# GENERALIZED TELEPORTATION PROTOCOL

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A generalized teleportation protocol (GTP) for $N$ qubits is presented, where the teleportation channels are non-maximally entangled and all the free parameters of the protocol are considered: Alice's measurement basis, her sets of acceptable results, and Bob's unitary operations. The full range of Fidelity ($F$) of the teleported state and the Probability of Success ($P_{suc}$) to obtain a given fidelity are achieved by changing these free parameters. A channel efficiency bound is found, where one can determine how to divide it between $F$ and $P_{suc}$. A one qubit formulation is presented and then expanded to $N$ qubits. A proposed experimental setup that implements the GTP is given using linear optics.

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The concept of entanglement is central in quantum information processing. One major breakthrough was obtained by Bennett *et al.* [1], who created the quantum teleportation protocol. Right after its proposal, Bouwmeester *et al.* [2] and Boschi *et al.* [3] experimentally implemented the teleportation protocol. Interesting extensions were subsequently proposed, specially those regarding the teleportation of more than one qubit [4]. All the previous proposals assume, nevertheless, that the quantum channels used to teleport the qubits are noiseless maximally entangled states. But in a realistic scenario noisy effects and decoherence decrease the entanglement of the channel. In this scenario, Agrawal and Pati [5] constructed a protocol where it is possible to achieve unity fidelity teleportation of one qubit using directly non-maximally entangled channels. The price to pay is that the protocol is no more deterministic.

This paper generalizes Agrawal and Pati [5] work and expand it to teleport N qubits using directly N non-maximally entangled channels. The two previous proposals, namely the standard protocol [1] and the probabilistic protocol [5] are generalized and written in one single formalism. In this generalization one can enhance $P_{suc}$, on expense of $F$, by using a different measurement basis. The total channel efficiency is bounded by the entanglement of the two qubit channels, but one can decide how to "divide" this bound between $F$ and $P_{suc}$ to obtain this fidelity, according to the system requirements.

In general, Alice may wish to teleport $N$ qubits. A $N$ qubit state has $2^N$ arbitrary unknown complex amplitudes. Let $\alpha_i$, where $i = 1, ..., 2^N$, represent these amplitudes. Alice has one channel per qubit to be teleported. We assume that each channel is composed of two entangled qubits, one with Alice and one with Bob. The channels need not be maximally entangled and their entanglement are parametrized by $n_k$, where $k = 1, ..., N$. The protocol involves measurements by Alice, meaning that she uses a specific measurement basis to project her $2N$ qubits ($N$ qubits she wishes to teleport plus $N$ qubits from the $N$ two qubit channels). The measurement basis is characterized by the parameters $m_s$, where $s = 1, ..., N$. The measurement yields different possible results $|R_j\rangle$, each with probability $P_j$, where $j = 1, ..., 2^{2N}$, due to the fact that the measurement is performed jointly on the qubits to be teleported and the channel qubits. Alice decides, beforehand, which results are acceptable, i.e. the protocol has succeeded, and which results are not, meaning the protocol has failed. The acceptable results, $|R_l\rangle$, where $\{l\} \subseteq \{j\}$, she transmits to Bob via a classical channel. In terms of the measurement basis, the total initial state can be written as

$$|\psi\rangle = \underbrace{\sum_{l} \beta_l |R_l^A\rangle|\psi_l^B\rangle}_{\text{Success}} + \underbrace{\sum_{j\neq l} \beta_j |R_j^A\rangle|\psi_j^B\rangle}_{\text{Failure}}, \qquad (1)$$

