We consider the QCD scattering amplitudes at high energies $\sqrt{s}$ and fixed momentum transfers $\sqrt{-t}$ in the leading logarithmic approximation at a non-zero temperature $T$ in the $t$-channel. It is shown that the BFKL Hamiltonian has the property of holomorphic separability. The holomorphic Hamiltonian for $n$-reggeized gluons at temperature $T$ is shown to be obtained from the $T = 0$ Hamiltonian by an unitary transformation. The connection with field quantization in accelerated frames and black hole backgrounds is discussed. The effect of the running of $\alpha_{QCD}$ on the pomeron eigenvalues is investigated.

1 Quark Gluon Plasma and Reggeons

Current theoretical understanding of the quark-gluon plasma (QGP) generated in heavy nucleus collisions suggests that the QGP thermalizes via parton-parton scattering. The QGP is understood to cool down by hydrodynamic expansion till the temperature reaches the hadronization scale $\sim 160$MeV. One interesting phenomena is the suppression of the $\psi$-meson production in the heavy nucleus collisions due to the disappearance of the confining potential between $q$ and $\bar{q}$ at high temperature. A similar effect should exist for glueballs constructed from gluons. Because the Pomeron is considered as a composite state of reggeized gluons, the influence of the temperature on its properties is of great interest. We constructed in ref. the BFKL equation at temperature $T$ in the center-mass system of the $t$ -channel (where $\sqrt{t} = 2\epsilon$) and show that the BFKL dynamics in a thermostat for composite states of $n$ reggeized gluons in multi-colour QCD is integrable.

Let us consider the Regge kinematics in which the total particle energy $\sqrt{s}$ is asymptotically large in comparison with the temperature $T$. In this case one can neglect the temperature effects in the propagators of the initial and intermediate particles in the direct channels $s$ and $u$. But the momentum transfer $|q|$ is considered to be of the order of $T$ (note, that $q_{\mu}$ is the vector orthogonal to the initial momenta $q_{\mu} \approx q_{\perp}^{\perp}$). As it is well known the particle wave functions $\psi(x_{\mu})$ at temperature $T$ are periodic in the euclidean time $\tau = i t$ with period $1/T$. 
We introduce the temperature $T$ in the center of mass frame of the $t$-channel. Thus, the euclidean energies of the intermediate gluons in the $t$-channel become quantized as

$$ k_4^{(l)} = 2\pi l T. $$

In the $s$-channel the invariant $t$ is negative and therefore the analytically continued 4-momenta of the $t$-channel particles can be considered as euclidean vectors. It means, that at temperature $T$, the wave functions for virtual gluons are periodic functions of the holomorphic impact-parameter $\rho = x + iy$ with imaginary period $iT$. Also, the canonically conjugated momenta $p$ have their imaginary part quantized,

$$ \rho \rightarrow \rho + \frac{i}{T}, \quad p = \text{Re} p + \pi i l T. $$

with integer $l$ (note that $p = (p_1 + ip_2)/2$).

## 2 Reggeon Hamiltonian at temperature $T$

The calculation of the Regge trajectory $1 + \omega(t)$ of the gluon at temperature $T$ in the $t$-channel, in one-loop approximation reduces to the integration over the real part $k_1$ of the transverse momentum $k$ of the virtual gluon and to the summation over its imaginary part $k_2 = l$. In such a way we obtain the following result for the trajectory having the separability property [cf. 2],

$$ \omega(-\vec{q}^2) = -\frac{g^2}{8\pi^2} N_c \Omega(-\vec{q}^2), \quad \Omega(-\vec{q}^2) = \Omega(q) + \Omega(q^*). $$

Here,

$$ \Omega(q) = \frac{\pi T}{\lambda} + \frac{1}{2} \left[ \psi(1 + \frac{iq}{2\pi T}) + \psi(1 - \frac{iq}{2\pi T}) - 2\psi(1) \right], $$

where we regularized the infrared divergence for the zero mode $l = 0$ introducing a mass $\lambda$ for the gluon (see 2).

