Kac-Moody algebras in gravity and M-theories\textsuperscript{1}

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Abstract

The formulation of gravity and M-theories as very-extended Kac-Moody invariant theories is reviewed. Exact solutions describing intersecting extremal brane configurations smeared in all directions but one are presented. The intersection rules characterising these solutions are neatly encoded in the algebra. The existence of dualities for all $G^{+++}$ and their group theoretical-origin are discussed.

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A theory containing gravity suitably coupled to forms and dilatons may exhibit upon dimensional reduction down to three dimensions a simple Lie group \( \mathcal{G} \) symmetry non-linearly realised. The scalars of the dimensionally reduced theory live in a coset \( \mathcal{G}/\mathcal{H} \) where \( \mathcal{G} \) is in its maximally non-compact form and \( \mathcal{H} \) is the maximal compact subgroup of \( \mathcal{G} \). A maximally oxidised theory is such a Lagrangian theory defined in the highest possible space-time dimension \( D \) namely a theory which is itself not obtained by dimensional reduction. These maximally oxidised actions have been constructed for all \( \mathcal{G} \) \(^\text{1}\) and they include in particular pure gravity in \( D \) dimensions and the low energy effective actions of the bosonic string and of M-theory.

\[ \text{Figure 1: } \text{The nodes labelled 1,2,3 define the Kac-Moody extensions of the Lie algebras. The horizontal line starting at 1 defines the 'gravity line', which is the Dynkin diagram of a } A_{D-1} \text{ subalgebra.} \]

It has been conjectured that these theories, or some extensions of them, possess the much larger very-extended Kac-Moody symmetry \( \mathcal{G}^{+++} \). \( \mathcal{G}^{+++} \) algebras are defined by the Dynkin diagrams depicted in Fig.1, obtained from those of \( \mathcal{G} \) by adding three nodes \(^2\). One first adds the affine node, labelled 3 in the figure, then a second node, 2, connected to it by a single line and defining the overextended \( \mathcal{G}^{++} \) algebra\(^2\), then a third one, 1, connected by a single line to the overextended node. Such \( \mathcal{G}^{+++} \) symmetries were first conjectured in the aforementioned particular cases \(^3\) and the extension to all \( \mathcal{G}^{+++} \) was proposed in \(^6\). In a different development, the study of the properties of cosmological

\(^3\)In the context of dimensional reduction, the appearance of \( E_8^{++} = E_{10} \) in one dimension has been first conjectured by B. Julia \(^3\).
solutions in the vicinity of a space-like singularity, known as cosmological billiards \[7\], revealed an overextended symmetry $G^{+++}$ for all maximally oxidised theories \[8\] \[9\].

The possible existence of this Kac-Moody symmetry $G^{+++}$ motivates the construction of a Lagrangian formulation explicitly invariant under $G^{+++}$ \[10\]. The action $S_{G^{+++}}$ is defined in a reparametrisation invariant way on a world-line, a priori unrelated to space-time, in terms of fields $\phi(\xi)$ living in a coset $G^{+++}/K^{+++}$ where $\xi$ spans the world-line. A level decomposition of $G^{+++}$ with respect to the subalgebra $A_{D-1}$ of its gravity line (see Fig. 1) is performed where $D$ is identified to the space-time dimension\(^4\). The subalgebra $K^{+++}$ is invariant under a ‘temporal’ involution which ensures that the action is $SO(1, D-1)$ invariant at each level where the index 1 of $A_{D-1}$ is identified to a time coordinate.

Each $G^{+++}$ contains indeed a subalgebra $GL(D)$ such that $SL(D) = A_{D-1} \subset GL(D) \subset G^{+++}$. The generators of the $GL(D)$ subalgebra are taken to be $K^a_b$ ($a, b = 1, 2, \ldots, D$) with commutation relations

$$[K^a_b, K^c_d] = \delta^c_b K^a_d - \delta^a_d K^c_b.$$  \hspace{1cm} (1)