where $P_j = |\beta_j|^2$. In this scenario, Alice has $P_{suc} = \sum_l P_l$ probability of success, meaning one of the acceptable results has been obtained. After measurement, the initial state collapses to one of the $|R_j^A\rangle|\psi^B\rangle$ states and Alice transmits to Bob her outcome, i.e. the value of $j$, conditioned on the restriction $j \in \{l\}$. Bob now performs a unitary transformation $\mathbf{U}_l$ on his $N$ qubits, which can be different for each one of Alice's measurement results. We assume Bob's unitary operations are local in his qubits: $\mathbf{U}_l = U_1 \otimes \cdots \otimes U_N$. A general unitary transformation on $N$ qubits can be represented by $4N$ parameters (four parameters for each local unitary operation) and Bob must decide beforehand what operations to do on his qubits conditioned on the information received from Alice. However, for each result Bob receives from Alice, he can choose among the $4N$ parameters. (These parameters are part of the protocol and cannot be changed during the teleportation.) After these transformations, Bob obtains the final state $|\phi_l^B\rangle$, with the accompanying

fidelity $F_l = |\langle \phi^A | \phi_l^B \rangle|^2$.

The quantities of interest here, i.e. probability and fidelity, are dependent on $|\alpha_i|^2$ and $|\alpha_i \alpha_j|^2$. However, we wish to get $\alpha_i$-independent results for the protocol. Since the input state is arbitrary and in general unknown, this is achieved by averaging over these quantities with the appropriate distribution function. This is done by using spherical coordinates in a $2^N$-dimensional real space, where $N$ is the number of qubits to be teleported. We thus find that $\langle |\alpha_i|^2 \rangle = 1/2^N$ and $\langle |\alpha_i \alpha_j|^2 \rangle = (2^{\delta_{ij} - N})/(1+2^N)$, which is all that is required in the following calculations. Here, $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

Alice's probabilities $P_j$ may depend on the particular state to be teleported ($\alpha_i$), thus we use the average probability $\langle P_j \rangle$. Bob gets the fidelities $F_l$ with probability $P_l$. Averaging over many implementations of the protocol we obtain the protocol efficiency $C^{pro}$, which can also be viewed from a different perspective by defining the protocol fidelity $F^{pro}$. The channel efficiency, $C^{channel}$, is defined as the maximal protocol efficiency, where the maximization is done over all the free parameters (which exclude $\{n_k\}$).

$$C^{pro} = \sum_l \langle P_l F_l \rangle. \tag{2}$$

$$F^{pro} = \frac{C^{pro}}{\langle P_{suc} \rangle} = \frac{\langle \sum_l P_l F_l \rangle}{\langle \sum_l P_l \rangle}. \tag{3}$$

$$C^{channel}(\{n_k\}) = \max_{m_s,\,\{l\},\,\mathbf{U}_l} C^{pro}. \tag{4}$$

The protocol efficiency can be interpreted as the average qubit transmission rate for a specific protocol choice and is the product of the probability of its success and its fidelity. For the specific case where Alice accepts all results: $P_{suc} = 1$ and $C^{pro} = F^{pro}$. Eq. (3) shows that Alice and Bob have the freedom to modify $F$ and $P_{suc}$ while maintaining the same protocol efficiency. For a given $C^{pro}$, they can get higher (lower) fidelity lowering (increasing) $P_{suc}$. The channel efficiency gives the maximal qubit teleportation rate for a given channel.

For the one qubit case, a quantum channel which is not maximally entangled (we consider pure states only) is given as [5] $|\Phi_n^+\rangle = (1/\sqrt{1+|n|^2})(|00\rangle + n|11\rangle)$. Here $n$ is a complex number in which $0 \le |n| \le 1$. The concurrence for this state, a well known entanglement monotone [6], is $c(n) = 2|n|/(1+|n|^2)$, which is a monotonically increasing function of $|n|$. (Throughout the paper when we talk about the degree of entanglement of a state we are referring to its concurrence.) Alice wishes to teleport the qubit $|\phi^A\rangle = \alpha_1|0\rangle + \alpha_2|1\rangle$, where $|\alpha_1|^2 + |\alpha_2|^2 = 1$. Alice's arbitrary Bell-measurement basis are $\{|R_j\rangle\} = \{|\Phi_m^+\rangle, |\Phi_m^-\rangle, |\Psi_m^+\rangle, |\Psi_m^-\rangle\}$, where $|\Phi_m^+\rangle = M(|00\rangle + m|11\rangle)$, $|\Phi_m^-\rangle = M(m^*|00\rangle - |11\rangle)$, $|\Psi_m^+\rangle = M(|01\rangle + m|10\rangle)$, and $|\Psi_m^-\rangle = M(m^*|01\rangle - |10\rangle)$. Here $M = 1/\sqrt{1+|m|^2}$ and $m^*$ is the complex conjugate of $m$.

