Non-perturbative renormalization in Light Front Dynamics with Fock space truncation
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Within the framework of the Covariant formulation of Light-Front Dynamics, we develop a general non-perturbative renormalization scheme based on the Fock decomposition of the state vector and its truncation. The explicit dependence of our formalism on the orientation of the light front is essential in order to analyze the structure of the counterterms and bare parameters needed to renormalize the theory. We present here a general strategy to determine the dependence of these quantities on the Fock sectors. We apply our formalism to QED for the two-body (one fermion and one boson) truncation and recover analytically, without any perturbative expansion, the renormalization of the electric charge according to the requirements of the Ward Identity.

1. INTRODUCTION

The knowledge of the hadron properties within the framework of QCD is one of the main issues in strong interaction physics. Several approaches have been pursued in the last twenty years, in particular lattice gauge calculations. Among the alternatives to these calculations, Light-Front Dynamics (LFD) is of particular interest \cite{1}. It has proven successful in many phenomenological applications involving few-body systems in particle and nuclear physics. However, the application of LFD to field theoretical calculations is still in its infancy. The main issue to be solved is the renormalization procedure. In perturbative calculations, the renormalization of the electron self-energy in QED, in standard LFD, is already non-trivial in the sense that it involves non-local counterterms. This unpleasant feature is however a direct consequence of the choice of a preferential direction, the $z$ axis, in the determination of the quantization plane. This can be well understood in the Covariant formulation of Light-Front Dynamics (CLFD) \cite{2,3}. In this formulation, the state vector is defined on the light-front plane given by the equation $\omega \cdot x = 0$, where $\omega$ is the four-vector with $\omega^2 = 0$. The particular case where $\omega = (1, 0, 0, -1)$ corresponds to standard LFD. We shall show in this study how one should determine, in a true non-perturbative way, the renormalization condition for systems composed of one fermion and at most $N-1$ bosons. The first calculation of the purely scalar system in CLFD, for $N = 3$, has been done in \cite{4}. Non-perturbative calculation in a gauge theory (for $N = 2$) is given in \cite{5} and in Yukawa theory (for $N = 3$) – in \cite{6}.

2. EQUATION OF MOTION IN TRUNCATED FOCK SPACE

Our starting point is the general eigenstate equation for the state vector $\phi(p)$:

$$\hat{P}^2 \phi(p) = M^2 \phi(p),$$

(1)

where $M$ is the mass of the physical state. The momentum operator $\hat{P}_\mu$ is decomposed into two parts: the usual free one and the interacting one given by

$$\hat{P}^\text{int}_\mu = \omega_\mu \int H^\text{int}(x) \delta(\omega \cdot x) \, d^4x$$

(2)

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where $\hat{H}^{\text{int}}$ is the Fourier transform of the interaction Hamiltonian $H^{\text{int}}(x)$. The state vector $\phi(p)$ is then decomposed in Fock components $\phi_i$. For convenience, it will be more appropriate to work with the vertex function defined by

$$\Gamma_i = 2(\omega)p\tau\phi_i = (s - M^2)\phi_i,$$

where $s$ is the square of the invariant mass of the $i$-particle system. The general equation of motion can thus be written, after some algebraic manipulation, as

$$\mathcal{G}(p) = -\int \hat{H}^{\text{int}}(\omega\tau) \frac{1}{\tau} \mathcal{G}(p) \frac{d\tau}{2\pi},$$

where $\mathcal{G}(p)$ is defined by:

$$\mathcal{G}(p) = 2(\omega)p\hat{\tau}\phi(p).$$

The operator $\hat{\tau}$ acts on a given Fock component $\phi_i$ to give $\tau\phi_i$. $\mathcal{G}(p)$ is thus decomposed in an infinite sum of the vertex functions $\Gamma_i$.

The interacting Hamiltonian $\hat{H}^{\text{int}}(\omega\tau)$ relates the component $n$ with components $n, n \pm 1, n \pm 2, \ldots$ depending on the system under consideration. It is expressed in terms of bare quantities like the bare coupling constant $g_0$, and/or physical quantities like the mass $m$ of the constituent fermion with a mass counterterm $\delta m$. In order to calculate physical observables, these bare quantities and counterterms should be fixed from physical conditions, in the limit where $M \to m$. If we restrict ourselves to QED in order to compare with well established results in perturbation theory, the mass counterterm is fixed to get $P^2 = M^2$ for the physical state, and $g_0 \equiv e_0$ is fixed to get the physical charge of the electron, with $\alpha = \frac{1}{137} = \frac{1}{4\pi}$.