A similar divergence appears in the Fourier transformation $G(\vec{\rho}_{12})$ of the effective gluon propagator $(\vec{k}_1^2 + \lambda^2)^{-1}$ contained in the product of the effective vertices $q_1 k^{-1} q_2^*$ for the production of a gluon with momentum $k_\mu$ (cf. 2),

$$ G(\vec{\rho}_{12}) = -\frac{\pi T}{\lambda} + \ln (2 \sinh \pi T \rho_{12}) + \ln (2 \sinh \pi T \rho_{12}^*). $$

Therefore, the divergence at $\lambda \rightarrow 0$ cancels in the sum of kinetic and potential contributions to the BFKL equation and the Hamiltonian $H_{12}$ for the Pomeron in a thermostat has the property of holomorphic separability with the holomorphic Hamiltonian given below (as it is the case for zero temperature 2)

$$ h_{12} = \sum_{r=1}^{2} \left[ \frac{1}{2} \psi(1 + \frac{ip_r}{2\pi T}) + \frac{1}{2} \psi(1 - \frac{ip_r}{2\pi T}) + \frac{1}{p_r} \ln (2 \sinh \pi T \rho_{12} p_r - \psi(1)) \right]. $$

We constructed in ref. 11 the eigenfunctions and eigenvalues of the hamiltonian eq. 5 in terms of hypergeometric functions.
3 Wavefunctions in coordinate space

The Pomeron wave function can be constructed directly in coordinate space. For this purpose we use the conformal transformation

$$\rho'_r = \frac{1}{2\pi T} e^{2\pi T \rho_r}$$

(6)

to map the zero temperature wave functions into the wavefunctions at temperature $T$.

Thus, the Pomeron wave function at non-zero temperature having the property of single-valuedness and periodicity takes the form

$$\Psi^{(m,\tilde{m})}(\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_0) = \left( \frac{\sinh[\pi T \rho_{12}]}{2\sinh[\pi T \rho_{10}] \sinh[\pi T \rho_{20}]} \right)^m \times \left( \frac{\sinh[\pi T \rho'_{12}]}{2\sinh[\pi T \rho'_{10}] \sinh[\pi T \rho'_{20}]} \right)^{\tilde{m}}. \quad (7)$$

The orthogonality and completeness relations for these functions can be easily obtained from the analogous results for $T = 0$ (see 2) using the above conformal transformation.

Moreover, the pair BFKL Hamiltonian $h_{12}$ can be expressed in terms of the BFKL Hamiltonian at zero temperature in the new variables

$$h_{12} = \ln(p'_1 p'_2) + \frac{1}{p'_1} \log(p'_1) p'_1 + \frac{1}{p'_2} \log(p'_2) p'_2 - 2\psi(1), \quad (8)$$

where $p'_r = i \frac{\partial}{\partial \rho'_r}$. We recognize here the zero temperature BFKL Hamiltonian 2. In the course of the derivation the following operator identity (see 2)

$$\frac{1}{2} \left[ \psi \left( 1 + z \frac{\partial}{\partial z} \right) + \psi \left( -z \frac{\partial}{\partial z} \right) \right] = \ln z + \ln \frac{\partial}{\partial z}$$

was used to transform the kinetic part as well as properties of the $\psi$-function.

In summary, the exponential mapping eq. (6) maps the reggeon dynamics from zero temperature to temperature $T$. This mapping explicitly exhibits a periodicity $\rho \rightarrow \rho + \frac{i}{T}$ for a thermal state. It must be noticed that such class of mappings are known to describe thermal situations for quantum fields in accelerated frames and in black hole background 3.