The initial three-qubit state (Alice's qubit and the channel qubits) can be projected onto Alice's two qubits arbitrary Bell-basis, with the appropriate probabilities. Alice transmits the result of her measurement via a classical channel to Bob, who, whereupon, performs a unitary transformation on his qubit. Bob has 16 free parameters, four for each of Alice's measurement result. We restrict ourselves, however, to only one free parameter ($\theta_j$) for each result. The unitary operations are $\{|R_j\rangle\} \to \exp(i\sigma_z\theta_j)O_j$, where $\{O_j\} = \{I, \sigma_z, \sigma_x, \sigma_z\sigma_x\}$. $I$ is the identity and $\sigma$ are the usual Pauli matrices.

Implementing the averaging procedure described above, the averaged probabilities and fidelities are found to be

$$\langle P_{\Phi_m^+} \rangle = \langle P_{\Psi_m^-} \rangle = \frac{1+|nm|^2}{2(1+|n|^2)(1+|m|^2)}, \tag{5}$$

$$\langle P_{\Phi_m^-} \rangle = \langle P_{\Psi_m^+} \rangle = \frac{|n|^2+|m|^2}{2(1+|n|^2)(1+|m|^2)}. \tag{6}$$

$$\langle F_{\Phi_m^+,\Psi_m^-} P_{\Phi_m^+,\Psi_m^-} \rangle = \frac{1+|nm|^2+|mn|\cos(\xi_{\Phi^+,\Psi^-})}{3(1+|n|^2)(1+|m|^2)}, \tag{7}$$

$$\langle F_{\Phi_m^-,\Psi_m^+} P_{\Phi_m^-,\Psi_m^+} \rangle = \frac{|n|^2+|m|^2+|mn|\cos(\xi_{\Phi^-,\Psi^+})}{3(1+|n|^2)(1+|m|^2)}, \tag{8}$$

where, using that $n = |n|e^{i\theta_n}$ and $m = |m|e^{i\theta_m}$, we have $\xi_{\Phi^\pm} = \theta_n - \theta_m - 2\theta_{\Phi^\pm}$ and $\xi_{\Psi^\pm} = \theta_n + \theta_m + 2\theta_{\Psi^\pm}$.

For the special case where Alice accepts all possible results, i.e. the protocol always succeed, $P_{suc} = 1$ and the protocol efficiency is:

$$C^{pro} = \frac{2}{3}\left(1 + \frac{|nm|\sum_{j=\Phi^\pm,\Psi^\pm}\cos(\xi_j)}{2(1+|n|^2)(1+|m|^2)}\right). \tag{9}$$

Looking at Eq. (9) we see that $C^{pro}$ is maximum if $\xi_j = 2\pi q_j$, $q_j$ integer. This can always be achieved if Bob properly adjusts his four free parameters $\theta_j$, which depend on the channel and measuring basis entanglement ($n_k$ and $m_s$, respectively, assumed to be known by Bob). This is equivalent to working with real $n$ and $m$, a scenario which, unless stated otherwise, is assumed throughout the rest of this paper. Therefore, Eq. (9) reads $C^{pro} = (2/3)(1 + c(n)c(m)/2)$, where $c$ is the concurrence. Note that Eq. (9) is invariant if we interchange the parameters $m$ and $n$. This remarkable result shows that $C^{pro}$ *is the same if we exchange the channel entanglement and the measuring basis entanglement*. We can easily see that the channel efficiency, i.e. maximal protocol efficiency when $m$ and $\xi_j$ are the free parameters, is achieved for all $n$ at $m = 1$ and all $\xi_j = 0$.

