Effects of a mixed vector-scalar kink-like potential for spinless particles in two-dimensional spacetime

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Abstract

The intrinsically relativistic problem of spinless particles subject to a general mixing of vector and scalar kink-like potentials ($\sim \tanh \gamma x$) is investigated. The problem is mapped into the exactly solvable Surm-Liouville problem with the Rosen-Morse potential and exact bounded solutions for particles and antiparticles are found. The behaviour of the spectrum is discussed in some detail. An apparent paradox concerning the uncertainty principle is solved by recurring to the concept of effective Compton wavelength.
1 Introduction

There has been a continuous interest for solving the Klein-Gordon (KG) equation in the four-dimensional space-time as well as in lower dimensions for a variety of potentials [1]-[13]. A few recent works has been devoted to the investigation of the solutions of the KG equation by assuming that the vector potential is equal to the scalar potential [6]-[9] whereas other works take a more general mixing [1]-[2],[10]-[13]. Although the KG equation can give relativistic corrections to the nonrelativistic quantum mechanics, in some circumstances it can present solutions not found in a nonrelativistic scheme. Undoubtedly such circumstances reveal to be a powerful tool to obtain a deeper insight about the nature of the KG equation and its solutions.

The Dirac equation in the background of the kink configuration of the $\phi^4$ model $(\tanh \gamma x)$ [14] is of interest in quantum field theory where topological classical backgrounds are responsible for inducing a fractional fermion number on the vacuum. Models of these kinds, known as kink models are obtained in quantum field theory as the continuum limit of linear polymer models [15]-[17]. In a recent paper the complete set of bound states of fermions in the presence of this sort of background has been addressed by considering a pseudoscalar coupling [18]. A peculiar feature of the kink-like potential is the absence of bounded solutions in a nonrelativistic theory because it gives rise to an ubiquitous repulsive potential. The spectrum of this problem was found analytically due to the fact that, apart from solutions corresponding to $|E| = mc^2$, the problem is reducible to the finite set of solutions of the nonrelativistic modified Pöschl-Teller potential.

In the present work the problem of a spinless particle in the background of a kink-like potential is considered with a general mixing of vector and scalar Lorentz structures. Apart from the intrinsic interest as new solutions of a fundamental equation in physics, the bound-state solutions of this system is related to a number of applications ranging from ferroelectric domain walls in solids, magnetic chains and Josephson junctions [19]. The problem is mapped into an exactly solvable Sturm-Liouville problem of a Schrödinger-like equation with an effective Rosen-Morse potential [20]-[24]. The whole spectrum of this relativistic problem is found analytically, if the particle is massless or not. Nevertheless, bounded solutions do exist only if the scalar coupling is stronger than the vector coupling. A remarkable feature of this problem is the possibility of trapping a particle with an uncertainty in the position that can shrink to zero for arbitrarily large values of the potential parameters. Due to the repulsive nature of the kink-like potential those solutions do not manifest in a nonrelativistic approach even though one can find $|E| \simeq mc^2$. Therefore, they are intrinsically relativistic solutions of the Klein-Gordon equation.
In the presence of vector and scalar potentials the 1+1 dimensional time-independent KG equation for a particle of rest mass $m$ reads

$$-\hbar^2 c^2 \frac{d^2\phi}{dx^2} + \left(mc^2 + V_s\right)^2 \phi = (E - V_v)^2 \phi$$  \hspace{1cm} (1)

where $E$ is the energy of the particle, $c$ is the velocity of light and $\hbar$ is the Planck constant. The vector and scalar potentials are given by $V_v$ and $V_s$, respectively. The subscripts for the terms of potential denote their properties under a Lorentz transformation: $v$ for the time component of the 2-vector potential and $s$ for the scalar term. Note that $\phi$ remains invariant under the simultaneous transformations $E \rightarrow -E$ and $V_v \rightarrow -V_v$. Furthermore, for $V_v = 0$, the case of a pure scalar potential, the negative- and positive-energy levels are disposed symmetrically about $E = 0$. It is remarkable that the KG equation with a scalar potential, or a vector potential contaminated with some scalar coupling, is not invariant under the simultaneous changes $V \rightarrow V + \text{const.}$ and $E \rightarrow E + \text{const.}$, this is so because only the vector potential couples to the positive-energies in the same way it couples to the negative-ones, whereas the scalar potential couples to the mass of the particle. Therefore, if there is any scalar coupling the the energy itself has physical significance and not just the energy difference.

