Optical generation of hybrid entangled state via entangling single-photon-added coherent state

Yan Li,1,2,3* Hui Jing,1,2 and Ming-Sheng Zhan1,2†
1 State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, P. R. China
2 Center for Cold Atom Physics, Chinese Academy of Sciences, Wuhan 430071, P. R. China
3 Graduate School of the Chinese Academy of Sciences, Beijing 100080, P. R. China

We propose a feasible scheme to realize the optical entanglement of single-photon-added coherent state (SPACS) and show that, besides the Sanders entangled coherent state, the entangled SPACS also leads to new forms of hybrid entanglement of quantum Fock state and classical coherent state. We probe the essential difference of two types of hybrid entangled state (HES). This HES provides a novel link between the discrete- and the continuous-variable entanglement in a natural way.

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The generation of quantum entangled states plays an important role in quantum information science, and many practical schemes have been investigated theoretically and experimentally by applying, e.g., light fields, trapped ions, cavity QED, and ultra-cold atomic ensembles, etc. Among these appealing schemes, the optical generations of entanglement are always of intense interests due to their versatile applications for various purposes, including the well-known discrete-variable and the so-called continuous-variable entangled states. Therefore a simple question may naturally arise: is it possible (and then useful) to optically generate an intermediate hybrid entanglement which, however, may not be reduced to any of these two types of entangled states?

Recently, Zavatta et al. in their beautiful experiment (Science 306 (2004) 660) generate a novel single-photon-added coherent state (SPACS) and then visualize the interesting evolution of quantum-to-classical transition. The key elements of their experiment are the BBO-crystal-based parametric down-conversion process and then the single-photon detection technique. This experiment is also a clear implementation of the original idea of Agarwal et al. of the photon-added coherent state (PACS) via successive elementary one-photon excitation of a classical coherent field, for example,

$$SPACS : |\alpha, 1\rangle = \frac{\hat{a}^\dagger |\alpha\rangle}{\sqrt{1 + |\alpha|^2}} ,$$

where $|\alpha\rangle$ is the ordinary Glauber coherent state and $\hat{a}^\dagger$ is the photon creation operator. These new states occupy an intermediate position between the fully quantum-mechanical single-photon Fock state and the classical coherent state, containing both the discrete and continuous variables in some sense.

In this paper, we propose a feasible scheme to optically realize the entangled SPACS (ESPACS) and show that it can lead to some hybrid entanglement of single-photon Fock state and classical coherent state. Our scheme is based on the combination of two elegant concepts: the SPACS and the well-known entangled coherent state (ECS) firstly proposed by Sanders. We will show that, by generating the

* Electronic address: liyan@wipm.ac.cn
† Electronic address: mszhan@wipm.ac.cn
ESPACs in the light field, the interesting entanglement between the discrete and continuous variables can be achieved, which is the so-called hybrid entangled state (HES). We note that very recently Feng et al. suggested to generate a special entanglement of the parity qubit (even or odd coherent states) and the spin qubit (discrete variables) via the technique of conditional joint measurement [21]. As an interesting comparison, the output HES in our scheme has some very different characteristics, e.g., the qubit itself is hybrid in the sense that one cannot clarify which is the spin or the parity qubit in the created entanglement. In addition, our scheme is feasible which only needs coherent input lights and the single-photon detections. Of course, the unattractive point is the use of some nonlinear mediums.

Turning now to Fig. 1 for an illustration of our schematic setup. We can divide this configuration into two parts: the Mach-Zehnder (MZ) interferometer (I) and the BBO crystals (II). Let us firstly consider the part I. Two beams of classical coherent fields are incident on the first beam splitter (BS$_1$). Within one arm of the interferometer a nonlinear Kerr medium is placed which we approximate as a nonlinear oscillator in a single-mode treatment. For simplicity we assume that the nonlinear interferometer is lossless. The dynamical description involves two input modes $a$ and $b$, with corresponding Bose annihilation operators $\hat{a}$ and $\hat{b}$. This indicates the two-mode initial state as the following

$$|\psi\rangle_{in} = |\alpha\rangle_a |\beta\rangle_b.$$  \hspace{1cm} (2)

The BS Hamiltonian generating linear mode-coupling is given by

$$\hat{H}_B = i(\lambda^* \hat{a}^\dagger \hat{b} - \lambda \hat{a} \hat{b}^\dagger),$$  \hspace{1cm} (3)

