Entropic bounds and continual measurements

Alberto Barchielli
Politecnico di Milano, Dipartimento di Matematica,
Piazza Leonardo da Vinci 32, I-20133 Milano, Italy.
E-mail: Alberto.Barchielli@polimi.it

Giancarlo Lupieri
Università degli Studi di Milano, Dipartimento di Fisica,
Via Celoria 16, I-20133 Milano, Italy.
E-mail: Giancarlo.Lupieri@mi.infn.it

November 3, 2005

Abstract

Some bounds on the entropic informational quantities related to a quantum continual measurement are obtained and the time dependencies of these quantities are studied.

1 Introduction

In the problem of information transmission through quantum systems, various entropic quantities appear which characterize the performances of the encoding and decoding apparatuses. Due to the peculiar character of a quantum measurement, many bounds on the informational quantities involved have been proved to hold [1–8]. In the case of measurements continual in time, these bounds acquire new aspects (family of measurements are now involved) and new problems arise. A typical question is about which of the various entropic measures of information is monotonically increasing or decreasing in time. We already started the study of this subject in Refs. [9, 10]; here we apply to the case of continual measurements the new techniques developed [6–8] for the time independent case.

1.1 Notations and preliminaries

We denote by $\mathcal{L}(\mathcal{A}; \mathcal{B})$ the space of bounded linear operators from $\mathcal{A}$ to $\mathcal{B}$, where $\mathcal{A}$, $\mathcal{B}$ are Banach spaces; moreover we set $\mathcal{L}(\mathcal{A}) := \mathcal{L}(\mathcal{A}; \mathcal{A})$.

Let $\mathcal{H}$ be a separable complex Hilbert space; a normal state on $\mathcal{L}(\mathcal{H})$ is identified with a statistical operator, $\mathcal{T}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ are the trace-class and the space of the statistical operators on $\mathcal{H}$, respectively, and $\|\rho\|_1 := \text{Tr}\sqrt{\rho^*\rho}$, $\langle \rho, a \rangle := \text{Tr}_\mathcal{H}\rho a$, $\rho \in \mathcal{T}(\mathcal{H})$, $a \in \mathcal{L}(\mathcal{H})$. 

1
More generally, if $a$ belongs to a $W^*$-algebra and $\rho$ to its dual $\mathcal{M}^*$ or predual $\mathcal{M}_*$, the functional $\rho$ applied to $a$ is denoted by $\langle \rho, a \rangle$.

### 1.1.1 A quantum/classical algebra

Let $(\Omega, \mathcal{F}, Q)$ be a measure space, where $Q$ is a $\sigma$-finite measure. By Theorem 1.22.13 of [11], the $W^*$-algebra $L^\infty(\Omega, \mathcal{F}, Q) \otimes \mathcal{L}(\mathcal{H})$ ($W^*$-tensor product) is naturally isomorphic to the $W^*$-algebra $L^\infty(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H}))$ of all the $\mathcal{L}(\mathcal{H})$-valued $Q$-essentially bounded weakly* measurable functions on $\Omega$. Moreover (11), Proposition 1.22.12), the predual of this $W^*$-algebra is $L^1(\Omega, \mathcal{F}, Q; \mathcal{T}(\mathcal{H}))$, the Banach space of all the $\mathcal{T}(\mathcal{H})$-valued Bochner $Q$-integrable functions on $\Omega$, and this predual is naturally isomorphic to $L^1(\Omega, \mathcal{F}, Q) \otimes \mathcal{T}(\mathcal{H})$ (tensor product with respect to the greatest cross norm — [11], pp. 45, 58, 59, 67, 68).

Let us note that a normal state $\sigma$ on $L^\infty(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H}))$ is a measurable function $\omega \mapsto \sigma(\omega) \in \mathcal{T}(\mathcal{H})$, $\sigma(\omega) \geq 0$, such that $\text{Tr}_{\mathcal{H}}\{\sigma(\omega)\}$ is a probability density with respect to $Q$.

### 1.2 Quantum channels and entropies

#### 1.2.1 Relative and mutual entropies

The general definition of the relative entropy $S(\Sigma|\Pi)$ for two states $\Sigma$ and $\Pi$ is given in [12]; here we give only some particular cases of the general definition.