What happens if we truncate the Fock expansion? We shall call $N$ the total number of Fock components considered, and $n$ a particular Fock component with $n$ particles in the intermediate state (one fermion and $n - 1$ bosons).

i) First Eq. 5 should be modified for $n = N$ or less since $\hat{H}^{\text{int}} \mathcal{G}$ should not connect states with $n > N$. This is done "by hand" in the coupled eigenvalue equations for the vertex parts $\Gamma_n^{(N)}$. We have explicitly mentioned the dependence of $\Gamma$ on $N$ by a superscript.

ii) When the bare coupling constant $g_0$ and mass counterterm $\delta m$ are fixed from physical observables, this is done within the approximation of Fock truncation. In each intermediate state, the total number, $l$, of particles in which a given state can fluctuate should be less or equal to $N$.

Let us look for example at the renormalization of the fermion propagator in first order perturbation theory. The three contributions to the physical fermion propagator are indicated in Fig[1] These are from left to right the free propagator, the self energy contribution, and the contribution from the mass counterterm $\delta m$. The sum of these three contributions should be equal, at $p^2 = m^2$, to the free propagator. This fixes $\delta m = -g_0^2\Sigma(p^2 = m^2)$. This equation links two diagrams with two different Fock components. To keep track of the number of Fock component the counterterm is associated to (here 2), we use the superscript $(l)$, and note $\delta m^{(l)}, g_0^{(l)}$. We shall see in the following section how this dependence should be calculated in practice.

![Figure 1. Renormalization of the bare fermion propagator](image)

iii) The equation of motion for the vertex parts involves a bare coupling constant. This bare coupling constant is determined to get a physical condition, in a given approximation, i.e. within the Fock truncation. However, it cannot be identified with the usual bare coupling constant which is determined from external meson coupling like
the electromagnetic form factor in the case of QED. In the determination of the state vector, the exchanged meson is indeed not an external particle and thus should participate to the counting rule of the total number of particles either in the initial or final state.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2.png}
\caption{Second order radiative corrections to the self-energy of a fermion}
\end{figure}

To illustrate this fact, let us write down for instance the first contributions to the self energy involving at most two mesons in flight. These are indicated in Fig. 2. The vertices (1) and (2) in Fig. 2(a) involve bare coupling constants in order to cancel divergences arising in Fig. 2(b) and (c). However, the vertex (3) in Fig. 2(b) should not be corrected in the same way as the vertex (4) since the vertex (3) can not be modified anymore by radiative corrections, while the vertex (4) can, as indicated in Fig. 2(c). The vertex (3) would however be corrected if one takes as bare coupling constant the one fixed from the calculation of the electromagnetic form factor, as indicated in Fig. 3. We shall denote by \( \tilde{g}_0^{(l)} \) the ”Amputated” Bare Coupling Constant (ABCC) to be used in the equation of motion.

The strategy to calculate \( \delta m^{(l)} \) and \( \tilde{g}_0^{(l)} \) is thus the following. Any calculation of the state vector to order \( N \) involves \( \delta m^{(l)} \) and \( \tilde{g}_0^{(l)} \) for \( l = 1, 2, \ldots, N \). These quantities are however not arbitrary. They are just the realization of the only two quantities \( \delta m \) and \( \tilde{g}_0 \) for successive approximate calculations. One should fix them by calculating the state vector for \( N = 1, 2, \ldots, N \) successively. The case \( N = 1 \) is trivial since \( \delta m^{(1)} = \tilde{g}_0^{(1)} = 0 \) because the state vector corresponds just to a free fermion. The case \( N = 2 \) is calculated in section 4 for QED.