The $K^a_b$ along with abelian generators $R_u$ ($u = 1 \ldots q$), which are present when the corresponding maximally oxidised action $S_5$ has $q$ dilatons\(^5\), are the level zero generators. The step operators of level greater than zero are tensors $R_{d_1 \ldots d_s}$ of the $A_{D-1}$ subalgebra. Each tensor forms an irreducible representation of $A_{D-1}$ characterised by some Dynkin labels. In principle it is possible to determine the irreducible representations present at each level \[12\] \[13\]. The lowest levels contain antisymmetric tensor step operators $R^{a_1 \ldots a_r}$ associated to electric and magnetic roots arising from the dimensional reduction of field strength forms in the corresponding maximally oxidised theory. They satisfy the tensor and scaling relations

$$\begin{align*}
[K^a_b, R^{a_1 \ldots a_r}] &= \delta^a_b R^{a_1 \ldots a_r} + \cdots + \delta^a_{a_r} R^{a_1 \ldots a_r-1}, \\
[R, R^{a_1 \ldots a_r}] &= -\frac{\epsilon_A A_{AA}}{2} R^{a_1 \ldots a_r},
\end{align*}$$

where $\epsilon_A$ is the dilaton coupling constant to the field strength form and $\epsilon_A$ is $+1$ ($-1$) for an electric (magnetic) root \[14\]. The temporal involution $\Omega_1$ generalises the Chevalley involution to allow identification of the index 1 to a time coordinate in $SO(1, D - 1)$. It is defined by

$$K^a_b \xrightarrow{\Omega_1} -\epsilon_a \epsilon_b K^b_a, \quad R \xrightarrow{\Omega_1} -R, \quad R_{d_1 \ldots d_s} \xrightarrow{\Omega_1} -\epsilon_{c_1} \ldots \epsilon_{c_r} \epsilon_{d_1} \ldots \epsilon_{d_s} R_{c_1 \ldots c_r},$$

with $\epsilon_a = -1$ if $a = 1$ and $\epsilon_a = +1$ otherwise. It leaves invariant a subalgebra $K^{+++}$ of $G^{+++}$. The fields $\varphi(\xi)$ living in the coset space $G^{+++}/K^{+++}$ parametrize the Borel group

\[^4\]Level expansions of very-extended algebras in terms of the subalgebra $A_{D-1}$ have been considered in \[11\] \[12\] \[13\].

\[^5\]All the maximally oxidised theories have at most one dilaton except the $C_{q+1}$-series characterised by $q$ dilatons. In the rest of the paper we omit the $u$ index.
built out of Cartan and positive step operators in \( G^{+++} \). Its elements \( \mathcal{V} \) are written as

\[
\mathcal{V}(\xi) = \exp(\sum_{a \geq b} h_{a} h_{b} \xi K_{a} - \phi(\xi) R) \exp(\sum_{r=1}^{s} A_{h_{a_{1}} \ldots h_{a_{r}}} (\xi) R_{a_{1 \ldots a_{r}}} b_{1 \ldots b_{r}} + \ldots),
\]

where the first exponential contains only level zero operators and the second one the positive step operators of levels strictly greater than zero. Defining

\[
d\tilde{v}(\xi) = \Omega^{-1}_{1} d\tilde{v}(\xi) \quad d\tilde{v}(\xi) = -\xi_{1} dv(\xi) \quad dv_{\text{sym}} = \frac{1}{2}(dv + d\tilde{v}),
\]

one obtains, in terms of the \( \xi \)-dependent fields, an action \( S_{G^{+++}} \) invariant under global \( G^{+++} \) transformations, defined on the coset \( G^{+++}/K^{+++} \)

\[
S_{G^{+++}} = \int d\xi \frac{1}{n(\xi)} \left( \frac{dv_{\text{sym}}(\xi)}{d\xi} \right)^{2},
\]

where \( n(\xi) \) is an arbitrary lapse function ensuring reparametrisation invariance on the world-line and \( <,> \) is the invariant bilinear form.

Writing

\[
S_{G^{+++}} = S_{G^{+++}}^{(0)} \sum_{A} S_{G^{+++}}^{(A)},
\]

where \( S_{G^{+++}}^{(0)} \) contains all level zero contributions, one obtains

\[
S_{G^{+++}}^{(0)} = \frac{1}{2} \int d\xi \frac{1}{n(\xi)} \left[ \frac{1}{2} (g^{\mu \nu} g^{\sigma \tau} - \frac{1}{2} g^{\mu \sigma} g^{\nu \tau}) \frac{dg_{\mu \sigma}}{d\xi} \frac{dg_{\nu \tau}}{d\xi} + \frac{d\phi}{d\xi} \frac{d\phi}{d\xi} \right],
\]

\[
S_{G^{+++}}^{(A)} = \frac{1}{2r!s!} \int d\xi \frac{e^{-2\lambda \phi}}{n(\xi)} \left[ \frac{DA_{\mu_{1} \ldots \mu_{r}}^{\nu_{1} \ldots \nu_{s}}}{d\xi} g^{\mu_{1} \mu_{1} \ldots} g^{\mu_{r} \mu_{r}} g_{\nu_{1} \nu_{1} \ldots} g_{\nu_{s} \nu_{s}} \frac{DA_{\nu_{1} \ldots \nu_{s}}^{\mu_{1} \ldots \mu_{r}}}{d\xi} \right].
\]

