Infrared instability from nonlinear QCD evolution

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Using the Balitsky–Kovchegov (BK) equation as an explicit example, we show that nonlinear QCD evolution leads to an instability in the propagation toward the infrared of the gluon transverse momentum distribution, if one starts with a state with an infrared cut-off. This effect takes the mathematical form of rapidly moving traveling wave solutions of the BK equation, which we investigate by numerical simulations. These traveling wave solutions are different from those governing the transition to saturation, which propagate towards the ultraviolet. The infrared wave speed, formally infinite for the leading order QCD kernel, is determined by higher order corrections. This mechanism could play a rôle in the rapid decrease of the mean free path in the Color Glass Condensate scenario for heavy ion collisions.

I. INTRODUCTION

The properties of perturbative Quantum Chromodynamics (QCD) at high energy and/or density presently receive much attention. On phenomenological grounds this domain is interesting both for deep inelastic scattering at HERA and hard scattering at the Tevatron and LHC, and for heavy-ion collisions with an initially large density of partons, e.g. at RHIC and LHC. On theoretical grounds the high energy and density domain is a challenge, since it implies a resummation of logarithms in the perturbative expansion at all orders of the strong coupling constant. This resummation goes beyond the one at the origin of the linear evolution equation in rapidity \(Y\) known as the Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation [1]. It implies non-linear terms which describe the damping of the scattering amplitudes due to the high density of partons (mainly gluons) produced during the evolution. This leads to a transition to saturation [2] where a new state of QCD matter, the “Color Glass Condensate” (CGC) is expected to appear [3]. Indeed, the CGC is proposed as the initial state formed in heavy ion collisions at high energy [4].

In its mean-field version, the QCD evolution towards saturation is described by the Balitsky–Kovchegov (BK) nonlinear equation [5, 6]. In the color dipole picture of QCD [7, 8], it governs the energy dependence of the onium–target amplitude, where the target is supposed to be interacting independently with each dipole resulting from the evolution of the onium wave function with energy. In parton language, and neglecting impact parameter dependence, the BK equation describes the rapidity \(Y\) evolution of the “unintegrated gluon distribution”, i.e. the distribution of gluon transverse momenta \(k\) in the target.

An interesting set of results on this distribution during the transition to saturation have already been obtained from the mathematical properties of the BK equation. Indeed, analytic asymptotic solutions of the non-linear equation have been found [9] in terms of traveling waves, i.e. scaling solutions \(T(k,Y) \sim T(k/Q_s(Y))\), where \(L \equiv \log(k^2/k_0^2)\) and \(Y\) play the rôle of space and time, respectively. In fact, the traveling wave pattern of the solutions of the BK equation, which is already present at pre-asymptotic energies [10], reflects the earlier proposed [11] “geometric scaling” of the \(\gamma^p\) cross section.

The saturation scale \(Q_s(Y)\) is directly related to the movement in time of the traveling wave and in fact \(d\log Q_s(Y)/dY \sim v\) \((v > 0)\) is nothing else than the speed of the wave which reaches a universal critical value at high energies, depending only on the kernel [9]. Hence, in physical terms the gluon transverse momenta propagate with rapidity towards higher and higher values, i.e. to the ultraviolet, the typical transverse momentum scale being the saturation scale \(Q_s\). This is the key property of the transition to saturation.

In fact the “final” result of this evolution—the Color Glass Condensate—is supposed to be a fully saturated state where all gluons have a momentum equal to (or possibly larger than) the saturation scale characterizing the medium. However, in the rapidity evolution from an onium–target scattering, this “final” state is not expected to occur easily at finite energy for the whole system. On the other hand, heavy ion collisions at high energies could, due to the

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high initial density of partons, be a good candidate for the physical occurrence of the CGC [4] in the early stages of the collision. However, it is known that difficulties occur in reconciling this appealing scenario with the rapid thermalization which seems to occur in the next stages of the process. More precisely the mechanisms by which the original CGC, being inherently perturbative, feeds the low momentum region [12] are currently under scrutiny [13].