\section{3. Specific Light Front Counterterms}

We have seen in \[47\] that the 2-Point Green Function (2PGF) is in general dependent on \( \omega \). This comes from the general \( \omega \)-dependence of the self-energy according to:

\[ M^{(2)}(p) = g^2 \left[ A(p^2) + B(p^2) \frac{p}{m} + C(p^2) \frac{m \hat{\omega}}{\omega p} \right] . \]

We have thus exhibited a new counterterm \( Z_\omega \) so that the physical 2PGF does not depend on \( \omega \). We found that it corresponds also to the counterterm needed to get a two-body component independent of \( \omega \) at \( s = M^2 \). According to the previous discussion, we shall denote this counterterm by \( Z_\omega^{(l)} \). Note that for appropriately chosen regularization schemes, one may find \( C = 0 \). Such regularization schemes (Pauli-Villars subtraction for instance) are thus clearly favoured.

For the same reason, one should also consider in principle new \( \omega \)-dependent vertex counterterms, denoted by

\[ \delta g_\omega = \sum_i \delta g_{\omega,i} O_i(\omega) . \]

with appropriate structures \( O_i(\omega) \) depending on the type of vertices (scalar or vector). These counterterms are needed to make sure that the physical 3-point Green function, like the electromagnetic form factor for instance, is independent.
of \( \omega \). These counterterms should also depend on \( i \).

For the calculation we consider in this work, we do not need to discuss higher n-point Green function.

4. THE CASE OF QED FOR N=2

We follow here very closely the definitions already presented in [3].

4.1. Decomposition of the state vector

In this section, the state vector is written as

\[
|\sigma\rangle = |1\rangle + |2\rangle
\]

The one-body wave functions are then determined from the equation of motion. In the approximation where one considers only one fermion and one boson state, these components are constants. This is however not the case in general.

In order to make a clear connection with the usual normalization factor \( Z_2^{(2)} \) we shall define

\[
\sqrt{Z_2^{(2)}} = 2m \phi_1 .
\]

The normalization factor \( Z_2^{(2)} \) is fixed from the condition [9].

4.2. Solution of the equation of motion

We shall only recall here the results obtained in [3], with the identification \( \delta m \rightarrow \delta m^{(2)}, g \rightarrow \bar{g}_0^{(2)}, Z_\omega \rightarrow Z_\omega^{(2)} \). The counterterm \( \delta m^{(2)} \) is fixed from the equation for \( \Gamma_2 \) in order to get an \( \omega \)-independent two-body wave function at \( s = M^2 \), i.e. \( b_2 = 0 \). This gives

\[
\delta m^{(2)} = \left[ \bar{g}_0^{(2)} \right]^2 (A + B + C) ,
\]

while \( Z_\omega^{(2)} \) is determined by solving the equation for \( \Gamma_1 \), so that

\[
Z_\omega^{(2)} = \left[ \bar{g}_0^{(2)} \right]^2 C .
\]

The two-body component \( a_2 \) is thus given by

\[
a_2 = 2m \phi_1 \bar{g}_0^{(2)} = \bar{g}_0^{(2)} \sqrt{Z_2^{(2)}} .
\]

The quantities \( A, B \), and \( C \) are given by \( A, B, \) and \( C \) at \( p^2 = M^2 \). Note that to simplify notations, we include in \( C \) the contribution from the boson loop with the contact fermionic interaction [3]. With this, one can completely determine the normalized wave function and find from the normalization condition [3]:

\[
Z_2^{(2)} = \frac{1}{1 + \left[ \bar{g}_0^{(2)} \right]^2 I(L)} ,
\]

with \( I(L) \) given by

\[
I(L) = \frac{1}{2(2\pi)^3} \int \frac{Tr[(k + m)(\not{p} + m)]}{(s - M^2)^2} \frac{d^3k}{2\varepsilon_k(1 - x)}
\]

where \( k \) is the photon momentum, and \( L \) is a cut-off needed to give a meaning to the integral.
For our purpose, we do not need to precise the regularization scheme we use. $I(L)$ is a logarithmically divergent quantity as a function of $L$.

### 4.3. Determination of the amputated bare coupling constant

In the spirit of our previous choice of $b_2 = 0$ at $s = M^2$, we can define the ABCC $\bar{g}_0^{(2)}$, so that the component $a_2$ at $s = M^2$ is identified with the physical coupling constant:

$$a_2(s = M^2) = \bar{g}_0^{(2)} \sqrt{Z_2^{(2)}} \equiv g . \quad (15)$$

It is a well defined, non-perturbative, condition which is very convenient to impose in any numerical calculation (apart from the fact that we should extrapolate the component to the non-physical point $s = M^2$; this is here trivial since $a_2$ is a constant).