FIG. 1: Configuration of the generation of the entangled single-photon-added coherent state. Part I is composed of the two beam splitters, one Kerr medium and two mirrors, while part II contains two BBO crystals and two single-photon detectors.
where \( \lambda \) denotes the coupling strength between the two modes and \( \arg(\lambda) \) denotes the relative phase shift between the modes imposed by the coupling. The unitary evolution operator is

\[
\hat{U}_B(\theta, \phi) = \exp[\theta(\hat{e}^{-i\phi} \hat{a}^\dagger \hat{b} - \hat{e}^{i\phi} \hat{a} \hat{b}^\dagger)],
\]

where \( \theta = \lambda t \), with \( t \) being the interaction time and \( \phi = \arg(\lambda) \). This unitary operator keeps the factorized structure of the state of the system by transforming a two-mode product coherent state (CS) into another CS, which means that the state after the first BS is

\[
\hat{U}_B(\theta, \phi)|\psi\rangle_{in1} = |\alpha \cos \theta + \beta e^{-i\phi} \sin \theta \rangle_a | -\alpha e^{i\phi} \sin \theta + \beta \cos \theta \rangle_b.
\]

For the case of \( \theta = \pi/4 \) and \( \phi = 3\pi/2 \), we get

\[
\hat{U}_B(\theta, \phi)|\psi\rangle_{in1} = \frac{1}{\sqrt{2}}(\alpha + i\beta)|\alpha\rangle_a |1\rangle_b.
\]

The Hamiltonian of nonlinear Kerr medium is \( \hat{H}_K = \chi (\hat{a}^\dagger \hat{a})^2 \), the unitary evolution operator is

\[
\hat{U}_K = \exp[-i\chi (\hat{a}^\dagger \hat{a})^2],
\]

where \( \chi \) is the nonlinearity coefficient which is proportional to the nonlinear coefficient \( \chi^{(3)} \) of the medium and the interaction length. For a CS \( |\alpha\rangle_a \), the state after the Kerr interaction is (\( \chi = \pi/2 \))

\[
\hat{U}_K|\alpha\rangle_a = \exp[-|\alpha|^2/2] \sum_{n=0}^{\infty} \frac{\exp[-i\pi/4]|\alpha|^{2n} |n\rangle}{\sqrt{n!}} = \frac{1}{\sqrt{2}} (e^{-i\pi/4}|\alpha\rangle + e^{i\pi/4}|\alpha\rangle).
\]

Combining the Kerr medium and two BS in the MZ interferometer, we obtain the Sanders ECS [20] as the output state of part I:

\[
|\psi\rangle_{out1} = \hat{U}_{MZ}|\psi\rangle_{in1} = \hat{U}_B \hat{U}_K \hat{U}_B |\alpha\rangle_a |\beta\rangle_b = \frac{1}{\sqrt{2}} (e^{-i\pi/4}|\beta\rangle_{a'} |\alpha\rangle_{b'} + e^{i\pi/4}|\alpha\rangle_a |\beta\rangle_{b'}). \tag{9}
\]

In our scheme the Sanders ECS \( |\psi\rangle_{out1} \) is used as the input state of Part II in which two nonlinear crystals provide the important further manipulations. For the parametric down-conversion of BBO crystal, one high-energy pump photon can induce two lower-energy photons in symmetrically oriented directions being called the signal and idler modes. Without other light being injected into the crystal, a pair of entangled photons with random but correlated phases is produced. The Hamiltonian of BBO crystal is \( \hat{H}_C = \kappa \hat{c}^\dagger \hat{a} \hat{b} + \kappa \hat{a}^\dagger \hat{b}^\dagger \hat{c} \), where the operator \( \hat{c} \) can be regarded as \( c \)-number \( i \gamma \) for strong (classical) pumping, thereby the corresponding evolution operator is

\[
\hat{U}_c = \exp[-i(\kappa \hat{c}^\dagger \hat{a} + \kappa \hat{a}^\dagger \hat{b}^\dagger \hat{c})t] = \exp[\kappa \gamma t (\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b})] = \exp[\kappa \gamma t (\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b})], \tag{10}
\]

where \( g = \kappa \gamma t \) can be regarded as an effective interaction time. The two-mode entanglement is obtained for the input CS signal and vacuum idler states (see Fig. 2). If the parametric gain is kept sufficiently low, i.e. \( |g| \ll 1 \), the form of output state can be written as

\[
\hat{U}_c|\alpha\rangle_s |0\rangle_i = \exp[\kappa \gamma t (\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b})]|\alpha\rangle_s |0\rangle_i \approx |\alpha\rangle_s |0\rangle_i + g \hat{a}^\dagger |\alpha\rangle_s |0\rangle_i = |\alpha\rangle_s |0\rangle_i + g |\alpha, 1\rangle_s |1\rangle_i. \tag{11}
\]

The output signal mode will contain the original CS except for the relatively rare single-photon detections in the idler output mode. These rare events stimulate emission of one photon in the CS
FIG. 2: The generation of entangled photon-pairs in the oriented directions via the parametric down-conversion of pump light $i\gamma$. Given coherent state input in one mode and the other vacuum state, when one photon is detected in the idler mode, the SPACS $|\alpha,1\rangle$ will be created in the signal mode.