The \( \xi \)-dependent fields \( g_{\mu \nu} \) are defined as \( g_{\mu \nu} = e_{a}^{a} e_{a}^{b} \eta_{ab} \) where \( e_{a}^{a} = (e^{-h(\xi)})^{a} \). The appearance of the Lorentz metric \( \eta_{ab} \) with \( \eta_{11} = -1 \) is a consequence of the temporal involution \( \Omega_{1} \). The metric \( g_{\mu \nu} \) allows a switch from the Lorentz indices \((a, b)\) of the fields appearing in Eq.\( (5) \) to \( \text{GL}(D) \) indices \((\mu, \nu)\). \( D/D\xi \) is a covariant derivative generalising \( d/d\xi \) through non-linear terms arising from non-vanishing commutators between positive step operators and \( \lambda \) is the generalisation of the scale parameter \(-\varepsilon_{a} a_{a}/2\) to all roots.

The \( G^{+++} \)-invariant actions \( S_{G^{+++}} \) leads to two distinct actions invariant under the overextended Kac-Moody algebra \( G^{+++} \). The first one \( S_{G^{+++}}^{(0)} \) is constructed from \( S_{G^{+++}} \) by performing a consistent truncation. The corresponding \( G^{+++} \) algebra is obtained from \( G^{+++} \) by deleting the node labelled 1 from the Dynkin diagram of \( G^{+++} \) depicted in Fig. 1. The truncation is achieved by putting to zero in the coset representative the field multiplying the Chevalley generator \( H_{1} \) and all the fields multiplying the positive step operators associated to roots whose decomposition in terms of simple roots contains the deleted root \( \alpha_{1} \). This theory carries a Euclidean signature and is the generalisation
to all $G^{++}$ of the $E_8^{++} = E_{10}$ invariant action first proposed in reference [14] in the context of M-theory and cosmological billiards. The parameter $\xi$ is then identified with the time coordinate and the action restricted to a defined number of lowest levels is equal to the corresponding maximally oxidised theory in which the fields depend only on this time coordinate. A second $G^{++}$-invariant action $S_{G^{++}}$ is obtained from $S_{G^{++}}$ by performing the same consistent truncation after conjugation by the Weyl reflection in the hyperplane perpendicular to the simple root corresponding to the node 1 of figure 1. The non-commutativity of the temporal involution with the Weyl reflection [15, 16] implies that after Weyl reflection the index 2 of $A_{D-1}$ is now identified to the time coordinate. Consequently the second action $S_{G^{++}}$ is inequivalent to the first one [17].

In $S_{G^{++}}$, $\xi$ is identified with a space-like direction and the action is characterised by a Lorentzian signature $(1, D - 2)$. This theory admits exact solutions which are identical to those of the corresponding maximally oxidised theory describing intersecting extremal brane configurations smeared in all directions but one [17, 10, 18]. For each of the $A = 1 \ldots N$ branes present in the intersecting brane configuration and characterised by $\lambda_1 \ldots \lambda_q$ longitudinal spacelike directions, one has one non-zero field component corresponding to one positive step operator associated with one positive real root $\alpha_A$.

\[
A_{2\lambda_1 \ldots \lambda_q A} = \epsilon_{2\lambda_1 \ldots \lambda_q A} [\frac{2(D - 2)}{\Delta_A}]^{1/2} H_A^{-1}(\xi) \quad A = 1 \ldots N,
\]

(11)

and

\[
p^a = \sum_{A=1}^{N} p^a_A = \sum_{A=1}^{N} \frac{\eta_A^a}{\Delta_A} \ln H_A(\xi) \quad a = 2, 3, \ldots, D
\]

(12)

\[
\phi = \sum_{A=1}^{N} \phi_A = \sum_{A=1}^{N} \frac{D - 2}{\Delta_A} \varepsilon_A a_A \ln H_A(\xi),
\]