We can note that according to our discussion in section 1, $\bar{g}_0^{(2)}$ can also be extracted from the calculation of the meson-nucleon vertex. In the $N = 2$ approximation, it is given by the contribution of Fig. 4 with the result:

$$g = \sqrt{Z_2^{(2)} \bar{g}_0^{(2)} \sqrt{Z_2^{(1)}}} = \sqrt{Z_2^{(2)} \bar{g}_0^{(2)}} . \quad (16)$$

![Figure 4. Calculation of the "amputated" bare coupling constant for $N = 2$. To this order, there is only one contribution to the electromagnetic amplitude photon + fermion → fermion.](image)

We recover exactly Eq. (15). In the r.h.s. of Fig. 4 the final state can fluctuate into a fermion plus a boson. It is thus given by $\Gamma_1^{(2)}$. This is however not the case for the initial state since there is already a boson in flight, so that it is given by $\Gamma_1^{(1)}$, i.e. a free fermion state with $Z_2^{(1)} = 1$.

Since the contribution from the r.h.s. of Fig. 4 is already $\omega$-independent, we should not consider any $\omega$-dependent counterterms, so that $\delta \bar{g}_0^{(2)} = 0$. This validates the use of an $\omega$-independent coupling constant in the $N = 2$ calculation of 3.

With this definition of $\bar{g}_0^{(2)}$, and with Eq. (14), we have

$$g = \frac{\bar{g}_0^{(2)}}{\sqrt{1 + \left[\bar{g}_0^{(2)}\right]^2 I(L)}} , \quad (17)$$

which gives

$$\left[\bar{g}_0^{(2)}\right]^2 = \frac{g^2}{1 - g^2 I(L)} , \quad (18)$$

and thus

$$Z_2^{(2)} = 1 - g^2 I(L) . \quad (19)$$

We would like to emphasize here that we did not do any perturbative expansion in order to express $Z_2^{(2)}$ in Eq. (19) starting from Eq. (14). The first one is expressed in terms of the physical coupling constant $g$ while the second one is a function of the ABCC $\bar{g}_0$.

### 4.4. Determination of the bare coupling constant

As already explained in [3], there are three $\omega$-dependent structures that we can think of in order to expand the electromagnetic form factors in QED, on top of the two physical ones $F_1$ and $F_2$. We recall here these three structures:

$$O_1^\rho = \left( \frac{m}{\omega p} - \frac{1}{1 + \eta} \right) \frac{(p + p')^\rho}{m} , \quad (20)$$

$$O_2^\rho = \frac{m}{\omega p} \omega^\rho , \quad (21)$$

$$O_3^\rho = \frac{m^2}{(\omega p)^2} \omega^\rho \quad (22)$$

with $\eta = Q^2/4m^2$, where $Q^2 = -q^2$. The most general decomposition of the total amplitude thus writes

$$\bar{u} \Gamma^\rho u = g \bar{u} \left[ F_1(Q^2) \gamma^\rho + F_2(Q^2) \frac{i}{2m} \sigma^{\mu\nu} q_{\mu} \right. + \sum_{i=1}^{3} B_i(Q^2) O_i^\rho \left. \right] u . \quad (23)$$
Each form factor can be easily extracted from $\Gamma^\rho$ according to the procedure detailed in [8].

The Bare Coupling Constant (BCC) $g(0)$ should thus be fixed to get the physical coupling constant defined by $g = g(0) F_1(0)$, i.e. $F_1(0) = 1$, while the $\omega$-dependent counterterms $\delta g^{(2)}_\omega$, defined in Eq.(7) should be fixed to get an $\omega$-independent amplitude, if needed.

$$g(0) = g,$$
$$\delta g^{(1)}_\omega = 0,$$

since the contribution from the r.h.s. of Fig.4 is independent of $\omega$.

For the case $N = 2$, the contributions to the form factor are indicated on Fig.6. Direct calculation of the contributions from fermionic contact interactions gives no contribution at $Q^2 = 0$ to the $\omega$-independent contribution $F_1$.