Hence, this asks the question of the propagation towards the infrared, when one starts with an initial state with a lower transverse momentum limit, or at least characterized by a typical high transverse momentum, as is the case for the CGC. This propagation has been considered from the point of view of a bottom-up mechanism [12], possibly helped by plasma instabilities [13]. However, it is worth considering the effect of energy evolution upon this type of dense initial phase already in the initial stage of the collision. This is our physical motivation for looking at the properties of the BK equation from a new point of view, the BK equation being considered as a prototype for non-linear perturbative QCD evolution. We expect most of our results to be valid also for more complete nonlinear evolution equations. In the present paper we essentially stick to the relevant mathematical properties of the BK evolution, leaving a more detailed analysis of the physics of heavy-ion collisions for the future.

We will thus discuss how the BK equation gives information on the propagation with energy of gluon momenta towards the ultraviolet and the infrared; namely, how the gluon momentum distribution evolves towards larger and smaller momenta, respectively, when starting with appropriate initial conditions. We will consider initial conditions with a “sharp” cut-off, either ultraviolet or infrared (see Fig.1), but the mathematical universality properties of traveling wave solutions will allow much more general initial conditions, provided the initial fronts are steep enough.

The plan of our paper is the following. In section II we recall the BK equation in transverse momentum space. In section III, we discuss the propagation of momenta towards the ultraviolet (i.e. large gluon momenta) and show that we recover the same traveling wave solutions and scaling properties as for the solution of the BK equation for the physical amplitude. In section IV, using appropriate initial conditions with an infrared cut-off, we derive a new type of scaling solutions of the BK equation for the propagation towards the infrared (i.e. towards small gluon momenta). In section V we solve the same equations numerically, confirming the analytical predictions and study in detail the solutions. Conclusions on the propagation properties of the BK equation and its possible relevance for the CGC and the initial stage of the thermalization process are discussed in section VI.

II. THE BALITSKY–KOVCHEGOV EQUATION

The BK equation for the onium–target S-matrix in position space reads

$$\frac{\partial S(x_{01},Y)}{\partial Y} = \int d^2 x_2 \, K(x_0,x_1;x_2) \left\{ S(x_{02},Y)S(x_{12},Y) - S(x_{01},Y) \right\} ,$$

(1)
where
\[ \mathcal{K}(x_0, x_1; x_2) = \frac{\alpha_s N_c}{\pi} \frac{x_{01}^2}{x_{02} x_{12}^2} \] (2)
is the BFKL kernel [1]. Eq. (1) is the impact-parameter independent form of the BK equation, which is what we consider in this work. \( \alpha_s \) is the QCD coupling constant, \( x_i \) are the endpoint coordinates of the QCD dipoles in two-dimensional transverse space, and \( x_{ij} = x_i - x_j \). To obtain the equation for the amplitude, perform the replacement \( S \rightarrow 1 - N \) to get
\[
\frac{\partial N}{\partial Y}(x_{01}, Y) = \int d^2 x_2 \mathcal{K}(x_0, x_1; x_2) \left\{ N(x_{02}, Y) + N(x_{12}, Y) - N(x_{01}, Y) - N(x_{02}, Y) N(x_{12}, Y) \right\}. \] (3)

Again assuming impact-parameter independence, it is well-known [6] that the Fourier transformation to momentum space leads to a rather simple form of the evolution equation which is suitable for the derivation of useful analytical properties [9]. Thus one introduces the function
\[ N(k, Y) = \int \frac{d^2 x_{01}}{2 \pi x_{01}^2} e^{-ik \cdot x_{01}} N(x_{01}, Y) = \int_0^\infty \frac{dr}{r} J_0(kr) N(r, Y), \] (4)
where \( r \equiv |x_{01}| \), and we have integrated over the azimuthal angle. Since \( N \) only depends on the magnitude \( r \) of the vector \( x_{01} \), \( N \) depends only on the gluon transverse momentum magnitude \( k \). \( N(k, Y) \) can be related to the unintegrated gluon distribution of the target, and thus gives a direct information on the distribution of transverse momenta of the gluons probed in the target when the total rapidity is \( Y \).