(13)

where $p^a \equiv -h^a, a$ and $h^b_a = 0$ if $a \neq b$. Here $\eta_A^a = q_A + 1$ or $-(D - 3 - q_A)$ depending on whether the direction $a$ is perpendicular or parallel to the $q_A$-brane and $\Delta_A = (q_A + 1)(D - 3 - q_A) + \frac{1}{2} a_A^2 (D - 2)$. The factor $\varepsilon_A$ is $+1$ for an electric brane and $-1$ for a magnetic one. Each of the branes in the configuration is thus described as electrically charged and is characterised by one positive harmonic function in $\xi$-space, namely one has

\[
\frac{d^2 H_A(\xi)}{d\xi^2} = 0 \quad A = 1 \ldots N.
\]

(14)

The consistent truncation yields for the spatial direction 1 the result

\[
p^1 = \sum_{A=1}^{N} \frac{q_A + 1}{\Delta_A} \ln H_A(\xi),
\]

(15)

identifying it to a direction transverse to all branes. The Eqs. (12), (13) and (15) are solutions provided the intersection rules [19]

\[
\bar{q} + 1 = \frac{(q_A + 1)(q_B + 1)}{D - 2} - \frac{1}{2} \varepsilon_A a_A \varepsilon_B a_B
\]

(16)
are satisfied.

The intersection rules Eq. (16) are neatly and elegantly encoded in the group structure \[18\]. They can indeed be expressed as an orthogonality condition between the real positive roots of \(G^{++}\) (and \(G^{+++}\)) for all branes present in the configuration \[18\] namely

\[
\alpha_A \cdot \alpha_B = 0 \quad \text{if } A \neq B = 1 \ldots N \tag{17}
\]

When \(G\) is not simply laced there is an additional condition in order to have a solution of \(S_{G^{++}}\), namely for each pair \(\alpha_A, \alpha_B\) with \(A \neq B\) one must have

\[
\alpha_A + \alpha_B \neq \text{root}. \tag{18}
\]

The conditions Eqs. (17) and (18) are in fact the input that permits the derivation of the exact solutions by allowing a reduction of \(S_{G^{++}}\) to quadratic terms. Furthermore there is a one-to-one correspondence between the exact solutions of \(S_{G^{++}}\) and the space-time intersecting extremal brane solutions of the corresponding maximally oxidised theory. Indeed, the configurations satisfying Eq. (17) and not Eq. (18) correspond in the maximally oxidised theory to configurations which are not solutions of the equations of motion because of the presence of Chern-Simons terms in the space-time action \[18\].

In the particular case of \(G^{++} = E_8^{++}\), it is well-known that the Weyl reflection generated by the root \(\alpha_{11}\) has an interpretation in terms of type IIA string duality. It correspond to a double T-duality in the direction 9 and 10 followed by an exchange of the two radii \[20, 21, 6\]. Furthermore the change of signatures which occur as a consequence of the non-commutativity of the temporal involution and the Weyl reflections \[15, 16\] are in agreement with the exotic phases of M-theory discussed in \[22, 23\].

The action of Weyl reflections generated by simple roots not belonging to the gravity line on the exact extremal brane solutions has been studied for all \(G^{+++}\)-theory \[10\]. The existence of Weyl orbits of extremal brane solutions similar to the U-duality orbits existing in M-theory has been discovered. This fact strongly suggests a general group-theoretical origin of ‘dualities’ for all \(G^{+++}\)-theories transcending string theories and supersymmetry. Furthermore, exotic phases of all the M-theories (all the \(G^{++}\) theories) related by ‘duality’ Weyl transformations to the conventional phase\(^6\) characterised by a signature \((1, D - 2)\) have been uncovered and classified \[24\].

We exemplify the existence of U-duality-like Weyl orbits of extremal branes in the case \(G_7^{+++} = E_7^{+++}\) whose corresponding maximally oxidised theory is gravity coupled to a 4- and a 2- form field strength with one dilaton in 9 space-time dimensions \[1\]. The Dynkin diagram of \(E_7^{+++}\) is depicted in Fig.1, which exhibits the two simple electric roots \(\alpha_{10}\) and \(\alpha_9\) corresponding respectively to the step operators \(R^7_{89}\) and \(R^6\) which couple to the electric potentials \(A_{789}\) and \(A_9\).

We take as input the electric extremal 2-brane \(e_{(8,9)}\) in the directions \((8, 9)\) associated with the 4-form field strength whose corresponding potential\(^7\) is \(A_{289}\) and submit it to

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\(^6\)The other orbits have been discussed in \[25\].