By definition, the vertex counterterms $\delta g^{(2)}_\omega$ are chosen to cancel the $\omega$ dependent contributions arising from the diagrams in Fig.6 so that $B_{1,2,3}$ in Eq.(23) are equal to zero. We do not detail here their calculation. They depend directly on the regularization scheme used in the evaluation of the loop diagrams.

One is thus left with one equation for the $\omega$-independent contribution which writes:

$$g = \sqrt{Z_2^{(2)} g_0^{(2)}} \sqrt{Z_2^{(2)}} + \sqrt{Z_2^{(2)} F^{(2)}(Q^2 = 0, L) g_1^{(1)} \sqrt{Z_2^{(2)}}},$$

where the factor $F^{(2)}$ is the contribution coming from the $\omega$-independent part of the loop diagram in Fig.6(b) for $N = 2$.

The difference between $g_0^{(2)}$ in Eq.(26) and $\bar{g}_0^{(2)}$ in Eq.(16) lies in the fact that the initial one-body state cannot fluctuate into a two-body part in the calculation of $\bar{g}_0^{(2)}$, so that the one body ($n = 1$) component should correspond to $N = 1$ and not to $N = 2$ like in the form factor calculation. One can easily check that for QED [9], we have

$$F^{(2)}(Q^2 = 0, L) = \left[\bar{g}_0^{(2)}\right]^2 I(L),$$

so that Eq.(26) writes with (16) and (24):

$$g = [1 - g^2 I(L)] g_0^{(2)} + g^3 I(L),$$

which gives finally

$$g_0^{(2)} = g.$$

This is in complete agreement with the Ward Identity. This originates from two particular features of our strategy: the ABCC from (16) which implies (19) instead of (14), and the definition of the Fock-dependent BCC in (26) and (24).

### 4.5. Calculation of the form factor

Let us now investigate the general form of the electromagnetic form factor for any $Q^2$. It is given by the contributions shown in Fig.6. The physical
form factors are thus extracted from our general decomposition \(23\).

The \(\omega\)-independent contributions from Fig. 6 have been already calculated in \([9]\). If we restrict ourselves for instance to the \(F_1\) form factor, we get

\[
gF_1(Q^2) = g_0^{(2)} z_2^{(2)} + g_0^{(1)} z_2^{(2)} F(2)(Q^2, L) \tag{30}
\]

with

\[
F(2)(Q^2, L) = \left[ g_0^{(2)} \right]^2 F(2)(Q^2, L) \tag{31}
\]

so that we have immediately with \(13\), \(10\), \(24\), and \(29\):

\[
F_1(Q^2) = 1 + g^2 \left[ \tilde{F}^{(2)}(Q^2, L) - \tilde{F}^{(2)}(Q^2 = 0, L) \right]
\]

We recover here exactly the perturbative calculation, as it is natural in the two-body approximation. Note that this result is obtained without any perturbative expansion.

5. CONCLUDING REMARKS AND PERSPECTIVES

We have presented in this contribution a systematic strategy to calculate physical observables for fermionic systems composed of a fermion and \(N - 1\) bosons. This implies to implement a renormalization scheme in a non-perturbative framework. Within the covariant formulation of Light Front Dynamics, we have shown how to fix the mass counterterm and bare coupling constant of the elementary Hamiltonian in a consistent way. As a check of our formalism, we treat the case of QED in the two body approximation. We have been able to recover, for the first time, the standard renormalization of the electromagnetic charge according to the Ward Identity (i.e. the bare charge should be equal to the physical one), without any perturbative expansion. This shows that no divergences are left uncancelled.

Our results have been made possible because of two important features of our formalism:

i) First we should extract the physical part of the two-body wave function at \(s = M^2\) in order to define the physical coupling constant to be used in the calculation of the state vector. This part is explicit in our formalism since it should be independent of \(\omega\).

ii) Since we truncate the Fock expansion, the physical content of the bare coupling constant to be used in the state vector should be different from the bare coupling constant calculated from external meson coupling (by looking at the electromagnetic form factor for instance if we deal with QED). These two couplings are of course the same in the limit of infinite Fock decomposition.

Our results are very encouraging in the perspective of doing true non-perturbative calculations of bound state systems in a field theoretical framework.

REFERENCES