We can consider the case of a dephased channel, where the quantum state describing it accumulates a relative phase. In the GTP notation, this amounts to $n = e^{i\theta_n}$. Assuming the dephasing rate is known, this obstacle can be overcome by an appropriate unitary operation performed by Bob. Let us assume, for example, $m = 1$. We see that by performing unitary transformations such that

$\theta_{\Phi^\pm} = \theta_n/2$ and $\theta_{\Psi^\pm} = -\theta_n/2$, we eliminate the dephasing and it results in a unity fidelity teleportation protocol (no averaging required). This result shows that *only the entanglement of the channel* is important for the teleportation protocol to succeed and *not which entangled state is used.*

In the standard protocol [1], Alice uses the standard Bell-basis (maximally entangled states) to implement her joint measurements. In the GTP formulation, this corresponds to $m = 1$ and all $\xi_j = 0$. This results in $P_{suc} = 1$, $C^{std} = F^{std} = (2/3)(1+n/(1+n^2))$. In the probabilistic quantum teleportation (PQT) protocol [5] Alice uses a special measurement basis, which in the GTP formalism corresponds to real $m = n$ and all $\xi_j = 0$. Also $\{|R_l\rangle\} = \{|\Phi_n^-\rangle, |\Psi_n^+\rangle\}$ which results in $P_{suc} = 2n^2/(1+n^2)^2$, $F^{PQT} = 1$, and $C^{PQT} = 2n^2/(1+n^2)^2$.

As seen from these examples, we can create a tradeoff between the fidelity of the protocol and the probability of its success. (We assume all $\xi_j = 0$). When Alice decides not to accept all possible results, i.e. not to transmit all the results to Bob, the protocol will have less than unity $P_{suc}$. However, as shown in the probabilistic quantum protocol, we gain unit fidelity when the teleportation does succeed under a special circumstance ($m = n$). It is noteworthy to consider the perturbed case of this protocol, i.e. $m \simeq n$. This requires averaging and results in less than unit fidelity. Fig. 1 show the perturbation in protocol fidelity $F^{PQT}$ (Eq. (3)), probability of success $P_{suc}$, and the protocol efficiency $C^{PQT}$ (Eq. (2)) as a function of $n$ and the perturbation from the Unity Fidelity Protocol (UFP), i.e. $n - m = 0$. As can be seen, we lose fidelity as the perturbation grows (Fig. 1(a)), but $P_{suc}$ is enhanced (Fig. 1(b)). The mean fidelity grows as $m \to 1$, as in the general case. We should note that this scenario is more realistic since the entanglement in the channel is not known completely, implying that the measurement basis cannot be set to $m = n$, but only as a close approximation.

The generalized teleportation protocol detailed above will now be expanded to $N$ qubits. (It can be seen as $N$ single qubit protocols implemented at once or in sequence. However, the overall fidelity and protocol efficiency are not trivial extensions of previous results.) The state Alice wants to teleport is the most general pure state for N qubits: $|\phi^A\rangle = \sum_{i=1}^{2^N} \alpha_i |\text{Bin}(i-1)\rangle$, where $\text{Bin}(i)$ is the binary representation of the integer $i$ with zeros padded to its left in order to leave all binary numbers with the same amount of digits. Now Alice needs N two-qubit channels, which is given by N Bell states with different degrees of entanglement (in general $n_i \neq n_j$, for $i \neq j$): $|\phi^{\text{channel}}\rangle = \bigotimes_{i=1}^N |\phi_{n_i}^+\rangle$. For each Bell state, one qubit is with Alice and the other one with Bob.

The rest of the protocol is similar to the one-qubit protocol: (a) Alice performs $N$ Bell measurements. The states expanding each basis she projects need not have the same degree of entanglement ($m_i \neq m_j$ in general); (b) Alice informs Bob of the acceptable results. At most she transmits $2N$ bits of classical information to Bob, two bits for each Bell measurement considered acceptable; (c) Bob performs unitary operations on his qubits according to the classical information received from Alice. Each qubit is subjected to one of the four possible transformations mentioned above.

