose, we first need to derive the equations of motion for the four-point Green’s function. As shown in Ref. [14], when we take the successive functional derivatives of the generating functional in Eq. (3) with respect to the field function $\bar{\psi}_\alpha(x_1)$ and sources $\bar{\eta}_\beta(x_2), \eta_\rho(y_1)$ and $\eta^c_\sigma(y_2)$ and finally setting all the sources to vanish, it can be found that
Similarly, successively differentiating the generating functional in Eq. (3) with respect to the field \( \overline{\psi}_b(x) \) and sources \( \overline{\mathcal{V}}_{a}(x_1), \eta_{\gamma}(y_1) \) and \( \eta^{\sigma}_{\nu}(y_2) \), and finally turning off all the sources, one may obtain

\[
[(i\partial_{x_1} - m_1 + \Sigma)G^b_{\alpha\beta\sigma}(x_1, x_2; y_1, y_2) = \delta_{\alpha\beta}\delta^4(x_1 - y_1)S_F^b(x_2 - y_2)_{\beta\sigma} \\
- (\Gamma^{\mu}_{\alpha\gamma})G^b_{\mu\alpha\beta\sigma}(x_1, x_2; y_1, y_2)_{\gamma\beta\sigma} + \int d^4z_1\Sigma(x_1, z_1)\gamma_{\alpha\gamma}G(z_1, x_2; y_1, y_2)_{\gamma\beta\sigma}.
\]  

(10)

In Eqs. (10) and (11), \( m_1 \) and \( m_2 \) are the quark and antiquark masses, \( S_F^c(x_1 - y_1) \) and \( S_F^c(x_2 - y_2) \) are the quark and antiquark propagators, \( \Sigma(x, y) \) and \( \Sigma^c(x, y) \) represent the quark and antiquark self-energies, respectively,

\[
(\Omega^{\mu\nu})_{\alpha\gamma} = g(\gamma^{\mu}T^{\alpha\nu})_{\alpha\gamma}, \quad (\overline{\Omega}^{\mu\nu})_{\beta\lambda} = g(\gamma^{\mu}\overline{T}^{\alpha\nu})_{\beta\lambda}
\]

(12)

and

\[
G^b_{\mu\nu}(x_i | x_1, x_2; y_1, y_2)_{\alpha\beta\sigma} = \langle 0^+ | T[A^b_{\mu\nu}(x_i)\psi_\gamma(x_1)\overline{\psi}_\beta(y_1)] | 0^- \rangle
\]

(13)

here \( i = 1, 2 \). Acting on Eq. (11) with the operator \( (i\partial_{x_2} - m_2 + \Sigma) \) (or equivalently, acting on Eq. (10) with \( (i\partial_{x_2} - m_2 + \Sigma^c) \)) and utilizing the equations satisfied by the quark and antiquark propagators

\[
[(i\partial_{x_1} - m_1 + \Sigma)S_F^b(x_2; y_1, y_2)_{\alpha\beta\sigma} = \delta_{\alpha\beta}\delta^4(x_1 - y_1) \Omega^{\mu\nu}_{\alpha\gamma}A^b_{\nu\alpha\beta\sigma}(x_2; y_1, y_2)_{\gamma\beta\sigma} \\
- (\Omega^{\mu\nu})_{\alpha\gamma}A^b_{\mu\nu\alpha\beta\sigma}(x_2; y_1, y_2)_{\gamma\beta\sigma} + \int d^4z_2\Sigma^c(x_2, z_2)_{\beta\gamma\sigma}G(z_2, x_2; y_1, y_2)_{\beta\gamma\sigma}
\]

(14)

where

\[
(\Sigma S_F^c)_{\alpha\beta}(x_1, y_1) = \int d^4z_1\Sigma(x_1, z_1)_{\alpha\gamma}S_F(z_1 - y_1)_{\gamma\beta} = (\Omega^{\mu\nu})_{\alpha\gamma}A^b_{\nu\alpha\beta\sigma}(x_1 | x_1, y_1)_{\gamma\beta}, \quad (\Sigma^c S_F^b)_{\beta\gamma}(x_2, y_2) = \int d^4z_2\Sigma^c(x_2, z_2)_{\beta\gamma\sigma}S_F^b(z_2 - y_2)_{\lambda\sigma} = (\overline{\Omega}^{\mu\nu})_{\gamma\beta}A^b_{\mu\nu\beta\gamma}(x_2 | x_2, y_2)_{\lambda\sigma}
\]

(15)

in which

\[
\Lambda^b_{\alpha\gamma}(x_1 | x_1, y_1)_{\gamma\beta} = \frac{1}{\epsilon}\langle 0^+ | T[A^b_{\mu\nu}(x_1)\psi_\gamma(x_1)\overline{\psi}_\beta(y_1)] | 0^- \rangle, \\
\Lambda^b_{\mu\nu}(x_2 | x_2, y_2)_{\lambda\sigma} = \frac{1}{\epsilon}\langle 0^+ | T[A^b_{\mu\nu}(x_2)\psi_\gamma(x_2)\overline{\psi}_\beta(y_2)] | 0^- \rangle
\]

(16)

and the equation obeyed by Green’s function \( G^b_{\mu\nu}(x_2 | x_1, x_2; y_1, y_2) \)

\[
[(i\partial_{x_1} - m_1 + \Sigma)G^b_{\alpha\beta\sigma}(x_2; x_1, x_2; y_1, y_2)_{\alpha\beta\sigma} = \delta_{\alpha\beta}\delta^4(x_1 - y_1)A^b_{\mu\nu}(x_2 | x_2, y_2)_{\lambda\sigma} \\
- (\Omega^{\mu\nu})_{\alpha\gamma}G^b_{\mu\nu\alpha\beta\sigma}(x_2; x_1, x_2; y_1, y_2)_{\gamma\beta\sigma} + \int d^4z_1\Sigma(x_1, z_1)_{\alpha\gamma}G^b_{\nu\alpha\beta\sigma}(x_2 | x_1, z_1; y_1, y_2)_{\gamma\beta\sigma}
\]

(17)

one may derive

\[
[(i\partial_{x_1} - m_1 + \Sigma)(i\partial_{x_2} - m_2 + \Sigma^c)G^b_{\alpha\beta\sigma}(x_1, x_2; y_1, y_2)_{\alpha\beta\sigma} = \delta_{\alpha\beta}\delta^4(x_1 - y_1)\delta^4(x_2 - y_2) \\
+ (\Omega^{\mu\nu})_{\alpha\gamma}(\overline{\Omega}^{\mu\nu})_{\beta\lambda}G^b_{\mu\nu\alpha\beta\sigma}(x_1 | x_1, x_2; y_1, y_2)_{\gamma\beta\sigma} \\
- (\Omega^{\mu\nu})_{\beta\lambda}G^b_{\alpha\nu\mu\beta\sigma}(x_1 | x_1, x_2; y_1, y_2)_{\gamma\beta\sigma} \\
\int d^2z_1d^2z_2\Sigma(x_1, z_1)_{\alpha\gamma}\Sigma^c(x_2, z_2)_{\beta\lambda}G(z_1, z_2; y_1, y_2)_{\gamma\beta\sigma}
\]

(18)

where

\[
G^b_{\mu\nu}(x_i | x_1, x_2; y_1, y_2)_{\alpha\beta\sigma} = \frac{1}{\epsilon}\langle 0^+ | T[A^b_{\mu\nu}(x_i)\psi_\gamma(x_1)\overline{\psi}_\beta(y_1)] | 0^- \rangle
\]

(19)

In the next section, it will be shown that the Green’s functions \( G^b_{\mu\nu}(x_1 | x_1, u_2; y_1, y_2) \) and \( G^b_{\mu\nu}(x_1, x_2 | x_1, x_2; y_1, y_2) \) are all B-S (two-particle) reducible. Therefore, we can write

\[
G^b_{\mu\nu}(x_1 | x_1, u_2; y_1, y_2)_{\gamma\lambda\sigma} = \int d^4z_1d^2z_2K^{(1)\mu}_{\nu\alpha\beta}(x_1, u_2; z_1, z_2)_{\gamma\lambda\sigma}G(z_1, z_2; y_1, y_2)_{\delta\tau\sigma},
\]

\[
G^b_{\mu\nu}(x_1, x_2 | x_1, x_2; y_1, y_2)_{\gamma\lambda\sigma} = \int d^4z_1d^2z_2K^{(2)\mu}_{\nu\alpha\beta}(x_1, x_2; z_1, z_2)_{\gamma\lambda\sigma}G(z_1, z_2; y_1, y_2)_{\delta\tau\sigma},
\]

\[
G^{ab}_{\mu\nu}(x_1, x_2 | x_1, x_2; y_1, y_2)_{\gamma\lambda\sigma} = \int d^4z_1d^2z_2K^{ab\mu}_{\nu\alpha\beta}(x_1, x_2; z_1, z_2)_{\gamma\lambda\sigma}G(z_1, z_2; y_1, y_2)_{\delta\tau\sigma}.
\]