Let us consider two quantum states $\sigma$, $\tau \in \mathcal{S}(\mathcal{H})$ and two classical states $q_k$ on $L^\infty(\Omega, \mathcal{F}, Q)$ (two probability densities with respect to $Q$). The quantum relative entropy and the classical one are

$$S_q(\sigma|\tau) = \text{Tr}_\mathcal{H}\{\sigma(\log \sigma - \log \tau)\}, \quad (1a)$$

$$S_c(q_1|q_2) = \int_\Omega Q(d\omega) q_1(\omega) \log \frac{q_1(\omega)}{q_2(\omega)}. \quad (1b)$$

We shall need also the von Neumann entropy of a state $\tau \in \mathcal{S}(\mathcal{H})$: $S_q(\tau) := -\text{Tr}\{\tau \log \tau\}$.

Let us consider now two normal states $\sigma_k$ on $L^\infty(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H}))$ and set $q_k(\omega) := \text{Tr}\{\sigma_k(\omega)\}$, $q_k(\omega) := \sigma_k(\omega)/q_k(\omega)$ (these definitions hold where the denominators do not vanish and are completed arbitrarily where the denominators vanish). Then, the relative entropy is

$$S(\sigma_1|\sigma_2) = \int_\Omega Q(d\omega) \text{Tr}_\mathcal{H} \{\sigma_1(\omega)(\log \sigma_1(\omega) - \log \sigma_2(\omega))\} \quad (2a)$$

$$= S_c(q_1|q_2) + \int_\Omega Q(d\omega) q_1(\omega) S_q(q_1(\omega)|q_2(\omega)). \quad (2b)$$

We are using a subscript “c” for classical entropies, a subscript “q” for purely quantum ones and no subscript for general entropies, eventually of a mixed character.

Classically a mutual entropy is the relative entropy of a joint probability with respect to the product of its marginals and this key notion can be generalized immediately to states on von Neumann algebras, every times we have a state on a tensor product of algebras [6–8].
1.2.2 Channels

**Definition 1.** ([12] p. 137) Let $M_1$ and $M_2$ be two $W^*$-algebras. A linear map $\Lambda^*$ from $M_2$ to $M_1$ is said to be a channel if it is completely positive, unital (i.e. identity preserving) and normal (or, equivalently, weakly* continuous).

Due to the equivalence [13] of $w^*$-continuity and existence of a preadjoint $\Lambda$, a channel is equivalently defined by: $\Lambda$ is a completely positive linear map from the predual $M_1^*$ to the predual $M_2^*$, normalized in the sense that $\langle \Lambda[\rho], 1_l \rangle^2 = \langle \rho, 1_l \rangle$, $\forall \rho \in M_{1s}$. Let us note also that $\Lambda$ maps normal states on $M_1$ into normal states on $M_2$.

A key result which follows from the convexity properties of the relative entropy is *Uhlmann monotonicity theorem* ([12], Theor. 1.5 p. 21), which implies that channels decrease the relative entropy.

**Theorem 1.** If $\Sigma$ and $\Pi$ are two normal states on $M_1$ and $\Lambda^*$ is a channel from $M_2 \to M_1$, then $S(\Sigma|\Pi) \geq S(\Lambda[\Sigma]|\Lambda[\Pi])$.

1.3 Continual measurements

Let us axiomatize the properties of a probability space where an independent-increment process lives and that ones of the $\sigma$-algebras generated by its increments. The probability measure $Q_1$ we are introducing will play the role of a reference measure.

**Assumption 1.** Let $(X, \mathcal{X}, Q_1)$ be a probability space with $(X, \mathcal{X})$ standard Borel. Moreover:

1. $\{\mathcal{X}_t^s, 0 \leq s \leq t\}$ is a two-times filtration of sub-$\sigma$-algebras: $\mathcal{X}_t^s \subset \mathcal{X}_t^r \subset \mathcal{X}$ for $0 \leq r \leq s \leq t \leq T$;
2. $\forall t \geq 0, \mathcal{X}_t^t$ is trivial;
3. $\mathcal{X}_t^s = \bigwedge_{T:T > t} \mathcal{X}_T^r$ for $0 \leq s \leq t$;
4. $\mathcal{X}_t^s = \bigvee_{r:s < r < t} \mathcal{X}_r^r$ for $0 \leq s < t$;
5. $\mathcal{X} = \bigvee_{t:T > 0} \mathcal{X}_t^0$;
6. for $0 \leq r \leq s \leq t \leq T$, $\mathcal{X}_r^r$ and $\mathcal{X}_t^t$ are $Q_1$-independent.