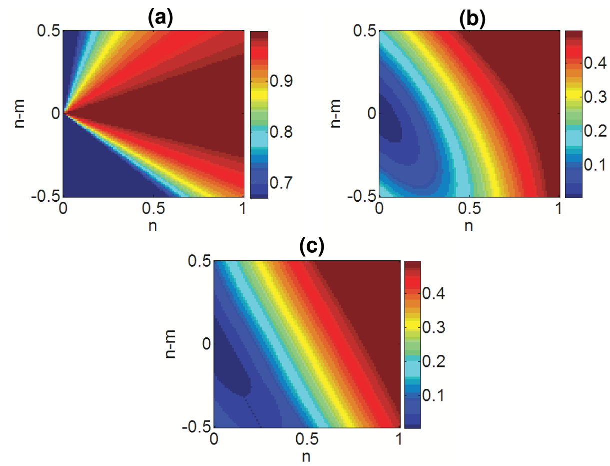


FIG. 1: (Color online) PQT attributes as a function of the dimensionless parameter $n$ and the perturbation from UFP, i.e. $n - m = 0$. (a) $F^{PQT}$; (b) $P_{suc}$; (c) $C^{PQT}$.

Building on the case for one qubit (and also for two and three qubits, which were analytically solved yet are too cumbersome to be detailed here), we were able to induce the channel efficiency for the N-qubit teleportation protocol:

$$C_N^{pro} = \frac{2}{2^N+1}\left(1 + \sum_{i=1}^{N} 2^{i-1}\text{Perm}_i^N\right), \qquad (10)$$

where $\text{Perm}_i^N$ is the sum of all permutations of the product of $i$ variables out of all $\{\chi_r\}$, where $r = 1, \ldots, N$, and $\chi_r = c(n_r)c(m_r)/2$. For example, the three qubit case gives $\text{Perm}_1^3 = \chi_1+\chi_2+\chi_3$, $\text{Perm}_2^3 = \chi_1\chi_2+\chi_1\chi_3+\chi_2\chi_3$, and $\text{Perm}_3^3 = \chi_1\chi_2\chi_3$. We can better understand Eq. (10) by analyzing specific qubits to be teleported. The contributions from $\text{Perm}_1^N$ appear when we try to teleport product states, without entanglement. When entangled qubits are teleported, the terms $\text{Perm}_{i>2}^N$ appear.

For the PQT of N qubits we see that $P_{suc}$ and thus the protocol efficiency are $C_N^{PQT} = P_{suc} = \prod_{i=1}^N 2n_i^2/(1+n_i^2)^2$. Note that $C_N^{PQT}$ decreases rapidly for a large number of qubits. This is due to the fact that only measurement results that project Alice's qubits on combinations of states given by $|\Phi_{m_i}^-\rangle$ and $|\Psi_{m_j}^+\rangle$ yield unity fidelity. All the other possible measurement outcomes are considered unacceptable in this protocol and are discarded (they do not give unity fidelity).