The equation (3) then takes the form
\[
\frac{\partial N(k, Y)}{\partial Y} = \frac{\alpha_s N_c}{\pi} \left[ K \otimes N(k, Y) - N^2(k, Y) \right] \] (5)
where the kernel \( K \) is the integral kernel of the BFKL equation in momentum space. It acts on the gluon distribution as
\[ K \otimes N(k, Y) = \int_0^\infty \frac{dk'^2}{k'^2} \left[ \frac{k'^2 N(k', Y) - k^2 N(k, Y)}{|k^2 - k'^2|} + \frac{k^2 N(k, Y)}{\sqrt{4k'^4 + k^4}} \right]. \] (6)

A comment useful for the further analysis is in order. It appears easier to discuss the propagation towards the ultraviolet and the infrared using formulas (5,6) in terms of gluon momenta. Indeed, this is the formulation for which the mathematical results in terms of traveling waves [9] are better known. However, the universality properties of the traveling wave regime in momentum space are expected to be valid in position space too. Hence, the propagation of momenta towards the ultraviolet (infrared) will correspond to the evolution of QCD dipoles towards smaller (larger) sizes.

The mathematical argument for this statement is, briefly, that it is easy to realize (e.g. by means of a Mellin transform [1, 7]), that the linear part of the equation is symmetric by Fourier transform, interchanging \( k, k' \) with \( 1/r, 1/r' \) in Eq.(6). It is then known from general mathematical properties [9, 14], that the traveling wave regime does not depend of the form of the non-linear term even if modified by Fourier transformation.

### III. Propagation Toward the Ultraviolet

Let us first consider the problem of the propagation of the distribution of momenta towards the ultraviolet, in an "universal way", that is in a way independent of the precise physical structure of the target. In order to formulate the problem, we want to consider initial conditions defined only by a sufficiently sharp ultraviolet cut-off. In fact, interestingly enough, we will see that the resulting traveling wave pattern is independent of the initial cut-off profile.

We start with a simple choice of initial condition, namely a \( \theta \)-function. As we shall see, the asymptotic solution is the same for any other initial distribution with a sharp enough cut-off\(^1\). Hence we choose
\[ T(k, Y_0) = \theta(-L) \] (7)

\(^1\) We check this point numerically in Section V.
where $L \equiv \log \left( k^2/k_0^2 \right)$, with $k_0$ being the initial ultraviolet cut-off. The notation $T$ is now for the solution of the BK equation starting from the $\theta$ cut-off (7). When the rapidity increases, the transverse momenta of the gluon distribution evolve towards larger and larger $k$. As we shall see, the solution takes the form of a traveling wave, the same that was found [9] for the physical amplitude $N(k,Y)$ itself. The wave front defines a critical region in $k$ moving forward with rapidity.

Let us sketch the arguments in the present case of an initial $\theta$-function. One introduces the Mellin transform with respect to the variable $k/k_0$. In the long-range tail of the distribution, where $T(Y) \sim 0$, the non-linear term in (5) will remain small when evolving in $Y$ and can at first be neglected. The solution of the linear part will thus remain valid. The solution is written

$$T(k,Y) \sim \int_C \frac{d\gamma}{2i\pi} \frac{1}{\gamma} \exp \left( -\gamma L + \omega(\gamma)(Y-Y_0) \right),$$

where

$$\omega(\gamma) \equiv \frac{\alpha_s N_c}{\pi} \chi(\gamma) = \frac{\alpha_s N_c}{\pi} \left[ 2\psi(1) - \psi(\gamma) - \psi(1) - \gamma \right]$$

is the Mellin transform of the kernel\(^2\) and the prefactor $1/\gamma$ comes from the initial $\theta$-function condition (7). We are expecting a wave front and thus search for a specific velocity $v_+$ characterizing its forward motion. Hence we define a comoving frame following the ultraviolet front and defined by the shift $L \rightarrow LWF \equiv L - v_+Y$. A priori, no special velocity $v_+$ is selected, since for each value of $\gamma$ one has a velocity $v_+(\gamma) = \frac{\omega(\gamma)}{\gamma}$ which stabilizes the value of $LWF$. However, it can be shown from general grounds [14] that, once the non-linearities are properly taken into account, and provided the initial conditions are steep enough in $L$, a critical velocity

$$v_+ = \frac{\omega(\gamma_c)}{\gamma_c} = \min_{\gamma} \frac{\omega(\gamma)}{\gamma} \equiv \omega'(\gamma_c)$$