(20)
where

\[ K_1(x_1, u_2; z_1, z_2) = \Omega^{u_2}_{\alpha \gamma} K^{(1)u}_{\alpha \gamma}(x_1, u_2; z_1, z_2) \gamma \lambda \delta \tau \]

\[ K_2(u_1, x_2; z_1, z_2) = \Omega^{u_1}_{\beta \gamma} K^{(2)u}_{\beta \gamma}(u_1, x_2; z_1, z_2) \gamma \lambda \delta \tau \]

\[ \mathcal{K}(x_1, x_2; z_1, z_2) = \Omega^{\alpha \gamma}_{\beta \gamma} \alpha \beta \gamma \delta \tau \]

Substituting Eq. (20) into Eq. (18) and defining

\[ \mathcal{K}(x_1, x_2; z_1, z_2) = \Omega^{\alpha \gamma}_{\beta \gamma} \alpha \beta \gamma \delta \tau \]

Eq. (18) will be written as a closed form

\[ \left[ (i \partial_{x_1} - m_1 + \Sigma)(i \partial_{x_2} - m_2 + \Sigma^c) \right] G(x_1, x_2; y_1, y_2) = \delta_{\alpha \beta} \delta_{\gamma \delta}(x_1 - y_1) \delta^4(x_2 - y_2) \]

\[ + \int d^4 z_1 d^4 z_2 K(x_1, x_2; z_1, z_2) \gamma \lambda \delta \tau G(z_1, z_2; y_1, y_2) \delta_{\rho \sigma} \]

where

\[ K(x_1, x_2; z_1, z_2) = \mathcal{K}(x_1, x_2; z_1, z_2) \gamma \lambda \delta \tau - \mathcal{K}_1(x_1, x_2; z_1, z_2) \gamma \lambda \delta \tau \]

\[ - \mathcal{K}_2(x_1, x_2; z_1, z_2) \gamma \lambda \delta \tau + \mathcal{K}_0(x_1, x_2; z_1, z_2) \gamma \lambda \delta \tau \]

is just the B-S interaction kernel in which

\[ \mathcal{K}_1(x_1, x_2; z_1, z_2) = \int d^4 u_2 \Sigma^c(x_2, u_2) \gamma \lambda \delta \tau G(x_1, u_2; y_1, y_2) \gamma \lambda \delta \tau \]

\[ \mathcal{K}_2(x_1, x_2; z_1, z_2) = \int d^4 u_1 \Sigma(x_1, u_1) \gamma \lambda \delta \tau G(u_1, x_2; y_1, y_2) \gamma \lambda \delta \tau \]

By making use of the Lehmann representation of the Green’s function \( G(x_1, x_2; y_1, y_2) \) or the well-known procedure proposed by Gell-Mann and Low [2], one may readily derive from Eq. (22) the B-S equation satisfied by B-S amplitudes describing the \( q\bar{q} \) bound states

\[ \left[ (i \partial_{x_1} - m_1 + \Sigma)(i \partial_{x_2} - m_2 + \Sigma^c) \right] \chi_{P\varsigma}(x_1, x_2) = \int d^4 y_1 d^4 y_2 K(x_1, x_2; y_1, y_2) \chi_{P\varsigma}(y_1, y_2) \]

where

\[ \chi_{P\varsigma}(x_1, x_2) = \langle 0^+ | T [\psi(x_1)\psi^\dagger(x_2)] | P\varsigma \rangle \]

represents the B-S amplitude in which \( P \) denotes the total momentum of a \( q\bar{q} \) bound state and \( \varsigma \) marks the other quantum numbers of the state. The above equation can be written in the form of an integral equation if we operating on the both sides of the above equation with the inverse \( (i \partial_{x_1} - m_1 + \Sigma)^{-1}(i \partial_{x_2} - m_2 + \Sigma^c)^{-1} \)

\[ \chi_{P\varsigma}(x_1, x_2) = \int d^4 z_1 d^4 z_2 d^4 y_1 d^4 y_2 S_F(x_1 - z_1) S^F_F(x_2 - z_2) K(z_1, z_2; y_1, y_2) \chi_{P\varsigma}(y_1, y_2). \]

III. B-S REDUCIBILITY OF THE GREEN’S FUNCTIONS

The aim of this section is to analyze the B-S reducibility of the Green’s functions \( G^{\mu}_\beta(x_1 | x_2; y_1, y_2) \), \( G^{\mu}_\nu(x_2 | u_1, x_2; y_1, y_2) \) and \( G^{\mu \nu}(x_1, x_2 | x_1, x_2; y_1, y_2) \) shown in Eq. (20). First, we start from the relation between the full \( q\bar{q} \) four-point Green’s function \( G(x_1, x_2; y_1, y_2) \) and its connected one \( G_c(x_1, x_2; y_1, y_2) \). In the case that the quark and the antiquark have different flavors, as derived in the beginning of Appendix, this relation is [15-17]

\[ G(x_1, x_2; y_1, y_2) = G_c(x_1, x_2; y_1, y_2) + S_F(x_1 - y_1) S^F_F(x_2 - y_2) \]

Here it is noted that since the flavors of the quark and the antiquark are different, the contraction between the quark and the antiquark vanishes, implying that the terms related to the quark-antiquark annihilation are absent in the above decomposition. A similar relation for the Green’s function \( G^{\mu}_\beta(x_1 | x_2; y_1, y_2) \) can be written from Eq. (A7) given in Appendix by setting the source \( J = 0 \) [15-17],

\[ G^{\mu}_\beta(x_1 | x_1, x_2; y_1, y_2) = G^{\mu}_\beta(x_1 | x_1, x_2; y_1, y_2) + \Delta^{\alpha}_\mu(x_1 | x_1; y_1) S^F_F(x_2 - y_2) \]

\[ + S_F(x_1 - y_1) \Lambda^{\alpha}_\mu(x_1 | x_2; y_2) \]
where \( i = 1, 2 \), \( G^a_{\mu}(x_i | x_1, x_2; y_1, y_2) \) is the connected part of the Green’s function \( G^a_{\mu}(x_i | x_1, x_2; y_1, y_2) \) and \( \Lambda^a_{\mu}(x_i | x_1; y_1) \) and \( \Lambda^{ca}_{\mu}(x_i | x_2; y_2) \) are the three-point Green’s functions as defined in Eq. (16).

Let us analyze the connected Green’s functions on the right-hand side of Eq. (30) through the technique of one-particle-irreducible decomposition of connected Green’s functions. The decompositions have been carried out in the Appendix. According to the decomposition in Eq. (A15), the three-point gluon-quark Green’s functions \( \Lambda^a_{\mu}(x_i | x_j; y_k) \) and \( \Lambda^{ca}_{\mu}(x_i | x_j; y_k) \) which are fully connected can be represented in the form

\[
\Lambda^a_{\mu}(x_i | x_j; y_k) = \frac{\delta}{i\delta J^{\mu}(x_i)} S^a_F(x_j - y_k)^J |_{J=0} = \int d^4z_1 \Sigma^a_{\mu}(x_i | x_j; z_1) S^a_F(z_1 - y_k)
\]

(30)

where

\[
\Sigma^a_{\mu}(x_i | x_j; z_1) = \int d^4u_1 d^4u_2 \Delta^{ab}_{\mu\nu}(x_i - u_1) S^a_F(x_j - u_2) \Gamma^{\nu\mu}(u_1 | u_2, z_1)
\]

(31)

and

\[
\Lambda^{ca}_{\mu}(x_i | x_j; y_k) = \frac{\delta}{i\delta J^{\mu}(x_i)} S^c_F(x_j - y_k)^J |_{J=0} = \int d^4z_2 \Sigma^{ca}_{\mu}(x_i | x_j; z_2) S^c_F(z_2 - y_k)
\]

(32)

where

\[
\Sigma^{ca}_{\mu}(x_i | x_j; z_2) = \int d^4u_1 d^4u_2 \Delta^{ab}_{\mu\nu}(x_i - u_1) S^c_F(x_j - u_2) \Gamma^{\nu\mu}(u_1 | u_2, z_2)
\]

(33)

In the above,

\[
\Delta^{ab}_{\mu\nu}(x_i - u_j) = \frac{1}{i} \langle 0^+ | T[\Lambda^a_{\mu}(x_i) \Lambda^{\nu b}(u_j)] | 0^- \rangle = \frac{1}{i} D^{ab}_{\mu\nu}(x_i - u_j)
\]