$|\alpha\rangle \otimes |0\rangle$, which generates the intermediate state $|\alpha,1\rangle_s$ in the correlated signal mode. Therefore, considering the second part of the device shown in Fig. 1, the input state for part II is: $|\psi\rangle_{in2} = |\psi\rangle_{out1} = \hat{U}_{MZ} |\alpha\rangle_a |\beta\rangle_b$, and the final output state after the BBO interactions for the two modes $a'$ and $b'$ can be obtained as

$$|\psi\rangle_{out} = \hat{U}_{ca} \hat{U}_{cb} \hat{U}_{MZ} |\alpha\rangle_a |\beta\rangle_b = \exp[g_1 (\hat{a}_a^\dagger \hat{a}_i^\dagger - \hat{a}_a \hat{a}_i)] \exp[g_2 (\hat{b}_b^\dagger \hat{b}_i^\dagger - \hat{b}_a \hat{b}_i)] \hat{U}_{MZ} |\alpha\rangle_a |\beta\rangle_b$$

$$= (1 + g_1 \hat{a}_a^\dagger \hat{a}_i^\dagger)(1 + g_2 \hat{b}_b^\dagger \hat{b}_i^\dagger) \frac{1}{\sqrt{2}} (e^{-i\pi/4} |\beta\rangle_{a's} |\alpha\rangle_{b's} |0\rangle_{a'i} |0\rangle_{b'i} + e^{i\pi/4} |\alpha\rangle_{a's} |\beta\rangle_{b's} |0\rangle_{a'i} |0\rangle_{b'i}).$$

For our present purpose we can simply assume that the whole device is lossless and let the effective interaction time of two BBO crystals just be equal, i.e. $g_1 = g_2$, then we reach the following output entangled state in a rather general hybrid form:

$$|\psi\rangle_{out} = \frac{1}{\sqrt{2}} [e^{-i\pi/4} |i\beta\rangle_{a's} |\alpha\rangle_{b's} |0\rangle_{a'i} |\beta\rangle_{b's} |0\rangle_{b'i} + e^{i\pi/4} |\alpha\rangle_{a's} |\alpha\rangle_{b's} |0\rangle_{a'i} |\alpha\rangle_{b's} |0\rangle_{b'i}]$$

$$+ \frac{1}{\sqrt{2}} g [e^{-i\pi/4} |i\beta\rangle_{a's} |\alpha\rangle_{b's} |0\rangle_{a'i} |\beta\rangle_{b's} |0\rangle_{b'i} + e^{i\pi/4} |\beta\rangle_{a's} |\alpha\rangle_{b's} |0\rangle_{a'i} |\beta\rangle_{b's} |0\rangle_{b'i}]$$

$$+ \frac{1}{\sqrt{2}} g^2 [e^{-i\pi/4} |i\beta\rangle_{a's} |\alpha\rangle_{b's} |0\rangle_{a'i} |\beta\rangle_{b's} |0\rangle_{b'i} + e^{i\pi/4} |\beta\rangle_{a's} |\alpha\rangle_{b's} |0\rangle_{a'i} |\beta\rangle_{b's} |0\rangle_{b'i}]$$

$$+ \frac{1}{\sqrt{2}} g^2 [e^{-i\pi/4} |i\beta\rangle_{a's} |\alpha\rangle_{b's} |0\rangle_{a'i} |\beta\rangle_{b's} |0\rangle_{b'i} + e^{i\pi/4} |\beta\rangle_{a's} |\alpha\rangle_{b's} |0\rangle_{a'i} |\beta\rangle_{b's} |0\rangle_{b'i}]$$

This indicates the conditional preparations of the different kinds of entangled single-photon-added CS (ESPACS) whenever a "click" is registered or not on the single-photon detectors placed in the output idler modes. For the concrete illustrations, we start to analyze in the following four different circumstances of this output entangled states:

First, if both of the two detectors do not detect one photon synchronously, we just get the entangled coherent state (ECS) of the output signal modes $20$, as it should be:

$$|\psi\rangle_{out} = \frac{1}{\sqrt{2}} [e^{-i\pi/4} |i\beta\rangle_{a's} |\alpha\rangle_{b's} + e^{i\pi/4} |\alpha\rangle_{a's} |\beta\rangle_{b's}].$$

Second, if one photon is registered in one detector (detector $a$ or $b$ as case $A$ or $B$) but not in the other one, we can get the generalized ECS, i.e., the entanglement between the SPACS and the CS:

$$|\psi\rangle_{out} = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{2}} g [e^{-i\pi/4} |i\beta,1\rangle_{a's} |\alpha\rangle_{b's} + e^{i\pi/4} |\alpha,1\rangle_{a's} |\beta\rangle_{b's}] & : \text{case A;} \\
\frac{1}{\sqrt{2}} g [e^{-i\pi/4} |i\beta,1\rangle_{a's} |\alpha\rangle_{b's} + e^{i\pi/4} |\alpha,1\rangle_{a's} |\beta\rangle_{b's}] & : \text{case B,}
\end{array} \right.$$
which can be termed as the type-II hybrid entangled state (HES), since it is essentially different from
the type-I HES obtained by Feng et al. (i.e., the entanglement of the parity qubit and the spin qubit)
via their BS-based conditional joint measurement way \[21\] or Chen et al. via non-optics methods \[23\].

Finally, if both of the detectors can capture one photon simultaneously, we can obtain the interesting
entanglement between the intermediate single-photon-added CS (ESPACS), i.e.,

\[
\text{ESPACS} : \frac{1}{\sqrt{2}} g^2 |e^{-i\pi/4}| \langle \beta \rangle \langle 1 \rangle_{a'b'} + e^{i\pi/4} | -\alpha \rangle \langle 1 \rangle_{a'b'}.
\]

This state, as a generalized form of ECS, realizes the entanglement of SPACS in a conceptually elegant
way. In particular, if we let \( \beta = 0 \) and assume \( \alpha \) being large enough, or equivalently the initial input
state is simple: \( |\psi\rangle_{m1} = |\alpha\rangle_o |0\rangle_b \), we can see that, by choosing suitable parameters, the simplified
form of ESPACS leads to the HES between discrete and continuous variables:

\[
|\Psi_{a,b}^{HES-II}\rangle = |1\rangle_a |\alpha\rangle_b + e^{i\omega} |1\rangle_a |\tilde{1}\rangle_b.
\]

We note that in two recent related works, Chen et al. probed the hybrid entangled state (HES) in
the trapped-ion and the atom-cavity systems \[23\], and Feng et al. investigated the mixed entangled
state (MES) via an optical scheme \[21\], these two states are in fact the same form, i.e.,

\[
|\Psi_{1,2}^{\pm}\rangle_{HES-I} = \xi (|\uparrow\rangle_1 |\alpha\rangle_o,2 \pm |\downarrow\rangle_1 |\alpha\rangle_e,2),
\]

where \( \xi \) is a normalized factor and \( |\alpha\rangle_o,e \) denotes the odd or even CS. Clearly, this is just the formally
discrete-variable entanglement of spin qubit and parity qubit. In other words, the physically CVs are
logically encoded into the formally DVs or parity qubit \[21, 23\]. However, in our type-II HES, we
cannot tell which is the spin or the parity qubit. The qubits itself are hybrid or mixed. In some sense,
we can view the type-II HES as the entangled Schrödinger’s cat state if we define the classical-world
CVs and the quantum-world DVs as the living-cat or the dead-cat states, respectively.

In conclusion, we propose a feasible scheme to achieve optical entanglement of SPACS and thereby
in a conceptually elegant way show that, besides the original Sanders ECS as the CV entanglement of
two CS, the generated ESPACS also leads to some new forms of HES of purely quantum Fock state
and classical CS. Our scheme is quite simple by combining two famous and experimentally accessible
techniques. It consists only of three kinds of familiar devices: two BSs – providing two-mode input-
output ports, nonlinear Kerr medium – generating CV entanglement of CS, and BBO crystals –
providing DV entanglement of photon pairs. We compared our new type of HES (type II HES) with
two previous related states (type-I HES) and pointed out their essential differences, i.e., whether or
not they can be written as the form of spin- and parity-qubit entanglement \[24\]. These new forms of
HES can also be expected to be realized in other non-optical systems. Besides providing a natural
link between the DV and the CV entanglements, these HES may serve as new entanglement resources
and their novel applications in quantum information science would be challenging for further studies.
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