\(^7\)We recall that in \(G^{++}\) the index 2 is identified to the time coordinate.
the non trivial Weyl reflection $W_{10}$ associated with the electric root $\alpha_{10}$ of Fig.1. We display below, both for $e_{(8,9)}$ and its transform, the vielbein components $p^a \equiv -h^a_a$ with $a = 2 \ldots 9$ and the the dilaton value $\phi$, of the brane solution Eqs.(12) and (13) as a nine-dimensional vector where the last component is the dilaton. We also indicate the transform of the step operator $R^{289}$ under the Weyl transformation. We obtain

$$(-4, 3, 3, 3, 3, -4, -4; 2\sqrt{7}) \frac{\ln H(\xi)}{14} e_{(8,9)} R^{289}$$

$$(-7, 0, 0, 0, 7, 0, 0; 0) \frac{\ln H(\xi)}{14} \downarrow W_{10}$$

$$(-4, 3, 3, -4, -4, 3, 3; 2\sqrt{7}) \frac{\ln H(\xi)}{14} e_{(5,6), R^{256}}$$

$$(-1, 6, 6, -1, -1, -1, -1, -1; 4\sqrt{7}) \frac{\ln H(\xi)}{14} m_{(5,6,7,8,9), R^{256789}}$$

The 2-brane transforms into a KK-wave in the direction 7 characterised by a non-zero $K^2_7$ \[10\]. This is reminiscent of a double T-duality in M-theory.

We now move the electric brane through Weyl reflections associated with roots of the gravity line to $e_{(5,9)}$ and submit it to the Weyl reflection $W_{10}$. We now find that the brane $e_{(5,9)}$ is invariant but moving it to the position $e_{(5,6)}$, we get

$$(-4, 3, 3, -4, -4, 3, 3; 2\sqrt{7}) \frac{\ln H(\xi)}{14} e_{(5,6), R^{256}}$$

$$(-1, 6, 6, -1, -1, -1, -1, -1; 4\sqrt{7}) \frac{\ln H(\xi)}{14} m_{(5,6,7,8,9), R^{256789}}$$

This is a magnetic 5-brane in the directions $(5,6,7,8,9)$ associated to the 2-form field strength! It is expressed in terms of its dual potential $A_{256789}$. Submit instead $e_{(5,9)}$ to the Weyl reflection $W_9$ associated with the electric root $\alpha_9$ of Fig.1. The 2-brane $e_{(5,9)}$ is again invariant, but moving it to the position $e_{(5,6)}$, we now get

$$(-4, 3, 3, -4, -4, 3, 3; 2\sqrt{7}) \frac{\ln H(\xi)}{14} e_{(5,6), R^{256}}$$

$$(-3, 4, 4, -3, -3, 4, 4, -3; -2\sqrt{7}) \frac{\ln H(\xi)}{14} m_{(5,6,9), R^{2569}}$$

This is a magnetic 3-brane in the directions $(5,6,9)$ associated to the 4-form field strength, expressed in terms of its dual potential $A_{2569}$.

Finally, let us submit the magnetic 5-brane $m_{(5,6,7,8,9)}$ obtained in Eq.(22) to the Weyl reflection $W_9$. One obtains

$$(-1, 6, 6, -1, -1, -1, -1, -1; 4\sqrt{7}) \frac{\ln H(\xi)}{14} m_{(5,6,7,8,9), R^{256789}}$$

$$(-3, 4, 4, -3, 3, 4, 4, -3; -2\sqrt{7}) \frac{\ln H(\xi)}{14} m_{(5,6,9), R^{2569}}$$

$$(-1, 3, 7, 0, 0, 0, 0, -7; 0) \frac{\ln H(\xi)}{14} m_{(1,3,4,9), R^{256789}}$$

$$(-4, 3, 3, 3, 3, -4, -4, -4; 2\sqrt{7}) \frac{\ln H(\xi)}{14} m_{(1,3,4,9), R^{256789}}$$
Eq. (24) describes, as in M-theory, a purely gravitational configuration, namely a KK-monopole with transverse directions (1,3,4) and Taub-NUT direction 9 in terms of a dual gravity tensor $h_{256789,9}^{10}$.

The approach based on Kac-Moody algebras constitutes certainly a very-exciting and innovative attempt to understand gravitational theories encompassing string theories which could lead to a completely new formulation of gravitational interactions where the structure of space-time is hidden somewhere in these huge algebras [14, 17, 26] or even huger ones [27].

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**References**