Continual measurements are a quantum analog of classical processes with independent increments [10, 14]. As any kind of quantum measurement, a continual measurement is represented by *instruments* [15–17], but, as shown in [7], instruments are equivalent to particular types of channels. Here we introduce continual measurements directly as a family of channels satisfying a set of axioms (cf. also [10, 18]).

**Assumption 2.** Let $\mathcal{H}$ be a separable complex Hilbert space. For all $s, t$, $0 \leq s \leq t$, we have a channel

$$\tilde{\Lambda}_t^s : L^1(X, \mathcal{X}_s^0, Q_1; T(\mathcal{H})) \to L^1(X, \mathcal{X}_t^0, Q_1; T(\mathcal{H}))$$

such that
1. $\tilde{\Lambda}_t^t = 1, \ t \geq 0$;
2. $\tilde{\Lambda}_r^t \circ \tilde{\Lambda}_s^r = \tilde{\Lambda}_t^s, \ 0 \leq r \leq s \leq t$;
3. $\forall \eta \in \mathcal{T} (\mathcal{H}), \tilde{\Lambda}_t^t [\eta]$ is $\mathcal{X}_t^t$-measurable, $0 \leq s \leq t$;
4. $\forall \eta \in \mathcal{T} (\mathcal{H}), \forall q \in L^1 (X, \mathcal{X}^0_s, Q_1), \tilde{\Lambda}_t^t [q \eta] = q \tilde{\Lambda}_t^t [\eta], \ 0 \leq s \leq t, (i.e. \tilde{\Lambda}_t^t [q \eta] (x) = q(x) \tilde{\Lambda}_t^t [\eta] (x) \ a.s.)$.

By points (3), (4) of Assumption 2 and (6) of Assumption 1 one gets: $\forall \sigma_s \in L^1 (X, \mathcal{X}^0_s, Q_1; \mathcal{T} (\mathcal{H})), 0 \leq s \leq t$,

$$\mathbb{E}_{Q_1} [\hat{\Lambda}_t^t [\sigma_s] | \mathcal{X}_t^t] = \hat{\Lambda}_t^t [\mathbb{E}_{Q_1} [\sigma_s]].$$

(3)

Here $\mathbb{E}_{Q_1}$ and $\mathbb{E}_{Q_1} [\cdot | \mathcal{X}_t^t]$ are the classical expectation and conditional expectation extended to operator-valued random variable.

Let us also define the evolution

$$U(t, s)[\tau] := \mathbb{E}_{Q_1} [\hat{\Lambda}_t^t [\tau]], \quad \tau \in \mathcal{T} (\mathcal{H}), \quad 0 \leq s \leq t; \quad (4)$$

$U(t, s)$ is a channel from $\mathcal{T} (\mathcal{H})$ into $\mathcal{T} (\mathcal{H})$. By points (2), (3), (4) of Assumption 2 for $0 \leq r \leq s \leq t, \sigma_s \in L^1 (X, \mathcal{X}^0_s, Q_1; \mathcal{T} (\mathcal{H})), we get

$$U(t, s) \circ U(s, r) = U(t, r), \quad \mathbb{E}_{Q_1} [\hat{\Lambda}_t^t [\sigma_s] | \mathcal{X}_t^t] = U(t, s)[\sigma_s].$$

(5)

The quantum continual measurements is represented by the operators $\tilde{\Lambda}_t^t$, in the sense that they give probabilities and state changes. If $\eta_0 \in \mathcal{S} (\mathcal{H})$ is the initial state at time 0 and $B \in \mathcal{X}_t^0$ is any event involving the output in the interval $(0, t)$, then $\int_B \text{Tr} \{ \tilde{\Lambda}_0^0 [\eta_0] (x) \} Q_1 (dx)$ is the probability of the event $B$ and $\tilde{\Lambda}_0^0 [\eta_0] (x)$ is the state at time t, conditional on the result $x$ (the a posteriori state). Instead, $U(t, 0)[\eta_0]$ represents the state of the system at time t, when the results of the measurement are not taken into account (the a priori state).