4 Integrability at non-zero temperature

As it is well known 5, the BFKL equation at $T = 0$ can be generalized to composite states of $n$ reggeized gluons. In the multi-colour limit $N_c \rightarrow \infty$ the BKP equations are significantly simplified thanks to their conformal invariance, holomorphic separability and integrals of motion 2. The generating function for the holomorphic integrals of motion coincides with the transfer matrix for an integrable lattice spin model 6. The transfer matrix is the trace of the monodromy matrix

$$t(u) = L_1(u) L_2(u) \ldots L_n(u),$$

satisfying the Yang-Baxter equations 6. The integrability of the $n$-reggeon dynamics in multi-colour QCD is valid also at non-zero temperature $T$, where, according to the above arguments we should take the $L$-operator in the form

$$L_k = \begin{pmatrix} u + p_k & e^{-2\pi T \rho_k} p_k \\ -e^{2\pi T \rho_k} p_k & u - p_k \end{pmatrix}. \quad (9)$$

In particular, the holomorphic Hamiltonian is the local Hamiltonian of the integrable Heisenberg model with the spins being unitarily transformed generators of the Möbius group (cf. 7)

$$M_k = \partial_k$$

(9)
\[ M_+ = e^{-2\pi T \rho_k} \partial_k \quad , \quad M_- = -e^{2\pi T \rho_k} \partial_k . \]

Because the Hamiltonian at non-zero temperature can be obtained by an unitary transformation
from the zero temperature Hamiltonian, the spectrum of the intercepts for multi-gluon states
is the same as for zero temperature and the wave functions of the composite states can be
calculated by the substitution \( \rho_k \to \frac{1}{2\pi T} e^{2\pi T \rho_k} \).

5 Running with the QCD coupling

Taking into account the running of \( \alpha_{QCD} \) to one-loop level \( \alpha_s(Q) = \frac{4\pi}{9 \log \frac{\Lambda^2}{\mu^2}} \) changes the previous
results. Now, the pomeron wave function must be an eigenfunction of the operator
\[ \mathcal{E}(Q) \equiv \alpha_s(Q) \chi \left( -\frac{i}{2} \frac{d}{d\ln |\rho|} \right) \quad , \quad \chi(\nu) = -\frac{6}{\pi} \text{Re} \left[ \gamma + \psi \left( \frac{1}{2} + i\nu \right) \right] \] (10)

Semiclassically, we have as quantization condition
\[ \phi(\nu_k) = k + \frac{1}{4} \quad , \quad k = 0, 1, 2, \ldots \] (11)

where
\[ \phi(\nu) \equiv \frac{4}{9 \alpha_s(Q)} \left[ \frac{1}{\chi(\nu)} \int_0^\nu dx \chi(x) - \nu \right] - \frac{1}{2\pi} \delta^T_{\nu}(\bar{Q}) \] (12)

and
\[ e^{i \delta^T_{m, \bar{m}}(\bar{Q})} = |Q|^{-4i\nu} \frac{\Gamma(m + iQ)}{\Gamma(1 - m + iQ)} \frac{\Gamma(\bar{m} - i\bar{Q}^*)}{\Gamma(1 - \bar{m} - i\bar{Q}^*)} \] .

The corresponding eigenvalue of the operator \( \mathcal{E}(Q) \) are given by
\[ \omega_k(Q) = \alpha_s(Q) \chi(\nu_k) \] (13)

The eigenvalues and eigenfunctions now depend parametrically on \( Q = \frac{Q_{\text{phys}}}{2\pi T} \). We recover
the zero temperature limit for \( Q \to \infty \). The characteristic scale in temperature is given by
\[ T_{\text{chara}} \equiv \frac{1}{\pi} e^{\frac{\beta_0}{2\pi T}} \] We find that the eigenvalue for \( k = 0 \) first grows with the temperature and
then goes down. In addition, we see that rotational invariance is recovered for \( T \to 0 \) since the
results for Im \( Q = 0 \) and 1 coincide in such limit.

References

3. See for example, N. D. Birrell, P. C. W. Davis, ‘Quantum Fields in Curved Space’, Cam-


8. L. N. Lipatov, Sov. Phys. JETP, 63, 904 (1986) [Zh. ETF, 90, 1536 (1986)].