The KG equation can also be written as

$$-\hbar^2 \frac{\phi''}{2} + \left(\frac{V_s^2 - V_v^2}{2c^2} + mV_s + \frac{E}{c^2} V_v\right) \phi = \frac{E^2 - m^2 c^4}{2c^2} \phi$$ \hspace{1cm} (2)

From this one can see that for potentials which tend to $\pm \infty$ as $|x| \rightarrow \infty$ it follows that the effective potential tends to $(V_s^2 - V_v^2) / (2mc^2)$, so that the KG equation furnishes a purely discrete (continuum) spectrum for $|V_s| > |V_v|$ ($|V_s| < |V_v|$). On the other hand, if the potentials remains finite as $|x| \rightarrow \infty$ the continuum spectrum is omnipresent but the necessary conditions for the existence of a discrete spectrum is not an easy task for general functional forms. In the nonrelativistic approximation (potential energies small compared to $mc^2$ and $E \simeq mc^2$) Eq. (2) becomes

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_v + V_s\right) \phi = \left(E - mc^2\right) \phi$$ \hspace{1cm} (3)

so that $\phi$ obeys the Schrödinger equation with binding energy equal to $E - mc^2$ without distinguishing the contributions of vector and scalar potentials.

It is well known that a confining potential in the nonrelativistic approach is not confining in the relativistic approach when it is considered as a Lorentz vector. It is
surprising that relativistic confining potentials may result in nonconfinement in the nonrelativistic approach. This last phenomenon is a consequence of the fact that vector and scalar potentials couple differently in the KG equation whereas there is no such distinction among them in the Schrödinger equation. This observation permit us to conclude that even a “repulsive” potential can be a confining potential. For both cases $V_v = V_s$ and $V_v = -V_s$, the KG equation reduces to Schrödinger-like equations with effective potentials with the same functional dependence as $V_s$. The case $V_v = -V_s$ can present bounded solutions in the relativistic approach, although it reduces to the free-particle problem in the nonrelativistic limit. The attractive vector potential for a particle is, of course, repulsive for its corresponding antiparticle, and vice versa. However, the attractive (repulsive) scalar potential for particles is also attractive (repulsive) for antiparticles. For $V_v = V_s$ and an attractive vector potential for particles, the scalar potential is counterbalanced by the vector potential for antiparticles as long as the scalar potential is attractive and the vector potential is repulsive. As a consequence there is no bounded solution for antiparticles. For $V_v = -V_s$ and a repulsive vector potential for particles, the scalar and the vector potentials are attractive for antiparticles but their effects are counterbalanced for particles. Thus, recurring to this simple standpoint one can anticipate in the mind that there is no bound-state solution for particles in this last case of mixing.

2 The mixed vector-scalar kink-like potential

Now let us focus our attention on scalar and vector potentials in the form

$$V_v = \hbar c \gamma g_v \tanh \gamma x, \quad V_s = \hbar c \gamma g_s \tanh \gamma x$$  \hspace{1cm} (4)

where the skew parameter, $\gamma$, and the dimensionless coupling constants, $g_v$ and $g_s$, are real numbers. The potentials are invariant under the change $\gamma \to -\gamma$ so that the results can depend only on $|\gamma|$. In this case the Sturm-Liouville problem corresponding to Eq. (2) becomes

$$H_{\text{eff}} \phi = -\frac{\hbar^2}{2m_{\text{eff}}} \phi'' + V_{\text{eff}} \phi = E_{\text{eff}} \phi$$  \hspace{1cm} (5)

where one can recognize the effective potential as the exactly solvable Rosen-Morse potential \cite{20}-\cite{24}

$$V_{\text{eff}} = -V_1 \sech^2 \gamma x + V_2 \tanh \gamma x$$  \hspace{1cm} (6)
The Rosen-Morse potential is a binding potential only if \( V_1 > 0 \) and \(|V_2| < 2V_1\) because only in this circumstance it presents a well structure with the possible discrete effective eigenenergies into the range

\[
-\left(V_1 + \frac{V_2^2}{4V_1}\right) < E_{\text{eff}} < -|V_2|
\]  

(9)

whereas \( E_{\text{eff}} > -|V_2| \) corresponds to the continuous part of the spectrum. Thus, in view of the interest in the bound-state solutions it is convenient to rewrite the coupling constants in terms of the variable \( \xi \) defined into the interval \((-1, 1)\) as

\[
g_s = g, \quad g_v = g \sin \left(\frac{\pi \xi}{2}\right)
\]  

(10)

in such a way that

\[
V_1 = \frac{\hbar^2 \gamma^2 g^2}{2m_{\text{eff}}} \cos^2 \left(\frac{\pi \xi}{2}\right), \quad V_2 = \frac{\hbar \gamma g}{m_{\text{eff}} c} \left[m c^2 + E \sin \left(\frac{\pi \xi}{2}\right)\right]
\]  

(11)