(34)

is the exact gluon propagator and \( \Gamma^{\nu\mu}(u_1 | u_2, z_1) \) and \( \Gamma^{\nu\mu}(u_1 | u_2, z_2) \) are the gluon-quark and gluon-antiquark three-line proper vertices respectively as defined in Eqs. (A17) and (A18). In the case of \( i = j \), the functions in Eqs. (31) and (33) give the quark and antiquark self-energies as shown in Eq. (15). When \( i \neq j \), the functions in Eqs. (31) and (33) are related to the one gluon exchange interactions. With the expressions in Eqs. (31) and (33), the last two terms in Eq. (29) can be represented as

\[
\Lambda^a_{\mu}(x_i | x_1; y_1) S^c_F(x_2 - y_2) + \Lambda^{ca}_{\mu}(x_i | x_2; y_2) S^c_F(x_1 - y_1)
\]

\[
= \int d^4z_1 d^4z_2 K_{0j}^{(i)a}(x_1, x_2; z_1, z_2) S^c_F(z_1 - y_1) S^c_F(z_2 - y_2)
\]

(35)

where

\[
K_{0j}^{(i)a}(x_1, x_2; z_1, z_2) = \Sigma^a_{\mu}(x_i | x_1; z_1) \delta^4(x_2 - z_2) + \Sigma^{ca}_{\mu}(x_i | x_2; z_2) \delta^4(x_1 - z_1)
\]

(36)

Now we turn to the irreducible decomposition of the first term in Eq. (29). As stated in Appendix, this decomposition may be derived from the functional differential of the Green’s function \( G_c(x_i, x_2; y_1, y_2) \) with respect to the source \( J^{\mu}(x_i) \) by using the one-particle irreducible decomposition of the function \( G_c(x_i, x_2; y_1, y_2) \). The latter decomposition whose derivation is sketched in Appendix is well-known [15-17] and can be represented in the form

\[
G_c(x_1, x_2; y_1, y_2) = \int \prod_{i=1}^2 d^4u_i d^4v_i S^c_F(x_i - u_i) S^c_F(x_i - v_i) \times \Gamma(u_1, u_2; v_1, v_2) S^c_F(v_1 - y_1) S^c_F(v_2 - y_2)
\]

(37)

where

\[
\Gamma(u_1, u_2; v_1, v_2) = \Gamma_1(u_1, u_2; v_1, v_2) + \Gamma_2(u_1, u_2; v_1, v_2)
\]

(38)

in which

\[
\Gamma_1(u_1, u_2; v_1, v_2) = - \int d^4z_1 d^4z_2 \Gamma^{\nu\mu}(z_1 | u_1, v_1) D^{\nu\mu}(z_1 - z_2) \Gamma^{\nu\mu}(z_2 | u_2, v_2)
\]

(39)

and \( \Gamma_2(u_1, u_2; v_1, v_2) \) defined in Eq. (A20) is the quark-antiquark four-line proper vertex. After substituting the expressions in Eqs. (37)-(39), which are now given in the presence of source \( J \), into Eq. (A8) and completing the
differentiation by using the differentials denoted in Eqs. (A21) and (A23), the one-particle irreducible decomposition of the Green’s function $G_{\mu \nu}^{a}(x_{i} | x_{1}, x_{2}; y_{1}, y_{2})$ will be found and, thereby, we can write

\[
G_{\mu \nu}^{a}(x_{i} | x_{1}, x_{2}; y_{1}, y_{2}) = \int d^{4} z_{1} d^{4} z_{2} K^{(i)\mu}_{\nu}(x_{1}, x_{2}; z_{1}, z_{2}) G_{c}(z_{1}, z_{2}; y_{1}, y_{2}) \\
+ \int d^{4} z_{1} d^{4} z_{2} [K^{(i)\mu}_{\nu}(x_{1}, x_{2}; z_{1}, z_{2}) + K^{(i)\mu}_{\nu}(x_{1}, x_{2}; z_{1}, z_{2})] S_{F}(z_{1} - y_{1}) S_{F}(z_{2} - y_{2}) + G_{\mu \nu}^{a}(x_{i} | x_{1}, x_{2}; y_{1}, y_{2}) \text{RE}
\]

(40)

where $K^{(i)\mu}_{\nu}(x_{1}, x_{2}; z_{1}, z_{2})$ was defined in Eq. (36),

\[
K^{(i)\mu}_{\nu}(x_{1}, x_{2}; z_{1}, z_{2}) = \int \prod_{j=1}^{2} d^{4} u_{j} d^{4} v_{j} S_{F}(x_{1} - u_{1}) S_{F}^{c}(x_{2} - u_{2}) \\
\times \left[ \Gamma(u_{1}, u_{2}; v_{1}, z_{2}) \Sigma_{\mu}^{2}(x_{i} | v; z_{1}) + \Gamma(u_{1}, u_{2}; v_{1}, z_{2}) \Sigma_{\mu}^{c}(x_{i} | v; z_{2}) \right],
\]

(41)

\[
K_{2\mu}^{(i)\mu}(x_{1}, x_{2}; z_{1}, z_{2}) = \int \prod_{j=1}^{2} d^{4} u_{j} S_{F}(x_{1} - u_{1}) S_{F}^{c}(x_{2} - u_{2}) \Gamma_{1R}^{\mu}(x_{i} | u_{1}, u_{2}; z_{1}, z_{2})
\]

(42)

and

\[
G_{\mu \nu}^{a}(x_{i} | x_{1}, x_{2}; y_{1}, y_{2}) \text{RE} = \int \prod_{j=1}^{2} d^{4} u_{j} d^{4} v_{j} d^{4} u S_{F}(x_{1} - u_{1}) S_{F}^{c}(x_{2} - u_{2}) \Gamma_{2R}^{\mu}(x_{i} | u_{1}, u_{2}; v_{1}, v_{2})
\]

(43)

where $\Gamma_{1R}^{\mu}(x_{i} | u_{1}, u_{2}; v_{1}, v_{2})$ and $\Gamma_{2R}^{\mu}(x_{i} | u_{1}, u_{2}; v_{1}, v_{2})$ are respectively the B-S irreducible and reducible parts of the vertex $\Gamma_{1R}^{\mu}(x_{i} | u_{1}, u_{2}; v_{1}, v_{2})$ defined in Eq. (A25). Here it has been considered that the vertex $\Gamma_{1R}^{\mu}(x_{i} | u_{1}, u_{2}; v_{1}, v_{2})$ is two-particle reducible (or say, B-S reducible) although it is one-particle irreducible. This vertex is specified in the following. Corresponding to Eq. (38), we have

\[
\Gamma_{1R}^{\mu}(x_{i} | u_{1}, u_{2}; v_{1}, v_{2}) = \Gamma_{1R}^{\mu}(x_{i} | u_{1}, u_{2}, v_{1}, v_{2}) + \Gamma_{2R}^{\mu}(x_{i} | u_{1}, u_{2}; v_{1}, v_{2})
\]

(44)

where

\[
\Gamma_{1R}^{\mu}(x_{i} | u_{1}, u_{2}; v_{1}, v_{2}) = \int d^{4} z D_{\mu}^{\nu}(x_{i} - z) \Gamma_{1R}^{\nu}(z | u_{1}, u_{2}; v_{1}, v_{2})
\]

(45)

in which

\[
\Gamma_{1R}^{\nu}(z | u_{1}, u_{2}, v_{1}, v_{2}) = \Gamma_{1R}^{\nu}(z | u_{1}, u_{2}, v_{1}, v_{2}) \Gamma_{1R}^{\nu}(z | u_{1}, u_{2}, v_{1}, v_{2})
\]

(46)

\[
\Gamma_{2R}^{\nu}(z | u_{1}, u_{2}, v_{1}, v_{2}) = \int d^{4} v_{1} d^{4} v_{2} D_{\mu}^{\nu}(z_{1} - v_{1}) \Gamma_{1R}^{\nu}(z_{1} - v_{1}) \Gamma^{\nu} D_{\mu}^{\nu}(z_{2} - v_{2}) + \Pi_{\rho \sigma}(z_{1}, z_{2}) \Gamma^{\nu}(z_{1} - v_{1}, v_{2}) + \Pi_{\rho \sigma}(z_{2}, z_{1}, z_{2}) \Gamma^{\nu}(z_{2} - v_{2}, v_{1})
\]

(47)

with

\[
\Pi_{\rho \sigma}(z_{1}, z_{2}) = \int d^{4} v_{1} d^{4} v_{2} D_{\rho \rho}^{\nu}(z_{1} - v_{1}) \Gamma_{1R}^{\nu}(z_{1} - v_{1}) \Gamma^{\nu} D_{\sigma}^{\nu}(z_{2} - v_{2})
\]

(48)

\[
\Pi_{\rho \sigma}(z_{2}, z_{1}, z_{2}) = \int d^{4} v_{1} d^{4} v_{2} D_{\rho \rho}^{\nu}(z_{2} - v_{2}) \Gamma_{1R}^{\nu}(z_{2} - v_{2}) \Gamma^{\nu} D_{\sigma}^{\nu}(z_{2} - v_{2})
\]