is selected, in analogy with the group velocity in wave physics [15, 16]. Here $\gamma_c \approx 0.6275$ is the critical anomalous dimension identical to that found for the BK equation [9, 17]. In fact the only condition [9] on the initial boundary dependence is $T(k,Y_0) < e^{-\gamma_c L}$. This can also be inferred from analyticity properties of the integrand of Eq. (8) in Mellin space [9], where the point is to check that the position of the prefactor pole $1/\gamma$ is situated to the left (in $\gamma$) of the critical point $\gamma_c = 0.6275$. This ensures that the overall behavior will be driven by the critical slope $\gamma_c$ and not by the one corresponding to the initial condition. These conclusions have been tested numerically in Ref. [18].

Indeed, the argument is essentially the same as for the transition matrix element, even if the initial condition is different. Recall that in that case, the initial conditions are dictated by the perturbative QCD property of “color transparency” $T(k,Y) \propto k^{-2}$ at large $k$ (i.e. color transparency, in terms of dipoles). In mathematical terms, all solutions verifying a sharp enough cut-off $T(k,Y_0) < e^{-\gamma_c L}$ lie in the same “universality class” and thus follow the same traveling wave pattern at high enough rapidity. For instance, both the physical amplitude $N$ and the solution $T$ with $\theta$-function initial condition lie in this “universality class”. Hence a unifying description of “propagation towards the ultraviolet” is correctly defined. We shall discuss the “propagation towards the infrared” within the same context in the next section IV.

There exists a critical value $k_c(Y)/k_0$ which characterizes the position of the moving wave front $T$ of momenta propagating towards the ultraviolet. The value of $k_c(Y)$ is defined as the value of $k$, at a specified $Y$, where the distribution $T$ has a fixed (and, for consistency, small) constant value. Since this is also the region where these distributions vary significantly with $L$, see Fig. 1, $k_c(Y)$ represents the most probable domain value for momenta as a function of $Y$. On general grounds, one finds

$$k_c(Y) \propto k_0 \exp \left( v_+Y - \frac{3}{2\gamma_c} \log Y \right),$$

where $\gamma_c$ and $v_+$ are given by the solution of the equation (10). The correction $\sim \log Y$ is a universal retardation effect due to the non-linearities. There exists also a third $O(Y^{-1/2})$ sub-asymptotic universal term in the exponential [9] which we neglect for our purposes. We also leave for further study the interesting question of pre-asymptotics and early scaling [10, 18].

\(^2\) In the expression (8) of the Mellin-transform one may see that it is symmetric in the Fourier transform from momentum to position space, allowing to extend to the latter the traveling wave properties of the former, as quoted in section II.
Together with the front velocity \( dk_c(Y)/dY \), it is possible to obtain some general knowledge of the front profile in the region next to the tail, including information on the diffusive approach to the scaling curve. This means that the gluon momenta do not reach instantaneously the scaling behavior of the traveling wave. There is a kind of diffusive retardation effect, violating geometric scaling. One finds

\[
T(k, Y) \sim \text{const.} \times \sqrt{\frac{2\pi}{N_c\alpha_s\chi''(\gamma_c)}} \log \left( \frac{k^2}{k_c^2(Y)} \right) \left( \frac{k^2}{k_c^2(Y)} \right)^{-\gamma_c} \exp \left( -\frac{\pi}{2N_c\alpha_s\chi''(\gamma_c)} \log^2 \left( \frac{k^2}{k_c^2(Y)} \right) \right).
\]

(12)

A physically important remark follows from the identification of the traveling wave characteristics of general ultraviolet evolution with the amplitude \( N \) with perturbative initial conditions. It means that the dynamics are dominated by gluon momenta of order of the saturation scale rather independently of the ultraviolet cut-off profile (if sharp enough). In other terms, the propagation towards the ultraviolet is essentially determined by the evolution of the saturation scale. These gluons are expected to eventually build the CGC phase.

### IV. PROPAGATION TOWARD THE INFRARED

Let us imagine that at \( Y = Y_0 \), instead of the boundary condition (7), one starts with boundary conditions for another solution \( \mathcal{U} \) of the BK equation, that describes an initial distribution of gluons with a low momentum cut-off \( k_0 \), see Fig. 1. As in the previous case, we shall discuss an ideal \( \theta \)-function condition, but our results will be valid for the whole class of initial conditions lying in the same “universality class” of sharp enough infrared cut-offs, to be defined with precision later on.