Normalizable KG wave functions, corresponding to bound-state solutions, are subject to the boundary condition \( \phi(\pm \infty) = 0 \) and the effective eigenenergy is given by (see Ref. [23])

\[
E_{\text{eff}} = \frac{E^2 - m_{\text{eff}} c^4}{2m_{\text{eff}} c^2} = - \left[\frac{\hbar^2 \gamma^2 a_n^2}{2m_{\text{eff}}} + \frac{m_{\text{eff}} V_2^2}{2\hbar^2 \gamma^2 a_n^2}\right]
\]  

\[
= - \frac{1}{2m_{\text{eff}} c^2} \left\{ \hbar^2 c^2 \gamma^2 a_n^2 + g^2 \left[m c^2 + E \sin \left(\frac{\pi \xi}{2}\right)\right]^2\right\}
\]  

(12)

with the effective mass, \( m_{\text{eff}} \), defined as

\[
m_{\text{eff}} = \sqrt{m^2 + \left[\frac{\hbar \gamma g}{c} \cos \left(\frac{\pi \xi}{2}\right)\right]^2}
\]  

(13)
In addition, \( a_n = s - n \) and
\[
s = \frac{1}{2} \left( -1 + \sqrt{1 + \frac{8m_{\text{eff}}V_1}{h^2\gamma^2}} \right) = \frac{1}{2} \left( -1 + \sqrt{1 + \left[ 2g \cos \left( \frac{\pi \xi}{2} \right) \right]^2} \right)
\] (14)

The quantum number \( n \) satisfies the constraint equation
\[
n = 0, 1, 2, \ldots \leq s - \sqrt{\frac{m_{\text{eff}}|V_2|}{h^2\gamma^2}} = s - \sqrt{\frac{|gmc^2 + E \sin \left( \frac{\pi \xi}{2} \right)|}{\hbar c |\gamma|}}
\] (15)

From Eqs. (12) and (13) one obtains the second-degree algebraic equation for the KG energies:
\[
\left[ a_n^2 + g^2 \sin^2 \left( \frac{\pi \xi}{2} \right) \right] E^2 + 2g^2mc^2 \sin \left( \frac{\pi \xi}{2} \right) E
+ m^2c^4 \left( g^2 - a_n^2 \right) + h^2c^2\gamma^2a_n^2 \left[ a_n^2 - g^2 \cos^2 \left( \frac{\pi \xi}{2} \right) \right] = 0
\] (16)

whose solutions are
\[
E = -\frac{mc^2g^2 \sin \left( \frac{\pi \xi}{2} \right)}{a_n^2 + g^2 \sin^2 \left( \frac{\pi \xi}{2} \right)}
\]
\[
\pm \frac{a_n c \sqrt{m^2c^2 \left[ a_n^2 - g^2 \cos^2 \left( \frac{\pi \xi}{2} \right) \right] + h^2\gamma^2 \left[ \frac{g^4}{4} \sin^2 (\pi \xi) + a_n^2 g^2 \cos (\pi \xi) - a_n^4 \right]}}{a_n^2 + g^2 \sin^2 \left( \frac{\pi \xi}{2} \right)}
\] (17)

These solutions present an intricate dependence on the parameters of the potential and per se do not tell the whole story because the KG energies have also to be in tune with the restrictions imposed by (15). Notice that the KG energies depend only on the absolute values of the skew parameter and of the coupling constant, and that the energy levels are symmetric about \( E = 0 \) for \( \xi = 0 \) (the case of a pure scalar coupling), as expected. The same happens for massless particles, independently of \( \xi \). This last symmetry is due to the fact that the effective eigenenergies of the Rosen-Morse potential as well as the maximum quantum number depend on the absolute value of \( V_2 \) and for \( m = 0 \) this dependence transmutes into the absolute value of \( E \). Anyhow, the energy levels corresponding to the bounded solutions are into the
range $|E| < m_{\text{eff}}^2$. Thus, one can see that the scalar coupling enlarges the bound-state gap and that the particle (antiparticle) energy levels enter from the upper (lower) continuum into the spectral gap. It is also clear that the capacity of the kink-like potential to hold bounded solutions increases as $|g|$ and $|\gamma|$ increase. The KG energies are plotted in Figs. 1, 2 and 3 as a function of $\xi$, $g$ and $\gamma$, respectively. The parameters were chosen for furnishing two bounded solutions, at the most. These figures show that the energy levels are into the range $|E| < m_{\text{eff}}^2$ and that, in general, both particle and antiparticle levels are members of the spectrum. They also show that neither there is crossing of levels nor the energy levels for particles (antiparticles) join the negative (positive) continuum. These facts imply that there is no channel of spontaneous particle-antiparticle creation so that the single-particle interpretation of the KG equation is ensured. Fig. 1 shows that both particle and antiparticle levels show their face for a pure scalar coupling ($\xi = 0$) and that they are symmetrically disposed about $E = 0$. Nevertheless, this symmetry is broken for $\xi \neq 0$ and there can be a region where the number of particle levels is different from the antiparticle levels. Furthermore, as $\xi \rightarrow \pm 1$ the energy levels tend to disappear one after another. Figs. 2 and 3 show that there are minimum values for both $|g|$ and $|\gamma|$ for the formation of a bound-state solution.