(49)
Since the vertex $\Gamma^\mu(x_1 \mid u_1, u_2; v_1, v_2)$ is B-S reducible, the function $G^\mu_{\alpha\beta}(x_1 \mid x_1, x_2; y_1, y_2)_{RE}$, as a part of the connected Green’s function, must be represented in a B-S reducible form such that

$$G^\alpha_{\epsilon\mu}(x_1 \mid x_1, x_2; y_1, y_2)_{RE} = \int d^4z_1 d^4z_2 \tilde{K}^{(i)}_{\mu}(x_1, x_2; z_1, z_2) G_c(z_1, z_2; y_1, y_2).$$

(50)

As argued in Ref. [15], the kernel $\tilde{K}^{(i)}_{\mu}(x_1, x_2; z_1, z_2)$ must be of the form

$$K^{(i)}_{\mu}(x_1, x_2; z_1, z_2) = K^{(i)}_{\mu}(x_1, x_2; z_1, z_2) + K^{(i)}_{\mu}(x_1, x_2; z_1, z_2) + K^{(i)}_{\mu}(x_1, x_2; z_1, z_2).$$

(51)

so as to make the B-S equation to be closed. With the above expression, when substituting Eq. (50) into Eq. (40) and then summing up the both expressions in Eqs. (35) and (40), noting Eq. (28), we obtain

$$G^\alpha_{\epsilon\mu}(x_1 \mid x_1, x_2; y_1, y_2) = \int d^4z_1 d^4z_2 K^{(i)}_{\mu}(x_1, x_2; z_1, z_2) G(z_1, z_2; y_1, y_2)$$

(52)

where

$$K^{(i)}_{\mu}(x_1, x_2; z_1, z_2) = K^{(i)}_{\mu}(x_1, x_2; z_1, z_2) + K^{(i)}_{\mu}(x_1, x_2; z_1, z_2) + K^{(i)}_{\mu}(x_1, x_2; z_1, z_2).$$

(53)

Eq. (52) with $i = 1, 2$ just gives the first two equalities in Eq. (20).

Now we proceed to discuss the B-S reducibility of the Green’s function $G^{ab}_{\mu\nu}(x_1, x_2 \mid x_1, x_2; y_1, y_2)$. The decomposition of this Green’s function into the connected ones can be given by the following calculation [16, 17]

$$G^{ab}_{\mu\nu}(x_1, x_2 \mid x_1, x_2; y_1, y_2) = D^{ab}_{\mu\nu}(x_1 - x_2) G(x_1, x_2; y_1, y_2) + \frac{\delta}{\delta J^{\mu\nu}(x_2)} G^\alpha_{\epsilon\mu}(x_1 \mid x_1, x_2; y_1, y_2) = 0$$

(54)

where $D^{ab}_{\mu\nu}(x_1 - x_2)$ is the exact gluon propagator defined in Eq. (34) and $G^\alpha_{\epsilon\mu}(x_1 \mid x_1, x_2; y_1, y_2)$ is the Green’s function defined in Eq. (13) in presence of the external source $J$. It is emphasized that the expression of the $G^\alpha_{\epsilon\mu}(x_1 \mid x_1, x_2; y_1, y_2)$ formally is the same as given in Eq. (29) or (52). On inserting Eq. (52) into Eq. (54), completing the differentiation and using Eq. (52) once again, one can get

$$G^{ab}_{\mu\nu}(x_1, x_2 \mid x_1, x_2; y_1, y_2) = \int d^4z_1 d^4z K^{ab}_{\mu\nu}(x_1, x_2; z_1, z_2) G(z_1, z_2; y_1, y_2)$$

(55)

where

$$K^{ab}_{\mu\nu}(x_1, x_2; z_1, z_2) = \sum_{i=0}^{2} K^{(i)ab}_{\mu\nu}(x_1, x_2; z_1, z_2)$$

(56)

with the definitions:

$$K^{(0)ab}_{\mu\nu}(x_1, x_2; z_1, z_2) = D^{ab}_{\mu\nu}(x_1 - x_2) \delta^4(x_1 - z_1) \delta^4(x_2 - z_2)$$

(57)

$$K^{(1)ab}_{\mu\nu}(x_1, x_2; z_1, z_2) = \frac{\delta}{\delta J^{\mu\nu}(x_2)} K^{(1)ab}_{\mu\nu}(x_1, x_2; z_1, z_2) = 0$$

(58)

which will be derived specifically in the next section and

$$K^{(2)ab}_{\mu\nu}(x_1, x_2; z_1, z_2) = \int d^4u_1 d^4u_2 K^{(1)ab}_{\mu\nu}(x_1, x_2; u_1, u_2) K^{(2)ab}_{\mu\nu}(u_1, u_2; z_1, z_2)$$

(59)

here $K^{(1)ab}_{\mu\nu}(x_1, x_2; u_1, u_2)$ and $K^{(2)ab}_{\mu\nu}(u_1, u_2; z_1, z_2)$ have been expressed in Eqs. (53), (36), (41) and (42). Eq. (55) precisely represents the B-S reducibility of the Green’s function $G^{ab}_{\mu\nu}(x_1, x_2 \mid x_1, x_2; y_1, y_2)$. 

7
IV. EXPRESSION OF THE B-S KERNEL

Substituting Eq. (53) with the concrete expressions given in Eqs. (36), (41) and (42) into the first two equalities in Eq. (21) and then inserting the $K_1(x_1, u_2; z_1, z_2)$ and $K_1(u_1, x_2; z_1, z_2)$ thus obtained into Eq. (24), we directly get the expressions of the kernels $\overline{K}_1(x_1, x_2; z_1, z_2)$ and $\overline{K}_2(x_1, x_2; z_1, z_2)$. As seen from Eq. (24) and the last equality in Eq. (21), the kernels $\overline{K}_1(x_1, x_2; z_1, z_2)$, $\overline{K}_2(x_1, x_2; z_1, z_2)$ and $\overline{K}_0(x_1, x_2; z_1, z_2)$ are all related to the self-energies of the quark and antiquark appearing in the B-S amplitude. It will be seen that these kernels play the role of cancelling the corresponding terms contained in the kernel $\overline{K}_1(x_1, x_2; z_1, z_2)$ defined in the third equality of Eq. (21). In view of Eqs. (56)-(59), the kernel $\overline{K}_1(x_1, x_2; z_1, z_2)$ can be written as

$$\overline{K}(x_1, x_2; z_1, z_2) = K^{(0)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} + K^{(1)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} + K^{(2)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma}$$

where

$$K^{(0)}(x_1, x_2; z_1, z_2) = (\Omega^{\mu\nu})_{\alpha\beta}(\Omega^{\mu
u})_{\beta\alpha}D_{\mu\nu}^{ab}(x_1 - x_2)\delta^4(x_1 - z_1)\delta^4(x_2 - z_2)$$

is the one-gluon exchange kernel for the t-channel interaction,

$$\overline{K}^{(1)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} = (\Omega^{\mu\nu})_{\alpha\gamma}(\Omega^{\mu
u})_{\beta\lambda}K^{(1)ab}_{\mu\nu}(x_1, x_2; z_1, z_2)\gamma\lambda\rho\sigma$$

and

$$\overline{K}^{(2)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} = (\Omega^{\mu\nu})_{\alpha\gamma}(\Omega^{\mu
u})_{\beta\lambda}K^{(2)ab}_{\mu\nu}(x_1, x_2; z_1, z_2)\gamma\lambda\rho\sigma$$

To prove the aforementioned cancellation, let us compute the functions $K^{(1)ab}_{\mu\nu}(x_1, x_2; z_1, z_2)$ and $K^{(2)ab}_{\mu\nu}(x_1, x_2; z_1, z_2)$ defined in Eqs. (58) and (59). According to the expressions in Eqs. (53) and (36), it is convenient to write

$$K^{(1)}_{\mu}(x_1, x_2; z_1, z_2) = \Sigma_{\mu}(x_1 | x_1; z_1)\delta^4(x_2 - z_2) + \overline{K}^{(1)}_{\mu}(x_1, x_2; z_1, z_2)$$

where

$$\overline{K}^{(1)}_{\mu}(x_1, x_2; z_1, z_2) = \Sigma_{\mu}(x_1 | x_2; z_2)\delta^4(x_1 - z_1) + K^{(1)}_{\mu}(x_1, x_2; z_1, z_2) + K^{(2)}_{\mu}(x_1, x_2; z_1, z_2)$$

When Eq. (64) is inserted into Eq. (59) and then Eq. (59) is inserted into Eq. (63), considering the second equality in Eq. (24), we get

$$\overline{K}^{(2)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} = \overline{K}_2(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} + \int d^4u_1d^4u_2(\Omega^{\mu\nu})_{\alpha\gamma}(\Omega^{\mu
u})_{\beta\lambda}K^{(1)ab}_{\mu\nu}(x_1, x_2; u_1, u_2)\gamma\lambda\rho\sigma\gamma\lambda\rho\sigma$$

In the above, we have considered that $\Sigma(x_1, z_1)_{\alpha\rho} = (\Omega^{\mu\nu})_{\alpha\gamma}\Sigma_{\rho}(x_1 | x_1; z_1)\gamma\rho$ is the quark self-energy.