We start with the initial condition given by

\[
\mathcal{U}(k, Y_0) = \theta(+L),
\]

(13)

where \( \mathcal{U}(k, Y) \) denotes the corresponding solution of the BK evolution equation. We will now demonstrate that under very general conditions such an initial condition leads to a gluon distribution that propagates rapidly “backwards”, i.e. towards the infrared.

However, a special treatment is needed for the BFKL kernel given by (2) in momentum space or (9) in Mellin space. The critical value of the speed of the traveling wave is obtained from the equation \( v = \omega'(\gamma_c) = \omega'(\gamma_c)/\gamma_c \), an equation that has one finite and one infinite solution when \( \chi \) is the leading logarithmic (LL) BFKL kernel. Indeed, \( \chi \) has one unique finite value of \( \gamma_c \) is \( \gamma_c = 0.6275 \), and one formally infinite one corresponding to the pole at \( \gamma = 0 \), see Fig. 2(a). The infinite value (formally for \( \gamma = 0 \)) is the indication of an infrared instability of the propagation which is to be regularized, as will become clear later on. The analysis of the asymptotic solutions of \( \mathcal{U} \) requires a particular treatment in this case.

#### A. Diffusive approximation

Let us proceed in steps and first go back to the original diffusive approximation of the BK equation where known mathematical properties are useful to address our problem. This approximation of the LL BFKL kernel, considered in Ref. [9], consists in expanding the kernel around \( \gamma = \frac{1}{2} \) and keeping terms up to second order,

\[
\chi(\gamma) \simeq \chi(\frac{1}{2}) + \chi'(\frac{1}{2})(\gamma - \frac{1}{2}) + \frac{1}{2} \chi''(\frac{1}{2})(\gamma - \frac{1}{2})^2 \quad (\text{diffusive approx.}).
\]

(14)

This brings [9] the BK equation into the form of the Fisher–Kolmogorov–Petrovsky–Piscounov equation [19]. In this kernel there are no poles and the equation for \( \gamma_c \) has two solutions, one \( (\gamma_+) \) with positive, the other \( (\gamma_-) \) with negative value \( v_- \) of the speed, see Fig. 2(b). The positive value yields the same forward traveling waves propagating with speed \( v_+ \) discussed in the previous section.

We now emphasize the significance of the second, negative, value. It will correspond to “backward” traveling waves moving with speed \( v_- \) towards the infrared.
FIG. 2: Graphical solutions of the equation $\chi(\gamma_c) = \gamma_c \chi'(\gamma_c)$, for (a) the LL BFKL kernel, Eq. (9), (b) the diffusive approximation, Eq. (14), and (c) the two-pole approximation, Eq. (16) for $\gamma_0 = 0.2$. For the BFKL kernel there is no solution giving a negative velocity (corresponding to negative $\gamma_c$), while for the diffusive and two-pole approximations there are both positive and negative velocity solutions. Note that the magnitude of the negative velocity (i.e. the slope of the tangent) is much larger than for the positive velocity (in the LL BFKL case, it is infinite).

The existence of two types of traveling wave solutions for the same equation has already been noticed in Refs. [20]. However, in that case they correspond to a statistical physics problem for which both waves propagate in the same direction\(^5\) but with different speed. In QCD evolution, the speed $v_- = \chi(\gamma_-)/\gamma_-$, with $\gamma_- < 0$ and $\chi(\gamma_-) > 0$ is negative. As in the forward case, there is a universality class of solutions with the same pattern, provided the initial condition for backward propagation is steep enough in the backward direction, namely $U(k, Y_0) < \gamma L, L < 0$. This universality class is characteristic of the propagation towards the infrared when using the diffusive approximation of the BFKL kernel.