The normalized KG wave function can be written as (see Ref. [23]):

$$\phi_n = N_n (1 - z)^{(a_n - b_n)/2} (1 + z)^{(a_n + b_n)/2} P_n^{(a_n - b_n, a_n + b_n)} (z)$$  \hspace{1cm} (18)

where $z = \tanh \gamma x$ and $P_n^{(\alpha, \beta)}$ is the Jacobi polynomial, a polynomial of degree $n$. Furthermore,

$$b_n = \frac{m_{\text{eff}} V_1}{\hbar^2 \gamma^2 (s - n)} = \frac{g^2}{2a_n} \cos^2 \left( \frac{\pi \xi}{2} \right)$$  \hspace{1cm} (19)

and the normalization constant is given by

$$N_n = 2^{-a_n} \sqrt{\frac{a_n^2 - b_n^2}{a_n \Gamma (2a_n + n + 1) \Gamma (n + 1)}} \frac{\Gamma (a_n + b_n + n + 1) \Gamma (n + 1)}{\Gamma (a_n - b_n + n + 1) \Gamma (a_n + b_n + n + 1)}$$  \hspace{1cm} (20)

Although it may seem strange, the KG wave functions are characterized just by $n$ so that when both particle and antiparticle levels are present in the spectrum a pair of bound-state solutions can share the same wave function. Indeed, this is not a riddle because the KG equation as expressed by Eq. (1) can not be seen as an eigenvalue problem in general. However, $\phi$ is an eigenfunction of the operator $H_{\text{eff}}$ defined in (5) with $E_{\text{eff}}$ as the corresponding eigenvalue. Therefore, a pair of KG energy levels corresponding to particle and antiparticle levels can share the same wave function.
if they possess the same effective eigenenergy, i.e., the same quantum number. It is noteworthy that the width of the position probability density, $|\phi|^2$, decreases as $|\gamma|$ or $|g|$ increases. As such it promises that the uncertainty in the position can shrink without limit. It seems that the uncertainty principle dies away provided such a principle implies that it is impossible to localize a particle into a region of space less than half of its Compton wavelength (see, for example, [25]). This apparent contradiction can be remedied by recurring to the concept of effective Compton wavelength defined as $\lambda_{\text{eff}} = \hbar / (m_{\text{eff}} c)$. Hence, the minimum uncertainty consonant with the uncertainty principle is given by $\lambda_{\text{eff}} / 2$. It means that the localization of a particle under the influence of the kink-like potential does not require any minimum value in order to ensure the single-particle interpretation of the KG equation, even if the trapped particle is massless. As $|\gamma|$ or $|g|$ increases the binding potential contributes to increase the effective mass of the particle in such a way that there is no energy available to produce particle-antiparticle pairs. The KG wave functions are displayed in Fig. 4. With the chosen parameters ($\hbar = c = m = \gamma = 1$, $g = 5$ and $\xi = 0$), a numerical calculation of the uncertainty in the position of the particle for the ground-state solution furnishes 0.475 whereas $\lambda_{\text{eff}} = 0.196$.

3 Conclusions

For short, the bound-state solutions of the KG equation for particles embedded in a mixed vector-scalar potential with the kink-like configuration of the $\phi^4$ model has been solved analytically. A half-half mixing produces no bound-state solution and in fact bounded solutions only can occur if the scalar coupling is stronger than the vector coupling. Although the Schrödinger equation for this sort of potential does not present any bounded solution at all, the KG equation can present a rich spectrum which might be useful for a better understanding of the localization of spinless particles in the regime of strong coupling.

Acknowledgments

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References


Figure 1: The KG energies for the kink-like potential as a function of $\xi$. The full thick line stands for for $n = 0$, the full thin line for $n = 1$ and the dashed line for $\pm m_{\text{eff}}$ ($\hbar = c = m = \gamma = 1$ and $g = 5$).
Figure 2: The same as in Fig. 1 as a function of $g$ for $\xi = 0$. 
Figure 3: The same as in Fig. 1 as a function of $\gamma$ for $\xi = 0$. 
Figure 4: The KG wave functions as a function of the $x$ for $\xi = 0$. The thick line stands for $n = 0$ and the thin line for $n = 1$ ($\hbar = c = m = \gamma = 1$ and $g = 5$).