Now, let us turn to compute $K^{(1)ab}_{\mu\nu}(x_1, x_2; z_1, z_2)$. Looking at the expressions in Eqs. (33), (41) and (42), it is seen that the function $\overline{K}^{(1)ab}_{\mu}(x_1, x_2; z_1, z_2)$ can be represented in the form

$$\overline{K}^{(1)ab}_{\mu}(x_1, x_2; z_1, z_2) = \int d^4u_2S_F^{a}(x_2 - u_2)Q_{\mu}^{a}(x_1 | x_1, u_2; z_1, z_2)$$

where

$$Q_{\mu}^{a}(x_1 | x_1, u_2; z_1, z_2) = Q_{\mu}^{(0)a}(x_1 | x_1, u_2; z_1, z_2) + Q_{\mu}^{(1)a}(x_1 | x_1, u_2; z_1, z_2) + Q_{\mu}^{(2)a}(x_1 | x_1, u_2; z_1, z_2)$$

in which

$$Q_{\mu}^{(0)a}(x_1 | x_1, u_2; z_1, z_2) = \int d^4u_1\Delta_{\mu\nu}(x_1 - u_1)\Gamma^{b
u}(u_1 | u_2, z_2)\delta^4(x_1 - z_1)$$

$$Q_{\mu}^{(1)a}(x_1 | x_1, u_2; z_1, z_2) = \int d^4u_1d^4uS_F(x_1 - u_1)\Gamma(u_1, u_2; z_2)\times\Sigma_{\mu}(x_1 | u_1, z_1) + \Gamma(u_1, u_2; z_1, u)\Sigma_{\mu}(x_1 | u, z_2)$$
and
\[
Q_{\mu}^{(2)\alpha}(x_1 | x_1, u_2; z_1, z_2) = \int d^4u_1 S_F(x_1 - u_1) \Gamma_{HR}^{\mu\nu}(x_1 | u_1, u_2; z_1, z_2).
\] (71)

When the expression in Eq. (64) with the expression in Eq. (67) is inserted into Eq. (58) and completing the differentiation, noticing Eq. (32), one can find
\[
\begin{align*}
K_{\mu\nu}^{(1)ab}(x_1, x_2; z_1, z_2) &= \Sigma_{\mu\nu}^{ab}(x_1, x_2 | x_1, z_1) \delta^4(x_2 - z_2) \\
&+ \int d^4v \Sigma_{\mu\nu}^{cb}(x_2 | x_2, v) K_{\mu\nu}^{(1)a}(x_1, v; z_1, z_2) \\
&+ \int d^4u_2 S_F(x_2 - u_2) Q_{\mu\nu}^{ab}(x_1, x_2 | x_1, u_2; z_1, z_2)
\end{align*}
\]
(72)

where
\[
\Sigma_{\mu\nu}^{ab}(x_1, x_2 | x_1, z_1) = \frac{\delta}{i\delta \mu\nu(x_2)} \Sigma_{\mu\nu}^{a}(x_1 | x_1, z_1)^J | J=0
\]
(73)

and
\[
\begin{align*}
Q_{\mu\nu}^{ab}(x_1, x_2 | x_1, u_2; z_1, z_2) &= \frac{\delta}{i\delta \mu\nu(x_2)} Q_{\mu}^{a}(x_1 | x_1, u_2; z_1, z_2)^J | J=0
\end{align*}
\]
(74)

which will be calculated in detail soon later. According to Eq. (64), the second term in Eq. (72) can be represented as
\[
\begin{align*}
&\int d^4v \Sigma_{\mu\nu}^{cb}(x_2 | x_2, v) K_{\mu\nu}^{(1)a}(x_1, v; z_1, z_2) \\
&- \Sigma_{\mu\nu}^{a}(x_1 | x_1, z_1)^J | J=0
\end{align*}
\]
(75)

Noticing this expression, when Eq. (72) is substituted into Eq. (62), we have
\[
\begin{align*}
&\tilde{K}^{(1)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} = (\Pi^{\mu\nu})_{\alpha\gamma} \Sigma_{\mu\nu}^{ab}(x_1, x_2 | x_1, z_1)_{\gamma\lambda\sigma\rho} (\Pi^{\mu\nu})_{\beta\delta} \delta^4(x_2 - z_2) \\
&+ \int d^4u_2 (\Pi^{\mu\nu})_{\alpha\gamma} (\Pi^{\mu\nu})_{\beta\delta} S_F(x_2 - u_2) \gamma\lambda\rho \delta^4(x_2 - z_2) \\
&+ \tilde{K}^{(1)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} - \tilde{K}^{(1)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma}
\end{align*}
\]
(76)

where the definitions given in Eqs. (21) and (24) as well as \(\Sigma^c(x_2, z_2)_{\beta\gamma} = (\Pi^{\mu\nu})_{\beta\lambda} \Sigma_{\mu\nu}^{c}(x_2 | x_2; z_2)_{\lambda\rho} \) which is the antiquark self-energy have been noted.

It is clear that when Eq. (60) is substituted into Eq. (23) and considering the expressions in Eqs. (66) and (76), we see, the last three terms in Eq. (23) are all cancelled out. As a result of the cancellation, we have
\[
\begin{align*}
&K(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} = K^{(0)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} \\
&+ (\Pi^{\mu\nu})_{\alpha\gamma} \Sigma_{\mu\nu}^{ab}(x_1, x_2 | x_1, z_1)_{\gamma\lambda\rho\sigma} (\Pi^{\mu\nu})_{\beta\delta} \delta^4(x_2 - z_2) \\
&+ \int d^4u_2 (\Pi^{\mu\nu})_{\alpha\gamma} (\Pi^{\mu\nu})_{\beta\delta} S_F(x_2 - u_2) \gamma\lambda\rho \delta^4(x_2 - z_2) \\
&+ \tilde{K}^{(1)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} - \tilde{K}^{(1)}(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma}
\end{align*}
\]
(77)

To give an explicit expression of the above kernel, we need to compute the functions \(\Sigma_{\mu\nu}^{ab}(x_1, x_2 | x_1, z_1)\) and \(Q_{\mu\nu}^{ab}(x_1, x_2 | x_1, u_2; z_1, z_2)\). Substituting Eq. (31) into Eq. (73) and employing the formulas as given in Eqs. (30), (32), (A21) and (A23) to complete the differentiation, it is easy to get
\[
\begin{align*}
\Sigma_{\mu\nu}^{ab}(x_1, x_2 | x_1, z_1) &= \int \prod_{i=1}^{3} d^4u_i \{ \Sigma_{\mu\nu}^{b}(x_2 | x_1, u_1) S_F(u_1 - u_2) \Delta_{\mu\nu}^{ac}(x_1 - u_3) \\
&\times \Gamma^{c\lambda}(u_3 | u_2, z_1) + S_F(x_1 - u_1) \Delta_{\mu\nu}^{bd}(x_2 - u_2) [\Pi^{\alpha\gamma}(u_2, x_1, u_3) \Gamma^{c\lambda}(u_3 | u_1, z_1) \\
&+ \Delta_{\mu\nu}^{ac}(x_1 - u_3) \Gamma^{dr, c\lambda}(u_2, u_3 | u_1, z_1)] \}
\end{align*}
\]
(78)

where \(\Pi^{\alpha\gamma}(u_2, x_1, u_3)\) was defined in Eq. (47) and \(\Gamma^{c\lambda}(u_2, u_3 | u_1, z_1)\) is a kind of gluon-quark four-line vertex defined in Eq. (A27).