B. Regularization by energy-momentum conservation

We now consider cases where the initial LL BFKL kernel gets modified, see Fig. 2(c). This is expected e.g. from energy momentum conservation constraints and other next-to-leading logarithmic (NLL) BFKL effects [21, 22]. The pole at $\gamma = 0$ gets shifted to a negative value (compare Fig. 2-a and -c). Indeed, the effective kernel function $\chi(\gamma)$ used in phenomenological calculations [22] gets modified along these lines and thus allows for a finite negative value $\gamma_-$. A simple argument for understanding the shifting of the pole is energy momentum conservation: In the relation $\omega = (\alpha_s N_c/\pi) \chi(\gamma)$, energy conservation translates into the condition that $\omega = 1$ when $\gamma = 0$. In the limit $\gamma \to 0$, the BFKL kernel behaves as $\chi \sim 1/\gamma$, so we may incorporate energy conservation by writing

$$\chi(\gamma) \sim \frac{1}{\gamma + \frac{2 \alpha_s N_c}{\pi}} \quad \text{(small } \gamma),$$

(15)

which moves the $\gamma = 0$ pole to $\gamma = -\alpha_s N_c/\pi$.

C. Regularization using the two-pole model

It is well-known that the LL BFKL kernel is well-approximated by a two-pole model, $\chi(\gamma) = 1/\gamma + 1/(1-\gamma)$, which reproduces the poles of the BFKL kernel. This kernel has also been studied in Ref. [18] in the context of saturation and is found to give results very similar to those from the full kernel.

In view of the above we use as a toy kernel

$$\chi(\gamma) = \frac{1}{\gamma + \gamma_0} + \frac{1}{1 - \gamma + \gamma_0} + 4 \left( \log 2 - \frac{1}{1 + 2 \gamma_0} \right),$$

(16)

\(^5\) We interpret this difference between the statistical physics problem and QCD by the fact that the former is a one-dimensional problem analogous to the random breaking of a segment into smaller pieces [20], while the latter is related to the cascading of dipoles in a two-dimensional position space (even if the equation under study is its one-dimensional projection).
which reproduces the main features of the BFKL kernel with the poles at $\gamma = 1$ and $\gamma = 0$ shifted by an amount $\gamma_0$ and $-\gamma_0$ respectively. We consider $\gamma_0$ as a free parameter of our model. This kernel can be viewed as an approximation of some next-to-leading effects, and is also simple enough to use in our reasoning. It will allow for analytical and numerical studies of the infrared propagation for given $\gamma_0$ and its instability when $\gamma_0 \to 0$. The constant term is chosen to reproduce the value of the full BFKL kernel at $\gamma = 1/2$.

For example, choosing the value $\gamma_0 = 0.2$ as illustrated in Fig. 2(c), one obtains two real solutions to the equation defining $\gamma_c$, $\gamma_c^+ = 0.766$ and $\gamma_c^- = -0.103$, leading to traveling wave solutions with both positive and negative velocities. All these examples, including the $\gamma_0 = 0$ case where the infrared instability appears, will now be discussed by comparison with numerical simulations.

V. NUMERICAL SIMULATIONS

The BK equation has been solved numerically by many groups [23] by direct solution of the integro-differential equation. Recently, in Ref. [18], it was solved using a discretization of the integrand through Chebyshev approximation and a Runge–Kutta method for the ensuing system of ordinary differential equation (see [18] for more details on the method). The solutions were studied in detail and compared to analytical results obtained using the traveling wave front method. In this paper we use the same numerical method to study the solutions for the distributions $T$ and $U$ in $k$-space. All results that follow are obtained with a fixed strong coupling constant, $\alpha_s N_c/\pi = 0.2$.

First, let us consider the simulation of equation (5) made using the original BFKL kernel (6), corresponding to the case of Fig. 2(a).

We observe that traveling wave solutions are generated in the forward direction, while the backward direction is characterized by a fixed (non-traveling) scaling curve, obtained almost immediately in the rapidity evolution. Moreover, the solution extends also immediately towards very negative values of $L \equiv \log(k^2/k_0^2)$, i.e. in the very far infrared. This is a clear sign of infrared instability of the evolution with the LL kernel.

In LL BFKL, hence, only forward traveling waves are present. The backward solution propagates instantly to high $Y$, and generates a fixed scaling solution. In a formal way, the speed gets infinite and the wave front is situated at $L = -\infty$. This situation is illustrated in Fig. 3.