A proposed experimental setup follows, using photon polarization as the encoding medium. In this setup the horizontal $|H\rangle$ and the vertical $|V\rangle$ polarizations of a photon encode $|0\rangle$ and $|1\rangle$, respectively. The features of the GTP are incorporated into current standard teleportation experiments by addition of simple linear photonic devices. The setup (Fig. 2) uses a Spontaneous Parametric Down-Conversion (SPDC) [7] which emits two polarization-entangled photon pairs. One pair is used as the teleporation channel, which can be at an arbitrary length and purity thus facilitating different channel entanglement $n$. The other pair is used for the teleported qubit. By using Faraday Rotators (FRs) and Single-Photon Detectors (SPhDs) Victor can measure one of the photons along a given polarization axis, thus ascertaining the other photon polarization and implement the averaging procedure. One photon from the channel and the qubit to be teleported arrive at Alice's, which has a complete Bell-State Analyzer (BSA). The latter was recently shown to be implemented in several manners [8]. (Since it uses linear optics it is a probabilistic BSA [9].) To transform it into a rotated-BSA (measures arbitrary Bell-basis) a FR is inserted in the path of the qubit to be teleported, introducing the free parameter $m$. The result of the BSA is then transferred classically to Bob, who operates unitarily on his qubit. This can be done by using Polarization Beam Splitters (PBS) and Phase Retardation (birefringent) Wave-plates (PRW). By using a PBS and two PRWs at the path of Bob's photon we add another degree of freedom, namely $\theta_{\Phi_\pm} = \theta_{\Psi_\pm} = \theta_B$. By measuring the resulting photon polarization via a SPhD Bob ends the teleportation protocol.

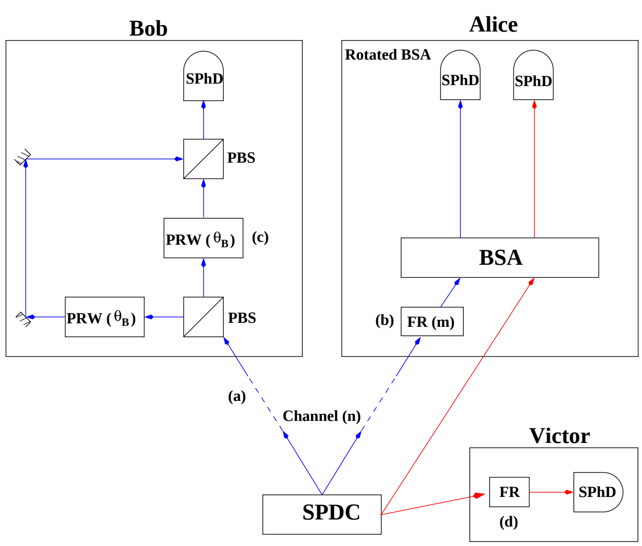


FIG. 2: (Color online) Scheme for experimental setup of the GTP. (a) Channel entanglement given by parameter $n$; (b) Measuring basis entanglement $m$; (c) Bob's unitary operation degree of freedom $\theta_B$; (d) Averaging capabilities by using different rotation angles. See text for further details.

The generalized teleportation protocol (GTP) developed here shows that the protocol efficiency depends solely on the channel and measuring basis entanglement in a quite interesting way: *interchanging the channel entanglement with the measurement basis entanglement leaves the protocol efficiency unchanged* (See Eq. (9)). This result is also valid for the N qubit protocol. The reason for this behavior is unclear but we suspect that it is related to an unknown information conservation law. Furthermore, the protocol efficiency increases either if the channel entanglement or the measurement basis entanglement are enhanced and a deterministically unity fidelity protocol occurs only if both quantities are maximum. We can understand this fact by noting that quantum teleportation is a genuine quantum task relying on entanglement. Therefore, as the availability of quantum resources (entanglement) are increased, a better performance of the protocol (higher output fidelity) is expected. On top of that, since the protocol efficiency is given as a function of the concurrences of the channels and of the measuring basis, we have provided an operational physical interpretation for the concurrence relating it to the efficiency of the teleportation protocol. Another result showed that for a dephased channel we can overcome the dephasing easily by selecting the proper unitary operation to counter it, implying that only the entanglement of the channel is important to achieve a given output fidelity and not which entangled state is used. Also, the generalization of the probabilistic quantum teleportation protocol showed that for a given channel entanglement, one can choose between different measurement basis in order to *divide the channel efficiency between fidelity and probability of success*. Finally, the extension of the protocol to $N$ qubits gave new insights to the understanding of quantum teleportation: the channel efficiency depends on the possible states to be teleported. For an unentangled state, it only depends on the individual channel concurrences, whereas for an entangled state it depends on the product of the concurrences of the channels used in the "entanglement teleportation".

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