In accordance with Eq. (68), the function \(Q_{\mu\nu}^{ab}(x_1, x_2 | x_1, u_2; z_1, z_2)\) in Eq. (77) can be written as
\[ Q^{ab}_{\mu \nu}(x_1, x_2 | x_1, u_2; z_1, z_2) = Q^{(0)ab}_{\mu \nu}(x_1, x_2 | x_1, u_2; z_1, z_2) + Q^{(1)ab}_{\mu \nu}(x_1, x_2 | x_1, u_2; z_1, z_2) + Q^{(2)ab}_{\mu \nu}(x_1, x_2 | x_1, u_2; z_1, z_2) \] (79)

The three terms on the right-hand side of Eq. (79) can easily be derived from the expressions written in Eqs. (69)-(71) by applying the formulas denoted in Eqs. (30), (32), (A21) and (A23). The results are displayed below.

\[ Q^{(0)ab}_{\mu \nu}(x_1, x_2 | x_1, u_2; z_1, z_2) = \frac{\delta}{i \delta \Gamma^{ab}_{\mu \nu}(x_2)} \frac{d^4 u}{d^4 x_2} Q^{(0)a}_{\mu}(x_1 | x_1, u_2; z_1, z_2) \bigg|_{J=0} \]
\[ = \int d^4 u_1 d^4 u_3 \Delta^{\mu \nu}(x_2 | x_1, u_1, u_2; z_1, z_2) \bigg|_{J=0} \]
\[ + \Gamma^{a}(u_3, u_2; z_1, u_1) \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ = \int d^4 u_1 d^4 u_3 d^4 u \Sigma^{ab}(x_2 | x_1, u_1) \bigg|_{J=0} \]
\[ + \Gamma^{a}(u_3, u_2; z_1, u_1) \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
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\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
\[ + \Sigma^{ab}(x_1 | u_2; z_2) \]
\[ + \bigg|_{J=0} \]
with $\Gamma_{\lambda\mu\nu(\sigma)}^{ab}(u, z, u_1, u_2)$ being the gluon four-line vertex, it is not difficult to derive from Eqs. (82) and (46) the expression of $\hat{\Gamma}_{\lambda\tau}^{cd}(u, u_3 | u_1, u_2; z_1, z_2)$ which is shown in the following:

$$
\hat{\Gamma}_{\lambda\tau}^{cd}(u, u_3 | u_1, u_2; z_1, z_2) = \Gamma_{1\lambda\tau}^{cd}(u, u_3 | u_1, u_2; z_1, z_2) + \Gamma_{2\lambda\tau}^{cd}(u, u_3 | u_1, u_2; z_1, z_2)
$$

(88)

where

$$
\begin{align*}
\Gamma_{1\lambda\tau}^{cd}(u, u_3 | u_1, u_2; z_1, z_2) &= - \int d^4v_1d^4v_2\Gamma_{e\lambda\tau}^{cd,\sigma}(u, v_1 | u_1, z_1)D_{\sigma\rho}^{\nu}(v_1 - v_2)\Gamma_{\nu\lambda\tau}^{c\rho}(v_2 | u_2, z_2) \\
+ &\Gamma_{\lambda\tau}^{cd,\sigma}(u, v_1 | u_1, z_1)[\Pi_{\lambda\tau}^{c\rho}(u_3, v_1, v_2)\Gamma_{\nu\lambda\tau}^{c\rho}(v_2 | u_2, z_2) + D_{\lambda\tau}^{\nu}(v_1 - v_2)\Gamma_{\nu\lambda\tau}^{c\rho}(v_2; u_2, z_2)] \\
+ &\Gamma_{\lambda\tau}^{cd,\sigma}(u, v_1 | u_1, z_1)\Gamma_{\nu\lambda\tau}^{c\rho}(v_1 | u_2, z_2) + \Gamma_{\lambda\tau}^{c\rho}(v_1 | u_1, z_1)\Pi_{\lambda\tau}^{c\rho}(u_3, v_1, v_2) \\
\times &\Gamma_{\nu\lambda\tau}^{c\rho}(v_2 | u_2, z_2) + D_{\sigma\rho}^{\nu}(v_1 - v_2)\Pi_{\lambda\tau}^{c\rho}(u_3, v_1, v_2) \\
\times &\Gamma_{\nu\lambda\tau}^{c\rho}(v_2 | u_2, z_2) + \Gamma_{\nu\lambda\tau}^{c\rho}(v_1 | u_1, z_1)\Pi_{\lambda\tau}^{c\rho}(u_3, v_1, v_2) \\
\times &\Gamma_{\nu\lambda\tau}^{c\rho}(v_2 | u_2, z_2)
\end{align*}
$$

(89)

$\Gamma_{2\lambda\tau}^{cd}(u, u_3 | u_1, u_2; z_1, z_2)$ is the six-point vertex $\Gamma_{\lambda\tau}^{cd}(u, u_3 | u_1, u_2; z_1, z_2)$ defined in Eq. (A30) and the other vertices can be read from Eq. (A24).

At last, we would like to note that the B-S kernel derived in Eqs. (77)-(89) with $\hat{K}_{\mu}^{(1)}(x_1, x_2; u_1, u_2)$ being given in Eq. (65) and $K_{\nu}^{(2)}(u_1, u_2; z_1, z_2)$ in Eq. (53) with $i = 2$ is symmetric with respect to quark and antiquark. This point can clearly be seen when all the terms in the kernel are represented by Feynman diagrams. For example, there are three terms in the second term in Eq. (77) which is given by the three terms in Eq. (78). The two terms given by the last two terms in Eq. (78) have an one-to-one correspondence with the two terms appearing in the third terms in Eq. (77) which are given by the two terms in Eq. (80). The remaining term given by the first term in Eq. (78) corresponds to such a terms included in the last term in Eq. (77) that it is given by the term $\Sigma_{\nu}^{\mu}(x_1 | x_2; u_2)\delta^{4}(x_1 - u_1)$ in $\hat{K}_{\mu}^{(1)}(x_1, x_2; u_1, u_2)$ and the term $\Sigma_{\nu}^{\mu}(x_2 | u_2; z_2)\delta^{4}(u_1 - z_1)$ in $K_{\nu}^{(2)}(u_1, u_2; z_1, z_2)$ and can be represented as

$$
\int \frac{3}{i=1} d^4u_1\Delta_{\mu\lambda}^{\nu}(x_1 - u_1)\Omega_{\mu\lambda}^{\nu}(x_2 - u_2)\Sigma_{\nu}^{\mu}(x_1 | u_2, u_3)\delta^{4}(x_1 - z_1) + \Sigma_{\nu}^{\mu}(x_2 | u_2, z_2)\delta^{4}(u_1 - z_1)
$$

(90)

The six terms mentioned above give a higher order correction to the vertices in the one-gluon exchange kernel.

V. SUMMARY AND REMARKS

In this paper, we have derived a new expression of the B-S interaction kernel for quark-antiquark bound states by means of the technique of irreducible decomposition of Green’s functions. The kernel given in the case that the quark and the antiquark have different flavors was expressed in Eq. (77). In Eq. (77), the $K^{(0)}(x_1, x_2; z_1, z_2)$ is the one-gluon exchange interaction kernel represented in Eq. (61), the function $\Sigma_{\nu}^{\mu}(x_1, x_2 | x_1, z_1)$ was described in Eq. (78), the function $\Sigma_{\nu}^{\mu}(x_1, x_2 | x_1, u_2; z_1, z_2)$ was formulated in detail in Eqs. (79)-(89) and the functions $\hat{K}_{\mu}^{(1)}(x_1, x_2; u_1, u_2)$ and $K_{\nu}^{(2)}(u_1, u_2; z_1, z_2)$ were specified in Eqs. (64), (65), (36), (41) and (42). In comparison with the previous expression derived in Ref. [14] which is compactly represented in terms of a few types of Green's functions, expression of the kernel derived in this paper is represented in terms of the quark, antiquark and gluon propagators and some kinds of quark, antiquark and/or gluon three, four, five and six-point vertices and therefore exhibits a more specific structure of kernel. It is noted that although the kernel given in this paper is limited to the case that the quark and the antiquark are of different flavors, it is sufficient to solve the problem of quark confinement because the strong interaction between the quark and antiquark which have the same flavors is the same as the one between the quark and antiquark which are of different flavors.

In previous investigations of meson spectrum within the framework of B-S equation [12,19], one often used the quark potential model in which besides the one-gluon exchange kernel denoted in Eq. (61), a phenomenological confining potential (for instance, the linear potential) is necessary to be introduced. As we see, the confining potential is used to simulate all the other terms included in the expression shown in Eq. (77). Obviously, the simulation is oversimplified. In the investigations within the framework of Dyson-Schwinger equation [20,21], since the equation is not closed, containing an infinite set of coupled equations, in practical calculations, one has to cut-off the equations involving higher order (more line) vertices, and only uses the truncated equations related to the lowest order vertices [22], it is apparent that use of the lowest order vertices is difficult to exactly consider the effect of the terms involving higher order vertices in Eq. (77). Since the kernel derived is exact, containing all the interactions taking place in
the bound states, clearly, one way of solving the quark confinement is to perform a nonperturbative calculation of the kernel presented in this paper or in our previous paper [14,15]. Even though the kernel given in this paper has a complicated structure, calculation of it is feasible. We suggest that in practical calculations, one may utilize the Ward-Takahashi identities to reduce the higher order vertices to the lower order ones and choose a special gauge to simplify the calculations.