In order to analyze in more detail this infrared instability, we considered the same problem with much smaller steps in rapidity, see Fig. 4. We see the evidence for a rapid diffusive evolution towards the scaling curve. Note also the far infrared region, which is immediately reached by the evolution. This is a feature related to the singularity of the canonical BFKL kernel at $\gamma = 0$, and, as we shall see now, is physically tamed by next-to-leading corrections. This sensitivity to next-to-leading QCD corrections is a specific and interesting feature of backward propagation towards the infrared.

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6 See Ref. [18] for the numerical implementation of the two-pole model.
FIG. 4: First steps of BFKL evolution towards the infrared. The rapidity steps are $\delta Y = 0.1$.

FIG. 5: Propagation of momenta for the two-pole model. Left: towards the infrared; Right: towards the ultraviolet. The curves are drawn for values $\gamma_0 = 0, 0.2, 0.4$ and the rapidity steps are $\delta Y = 1$.

In Fig. 5, we show the similar plot for the non-linear evolution using the two-pole kernel (16) with different values of the parameter $\gamma_0$ which, in our toy model, schematically characterize the amount of next-leading corrections. Interestingly enough, the toy model results for $\gamma_0 = 0$ reproduce the same features as for the BFKL kernel, with a scaling curve in the backward propagation. For non-zero $\gamma_0 = 0.2, 0.4$, left-moving traveling waves appear, in agreement with the analytical predictions. In all cases the speed $v_-$ is larger than for the forward propagation (i.e. larger than the ordinary forward BK traveling waves). Hence the infrared instability is made finite but the propagation towards the infrared is still much more rapid. In terms of QCD dipoles, one may say that dipoles with large sizes are easily produced during the energy evolution, in the “universality class” of backward propagation. The comparison between the forward and backward evolutions is exemplified in Fig. 6.

To round up the numerical results we give an example of the (in)dependence of the traveling wave regime from the precise shape of the cut-offs in the initial conditions provided they are sharp enough, either in the forward or the backward direction respectively, to satisfy the mathematical requirements for creating traveling waves. We thus
**FIG. 6:** Comparison of forward and backward propagation for $\gamma_0 = 0.4$. The rapidity steps are $\delta Y = 1$.

**FIG. 7:** Propagation of momenta for different initial conditions. Left: two-pole model towards the infrared; Right: BFKL kernel towards the ultraviolet. Continuous lines: initial step functions. Dotted lines: initial smoother cut-off (see text). The curves are drawn for the BFKL kernel and the rapidity steps are $\delta Y = 1$.

compare initial step functions with an initial form [4] of the cut-off profile which is smoother than a $\theta$-function, while satisfying the sharpness condition. As typical examples (see Fig. 7) we considered the backward waves obtained for the toy model and the forward waves for the original BFKL kernel. Finally, we confirm that the speed of the front propagation is correctly given by the analytical formula for asymptotically large rapidities, i.e., we compare the rapidity dependence of the saturation scale of the forward and backward traveling waves obtained from the numerical solution to the analytical formula for asymptotically large rapidities.

The saturation scale $k_c(Y)$ is defined as the momentum, at a certain rapidity, where the amplitude reaches a particular value $\kappa$, i.e., $T(k_c(Y), Y) = \kappa$. The overall normalization therefore depends on the chosen value of $\kappa$ and it is most convenient to study the logarithmic derivative of $k_c(Y)$, where the normalization cancels. Because the shape of the front is not completely fixed as it propagates in rapidity, the functional form of $\partial \log k_c(Y)/\partial Y$ will also depend on $\kappa$, but this dependence is a subleading effect in $Y$ and vanishes as $Y$ becomes large. Here we choose $\kappa = 0.01$.  

\[ 1 - \exp \left( -r^2 Q^2_s(Y)/4\log(e + 1/(r^2 \Lambda^2)) \right) . \] (17)

\[ \text{See [18] for examples and a thorough discussion.} \]
The analytical result given by (11) is

\[
\frac{\partial \log k_c(Y)}{\partial Y} = v_+ - 3 \frac{2}{\gamma c Y},
\]

where subleading \(\mathcal{O}(Y^{-3/2})\) terms have been omitted for sake of simplicity. Fig. 8 shows the results of the comparison between the simulations and expression (18) for the two-pole toy kernel (16) with \(\gamma_0 = 0.4\) (the constants are given by Eq. (10): \(v_+ = 0.7372\), \(v_- = -6.112\), \(\gamma_+ = 0.9157\), and \(\gamma_- = -0.2202\)). The results indicate that the two results tend to the same asymptotic values \(v_\pm\) of the derivative, while there are still appreciable subasymptotic effects [18].