VI. ACKNOWLEDGMENT

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VII. APPENDIX: ONE-PARTICLE IRREDUCIBLE DECOMPOSITIONS OF THE CONNECTED GREEN’S FUNCTIONS

Let us begin with the relation between the generating functional for full Green’s functions \( Z[J, \bar{\eta}, \eta, \xi, \bar{\xi}] \) and the one for connected Green’s functions \( W[J, \bar{\eta}, \eta, \xi, \bar{\xi}] \) [16,17]

\[
Z[J, \bar{\eta}, \eta, \xi, \bar{\xi}] = \exp \{ iW[J, \bar{\eta}, \eta, \xi, \bar{\xi}] \} \quad (A1)
\]

Taking the derivatives of Eq. (A1) with respect to the sources \( \bar{\eta}(x_1), \eta(x_2), \eta(y_1) \) and \( \eta^c(y_2) \) and then setting all the sources except for the source \( J \) to be zero, in the case that the quark and antiquark are of different flavors, one may obtain the following decomposition

\[
G(x_1, x_2; y_1, y_2)^J = G_c(x_1, x_2; y_1, y_2)^J + S_F(x_1 - y_1)^J S_F^c(x_2 - y_2)^J \quad (A2)
\]

where \( G(x_1, x_2; y_1, y_2)^J, G_c(x_1, x_2; y_1, y_2)^J, S_F^c(x_2 - y_2)^J \) and \( S_F(x_1 - y_1)^J \) are defined by

\[
G(x_1, x_2; y_1, y_2)^J = \frac{\delta^4 Z[J, \bar{\eta}, \eta, \xi, \bar{\xi}]}{\delta \bar{\eta}(x_1) \delta \bar{\eta}(x_2) \delta \eta(y_1) \delta \eta^c(y_2)} \bigg|_{\bar{\eta} = \bar{\xi} = \xi = 0} \quad (A3)
\]

\[
G_c(x_1, x_2; y_1, y_2)^J = i \frac{\delta^4 W[J, \bar{\eta}, \eta, \xi, \bar{\xi}]}{\delta \bar{\eta}(x_1) \delta \bar{\eta}(x_2) \delta \eta(y_1) \delta \eta^c(y_2)} \bigg|_{\bar{\eta} = \bar{\xi} = \xi = 0}, \quad (A4)
\]

\[
S_F(x_1 - y_1)^J = \frac{\delta^2 Z[J, \bar{\eta}, \eta, \xi, \bar{\xi}]}{i \delta \bar{\eta}(x_1) \delta \eta(y_1)} \bigg|_{\bar{\eta} = \bar{\xi} = \xi = 0} \quad (A5)
\]

and

\[
S_F^c(x_2 - y_2)^J = \frac{\delta^2 Z[J, \bar{\eta}, \eta, \xi, \bar{\xi}]}{i \delta \bar{\eta}(x_2) \delta \eta^c(y_2)} \bigg|_{\bar{\eta} = \bar{\xi} = \xi = 0} \quad (A6)
\]

When we set \( J = 0 \), Eq. (A2) will go over to the decomposition shown in Eq. (28). Differentiating Eq. (A2) with respect to the source \( J^{\alpha\mu}(x_1) \), we have

\[
G^{\alpha}_{c\mu}(x_1 | x_1, x_2; y_1, y_2)^J = G^{\alpha}_{c\mu}(x_1 | x_1, x_2; y_1, y_2)^J + \Lambda^{\alpha}_{C\mu}(x_1 | x_1; y_1)^J S_F^c(x_2 - y_2)^J + S_F(x_1 - y_1)^J \Lambda^{\alpha}_{C\mu}(x_1 | x_2; y_2)^J \quad (A7)
\]

where

\[
G^{\alpha}_{c\mu}(x_1 | x_1, x_2; y_1, y_2)^J = \frac{\delta}{i \delta J^{\alpha\mu}(x_1)} G_c(x_1, x_2; y_1, y_2)^J \quad (A8)
\]

\[
\Lambda^{\alpha}_{C\mu}(x_1 | x_1; y_1)^J = \frac{\delta}{i \delta J^{\alpha\mu}(x_1)} S_F(x_1 - y_1)^J \quad (A9)
\]
\[ \Lambda_{\mu}^{\alpha}(x_i | x_2; y_2)^J = \frac{\delta}{i \delta J^{\alpha\mu}(x_i)} S_{F}^{c}(x_2 - y_2)^J \]  

(A10)

Upon setting \( J = 0 \), Eq. (A7) immediately gives rise to the decomposition in Eq. (29).

Now, let us proceed to carry out one-particle-irreducible decompositions of the connected Green’s functions on the right-hand side of Eq. (29). The decompositions are easily performed with the help of the Legendre transformation which is described by the relation between the generating functional of proper vertices \( \Gamma \) and the one for connected Green’s functions \( W \) [16,17]

\[ \Gamma[A^{\alpha}_{\mu}, \bar{\psi}, \psi, \overline{C}^{\alpha}, C^{\alpha}] = W[J, \bar{\eta}, \eta, \xi, \xi] - \int d^4 x [J^{\alpha\mu} A^{\alpha}_{\mu} + \bar{\psi} \eta + \psi \eta + \xi C + \overline{C} \xi] \]  

(A11)

and the relations between the field functions and the external sources

\[ \psi(x) = \frac{\delta W}{\delta \eta(x)} \chi(x) = \frac{\delta W}{\delta J^{\alpha\mu}(x)} A^{\alpha}_{\mu}(x), C^{\alpha}(x) = \frac{\delta W}{\delta \xi^{\alpha}(x)} \overline{C}^{\alpha}(x) = - \frac{\delta W}{\delta \xi^{\alpha}(x)} \]  

(A12)

\[ \eta(x) = - \frac{\delta \Gamma}{\delta \psi(x)} \eta(x) = \frac{\delta \Gamma}{\delta \psi(x)} J^{\alpha}_{\mu}(x), \xi^{\alpha}(x) = - \frac{\delta \Gamma}{\delta C^{\alpha}(x)} \xi^{\alpha}(x) = - \frac{\delta \Gamma}{\delta C^{\alpha}(x)} \]  

(A13)

where the field functions in Eq. (A12) are all functionals of the external sources in Eq. (A13) and, simultaneously, the sources in Eq. (A13) are all functionals of the field functions in Eq. (A12).

Taking the derivative of both sides of the first equality in Eq. (A12) with respect to \( \psi(y) \) and employing the first relation in Eq. (A13), one may get

\[ \int d^4 z \frac{\delta^2 \Gamma}{\delta \psi(y) \delta \psi(z)} \frac{\delta^2 W}{\delta \eta(x) \delta \eta(z)} = \int d^4 z \frac{\delta^2 W}{\delta \psi(y) \delta \psi(z)} \frac{\delta^2 \Gamma}{\delta \psi(y) \delta \psi(z)} = - \delta^4(x - y) \]  

(A14)

where we only keep the term on the right-hand side of Eq. (A14) which is nonvanishing when the sources are set to vanish. In order to find the one-particle-irreducible decomposition for the quark-gluon three-point Green’s functions, one may differentiate Eq. (A14) with respect to the source \( J^{\alpha\mu}(x_i) \) and then using Eq. (A14) once again. By this procedure, it can be derived that

\[ \frac{\delta^3 W}{\delta J^{\alpha\mu}(x_i) \delta J^{\beta\nu}(x_j) \delta \eta(y_k)} = \int d^4 z d^4 u_1 d^4 u_2 \frac{\delta^2 W}{\delta J^{\alpha\mu}(x_i) \delta J^{\beta\nu}(x_j)} \frac{\delta^2 W}{\delta \eta(z) \delta \eta(u_2)} \]  

(A15)

where the coordinates in Eq. (A14) have been appropriately changed. When all the sources are set to be zero, noticing the definitions given in Eq. (A5) where the \( Z \) is replaced by \( i W \) and in Eq. (A9) as well as

\[ \Delta_{\mu\nu}^{ab}(x_i - y_j) = \frac{\delta^2 W}{\delta J^{\mu\nu}(x_i) \delta J^{\alpha\beta}(y_j)} \bigg|_{J=0} \]  

(A16)

\[ \Gamma^{b\nu}(u_1 | u_2, z) = i \frac{\delta^3 \Gamma}{\delta A^{b}_{\mu}(u_1) \delta J^{\alpha\nu}(y_j)} \frac{\delta^2 W}{\delta \eta(z) \delta \eta(u_2)} \bigg|_{A=\bar{\psi}=\psi=0} \]  

(A17)

the decomposition shown in Eqs. (30) and (31) straightforwardly follows from Eq. (A15). Analogously, if we replace \( \bar{\eta}(x_j) \) and \( \eta(y_k) \) by \( \bar{\eta}'(x_j) \) and \( \eta'(y_k) \) in Eq. (A15) and noticing

\[ \Gamma_c^{b\nu}(u_1 | u_2, z) = i \frac{\delta^3 \Gamma}{\delta A^{b}_{\mu}(u_1) \delta \bar{\psi}'(u_2) \delta \psi'(z)} \bigg|_{A=\bar{\psi}=\psi=0} \]  

(A18)

the decomposition shown in Eq. (32) and (33) will be derived. This decomposition may also be derived from Eq. (A15) by the charge conjugation transformation for the quark fields.