A study of subasymptotic effects and early scaling, which we postpone for future work, could take benefit from the recent work in Ref. [10].

VI. CONCLUSIONS AND PHYSICAL IMPLICATIONS

Let us summarize the results obtained in our study of the gluon momentum propagation (or equivalently, dipole size evolution) during nonlinear QCD evolution at high density/energy:

(i) The propagation of gluon momenta towards the ultraviolet and infrared with energy correspond to two different “universality classes” of solutions of the non-linear Balitsky–Kovchegov evolution equation. They are selected by the existence of an ultraviolet (resp. infrared) cut-off in the initial conditions but largely independent of the cut-off profile, if sharp enough.

(ii) The propagation towards the ultraviolet is characterized by asymptotic traveling wave solutions which are independent of the (sharp enough) initial distribution. In particular, they are the same as those exhibited by the onium-target amplitude with initial conditions defined by the QCD transparency property. Hence, the rate of evolution towards the ultraviolet is universally governed by the saturation scale, generalizing the behavior of the BK dipole-target amplitude.

(iii) The propagation of sizes towards the infrared shows new and distinctive features which have not been discussed, to our knowledge, in the literature previously. In the case of the LL BFKL kernel, i.e. in the leading log approximation of perturbative QCD, one obtains a scaling distribution (after a short diffusive interlude) which extends almost immediately towards the far infrared region, revealing an infrared instability. This phenomenon is due to the singularity of the BFKL kernel at \(\gamma = 0\).

(iv) As soon as this singularity is shifted or smoothed out, new traveling wave solutions appear, with a (large) speed of backward propagation towards the infrared. In particular, next-to-leading corrections to the BFKL kernel should give rise to such a phenomenon and determine the value of the critical speed, as tested in different examples.

(v) The analytical predictions of the traveling wave pattern in both directions (ultraviolet and infrared) have been verified by numerical solutions of the evolution equations.
Our findings of the new traveling wave pattern towards the infrared within a purely perturbative evolution framework could have sizable physical implications. An example we have in mind is the problem of thermalization of a Color Glass Condensate formed in the first stages of a heavy-ion collision.

Indeed, it is well known that it is not easy to reconcile the formation of a perturbative CGC (at weak coupling) as an initial compound state in a heavy ion collision, with the hydrodynamic features of the final state which probably require a fast thermalization or, at least, a rapid decrease of the mean free path in the medium. We think that a discussion of this mechanism would be interesting for the discussion of thermalization of an initial Color Glass Condensate phase during a heavy ion collision process.

As a further outlook, it will also be worth examining the effect of the growth of the coupling constant during the propagation towards the infrared. Our present results have been obtained for simplicity assuming a fixed value of the coupling constant. They will have a natural extension to running coupling, using the method of Ref. [9] for solutions of the BK equation with running coupling. We expect qualitatively similar results but the quantitative analysis deserves to be performed.

These essentially geometric features may also be modified by a better description of the interaction with the target. It is nowadays emphasized [18, 24] that fluctuations could modify the saturation solutions, violate geometric scaling or even exert a back-reaction onto the linear regime. What we obtain from our analysis of the “universality class” of propagation towards the ultraviolet will probably survive the implication of fluctuations. Even more interesting is the investigation of fluctuation effects on the infrared propagation. This will be the subject of forthcoming studies. Properties borrowed from statistical physics could help in attacking this problem with new tools.

Acknowledgments

We thank S.N. Majumdar for an inspiring seminar on statistical physics and A. Bialas and K. Golec-Biernat for fruitful discussions in a first stage of this work. C. Marquet and G. Soyez are to be thanked for their clever remarks and careful reading of the manuscript. R.E. thanks the Service de Physique Théorique of CEA/Saclay for hospitality when parts of this work was done.