The one-particle-irreducible decomposition of the connected Green’s function \( G_c(x_1, x_2; y_1, y_2) \) can be derived by the same procedure as obtaining Eq. (A15). On differentiating Eq. (A14) with respect to \( \bar{\eta}'(x_1) \) and \( \eta'(y_2) \) and setting all the sources but the source \( J \) to vanish, one may obtain
\[ G_c(x_1, x_2; y_1, y_2)^J = \int \prod_{i=1}^2 d^4u_i d^4v_i S_P(x_1 - u_1)^J S_P^c(x_2 - u_2)^J \times \Gamma(u_1, u_2; v_1, v_2)^J S_P(v_1 - y_1)^J S_P(v_2 - y_2)^J \]  

(A19)

where the four-point connected Green’s function and the propagators given in presence of the sources were defined before and the function \( \Gamma(u_1, u_2; v_1, v_2)^J \) is formally the same as that defined in Eqs. (38) and (39). When the source \( J \) is turned off, Eq. (A19) directly goes over to the decomposition in Eq. (37)-(39) being defined in Eqs. (A17) and (A18) and in the following:

\[ \Gamma_2(u_1, u_2; v_1, v_2) = i \frac{\delta^4 \Gamma}{\delta \psi(u_1) \delta \psi^\dagger(u_2) \delta \psi(v_1) \delta \psi^\dagger(v_2)} |_{\psi = \psi^\dagger = \psi'^\dagger = 0} \]  

(A20)

which is the quark-antiquark four-line proper vertex. It is emphasized here that the decomposition of the function \( G_c(x_1, x_2; y_1, y_2) \) in absence of the source \( J \) has the same form as that given in the presence of \( J \). This is because the Green’s function is defined only by the differentials with respect to the fermion fields as indicated in Eq. (A4).

The one-particle irreducible decomposition of the Green’s function \( G_{\alpha\beta}^{\mu\nu}(x_i | x_1, x_2; y_1, y_2) \) may be derived by starting from the expression given in Eq. (A15) with \( j, k = 1 \). By differentiating the both sides of Eq. (A15) with respect to the sources \( \mathcal{T}(x_2) \) and \( \eta^\dagger(y_2) \) and then turning off all the external sources, one may obtain the decomposition of the function \( G_{\alpha\beta}^{\mu\nu}(x_i | x_1, x_2; y_1, y_2) \) as shown in Eqs. (40)-(48). Alternatively, the decomposition may also be obtained by starting with the expression written in Eq. (A19). Substituting Eq. (A19) into Eq. (A8), then completing the differentiation with respect to the source \( J^{\alpha\mu}(x_i) \) and finally setting the source to vanish, one may also derive the irreducible decomposition of the function \( G_{\alpha\beta}^{\mu\nu}(x_i | x_1, x_2; y_1, y_2) \). In doing this, it is necessary to perform the differentiations of the fermion propagators with respect to the source \( J^{\alpha\mu}(x_i) \) as shown in Eqs. (A9) and (A10) and use their decompositions presented in Eqs. (30)-(33). In addition, we need to carry out the differentiations of the gluon propagator and some vertices with respect to the source \( J^{\alpha\mu}(x_i) \) as shown below. For the gluon propagator defined in Eq. (A16), from its representation in presence of the external source \( J \), in the same way as deriving the decomposition represented in Eqs. (A15), (30) and (32), one may obtain the one-particle irreducible decomposition of the gluon three-point Green’s function as follows:

\[ \Lambda^{\alpha\beta cd}(x_i, x_1, x_2) = \frac{\delta}{\delta J^{\alpha\mu}(x_i)} \Lambda_{\alpha\beta cd}(z_1 - z_2)^J |_{J=0} = \int d^4z D^{\alpha\beta cd}(x_i - z) \Pi^{cd\nu}_{\rho\sigma}(z, z_1, z_2) \]  

(A21)

where \( D^{\alpha\beta cd}(x_i - z) \) was defined in Eq. (34) and \( \Pi^{cd\nu}_{\rho\sigma}(z, z_1, z_2) \) was represented in Eq. (47) with the gluon three-line proper vertex defined by

\[ \Gamma^{\rho\sigma cd}_{\mu\nu}(z, u_1, u_2) = i \frac{\delta^3 \Gamma}{\delta A^\rho_{\mu}(z) \delta A^\sigma_{\nu}(u_1) \delta A^c_{\nu}(u_2)} |_{A=0} \]  

(A22)

For other proper vertices related to fermions, we use the notation \( \Gamma_{\nu\lambda\tau\cdots}^{\mu\nu\cdots}(z_1, x_1, z_2, \cdots) \) to represent them where color and Lorentz indices and the coordinates on the left-hand side of the vertical line belong to gluons and the coordinates on the right-hand side of the vertical line are attributed to fermions, their derivative with respect to the source \( J^{\alpha\mu}(x_i) \) can be represented as

\[ \frac{\delta}{\delta J^{\alpha\mu}(x_i)} \Gamma_{\nu\lambda\tau\cdots}^{\mu\nu\cdots}(z_1, x_1, z_2, \cdots)^J |_{J=0} = \int d^4z D^{\alpha\mu}_{ab}(x_i - z) \Gamma_{\nu\lambda\tau\cdots}^{\mu\nu\cdots}(z_1, z_1, z_2, \cdots | x_1, x_2, \cdots) \]  

(A23)

where

\[ \Gamma_{\nu\lambda\tau\cdots}^{\mu\nu\cdots}(z, z_1, z_2, \cdots | x_1, x_2, \cdots) = \frac{\delta}{\delta A^\nu_{\mu}(z)} \Gamma_{\nu\lambda\tau\cdots}^{\mu\nu\cdots}(z_1, z_2, \cdots | x_1, x_2, \cdots)^J |_{J=0} \]  

(A24)

Especially, for the differential of the vertex \( \Gamma(u_1, u_2; v_1, v_2) \) in Eq. (38) with respect to the source \( J^{\alpha\mu}(x_i) \), we write

\[ \Gamma^\mu_{\nu}(x_i | u_1, u_2; v_1, v_2) = \frac{\delta}{\delta J^{\alpha\mu}(x_i)} \Gamma(u_1, u_2; v_1, v_2)^J |_{J=0} \]  

(A25)

Similarly, we define

\[ \Gamma^{ab}_{\mu\nu}(x_1, x_2 | u_1, u_2; v_1, v_2) = \frac{\delta^2}{\delta J^{\alpha\mu}(x_1) \delta J^{\nu\rho}(x_2)} \Gamma(u_1, u_2; v_1, v_2)^J |_{J=0} \]  

(A26)
the explicit expression given by the above differentiation can be obtained by substituting Eqs. (38), (39) and (A20) into Eq. (A25) and making use of the formulas in Eqs. (A21) and (A23). The results are shown in Eqs. (44)-(48). According to the procedure stated above, it is not difficult to derive the expressions described in Eqs. (79)-(89). In the expressions, the vertices are defined as in Eq. (A24). Some examples are listed below,

\[ \Gamma^{ab}_{\mu\nu}(z_1, z_2 \mid u_1, v_1) = i \frac{\delta^4 \Gamma}{\delta A^{\mu}(z_1) \delta A^{\nu}(z_2) \delta \bar{\psi}(u_1) \delta \psi(v_1)} \mid_{A = \bar{\psi} = \psi = 0} \]  

(A27)

\[ \Gamma^{ab}_{\epsilon\mu\nu}(z, z_1 \mid u_1, v_1) = i \frac{\delta^4 \Gamma}{\delta A^{\epsilon}(z_1) \delta A^{\nu}(z_2) \delta \bar{\psi}(u_1) \delta \psi(v_1)} \mid_{A = \bar{\psi} = \psi = 0} \]  

(A28)

\[ \Gamma^{a}_{\mu}(z \mid u_1, u_2; v_1, v_2) = i \frac{\delta^4 \Gamma}{\delta A^{\mu}(z) \delta \bar{\psi}(u_1) \delta \bar{\psi}(u_2) \delta \psi(v_1) \delta \psi(v_2)} \mid_{A = \bar{\psi} = \psi = 0} \]  

(A29)

and

\[ \Gamma^{ab}_{\mu\nu}(z_1, z_2 \mid u_1, u_2; v_1, v_2) = i \frac{\delta^4 \Gamma}{\delta A^{\mu}(z_1) \delta A^{\nu}(z_2) \delta \bar{\psi}(u_1) \delta \bar{\psi}(u_2) \delta \psi(v_1) \delta \psi(v_2)} \mid_{A = \bar{\psi} = \psi = 0}. \]  

(A30)

VIII. REFERENCES