2 The initial state and the measurement

2.1 Ensembles

In quantum information theory, not only single states are used, but also families of quantum states with a probability law on them, called ensembles. An ensemble $\{ \mu, \rho \}$ is a probability measure $\mu (dy)$ on some measurable space $(Y, Y)$ together with a random variable $\rho : Y \rightarrow \mathcal{S} (\mathcal{H})$. Alternatively, an ensemble can be seen as a quantum/classical state of the type described in Section 1.1.1.

Given an ensemble, one can introduce an average state $\overline{\rho} \in \mathcal{S} (\mathcal{H})$

$$\overline{\rho} := \mathbb{E}_\mu [\rho] = \int_Y \mu (dy) \rho (y);$$

(6)

the integrals involving trace class operators are always understood as Bochner integrals. Finally, the average relative entropy of the states $\rho (y)$ with respect to $\overline{\rho}$ is called the “$\chi$-quantity” of the ensemble:

$$\chi (\mu, \rho) := \int_Y \mu (dy) S_q (\rho (y) | \overline{\rho}) = \mathbb{E}_\mu [S_q (\rho | \overline{\rho})].$$

(7)
This new quantity plays an important role in the whole quantum information theory \[3, 20\] and can be thought as a measure of some kind of quantum information stored in the ensemble.

2.2 The letter states

Let us consider the typical setup of quantum communication theory. A message is transmitted by encoding the letters in some quantum states, which are possibly corrupted by a quantum noisy channel; at the end of the channel the receiver attempts to decode the message by performing measurements on the quantum system. So, one has an alphabet \(A\) and the letters \(\alpha \in A\) are transmitted with some a priori probabilities \(P_i\). Each letter \(\alpha\) is encoded in a quantum state and we denote by \(\rho_i(\alpha)\) the state associated to the letter \(\alpha\) as it arrives to the receiver, after the passage through the transmission channel. While it is usual to consider a finite alphabet, also general continuous parameter spaces are acquiring importance \[19, 20\].

**Assumption 3.** Let \((A, \mathcal{A}, Q_0)\) be a probability space with \((A, \mathcal{A})\) standard Borel and let \(\sigma_i\) be a normal state on \(L^\infty(A, \mathcal{A}, Q_0; \mathcal{L}(\mathcal{H}))\).

Let us set

\[
\begin{align*}
q_i(\alpha) &:= \text{Tr}\{\sigma_i(\alpha)\}, \quad \rho_i(\alpha) := \frac{\sigma_i(\alpha)}{q_i(\alpha)}; \quad P_i(\alpha) := q_i(\alpha)Q_0(\alpha); \\
\end{align*}
\]

(8)

\(q_i\) is a probability density and \(\{P_i, \rho_i\}\) is the initial ensemble. The average state and the \(\chi\)-quantity of the initial ensemble are

\[
\begin{align*}
\eta_0 &:= \mathbb{E}_{Q_0}[\sigma_i] = \int_A P_i(\alpha) \rho_i(\alpha), \\
\chi\{P_i, \rho_i\} &:= \int_A P_i(\alpha) S_q(\rho_i(\alpha)|\eta_0).
\end{align*}
\]

(9)

(10)

The quantity \(\chi\{P_i, \rho_i\}\) is known also as Holevo capacity \[3, 20\].

2.3 Probabilities and states derived from \(\eta_0\)

For \(0 \leq r \leq s \leq t\) we define:

\[
\begin{align*}
\eta_t &:= \mathcal{U}(t, 0)[\eta_0], \quad \bar{\sigma}_t := \bar{\Lambda}_t[\eta_t], \\
\bar{q}_t &:= \|\bar{\sigma}_t\|_1, \quad \bar{q}_t^* := \frac{\bar{\sigma}_t}{\bar{q}_t^*}.
\end{align*}
\]

(11)

Then, \(\eta_t\) and \(\bar{q}_t(x)\) are states on \(\mathcal{L}(\mathcal{H})\), \(\bar{q}_t^*\) is a state on \(L^\infty(X, \mathcal{X}_t^0, Q_1)\) and \(\bar{\sigma}_t^*\) a state on \(L^\infty(X, \mathcal{X}_t, Q_1; \mathcal{L}(\mathcal{H}))\). We have also

\[
\begin{align*}
\mathbb{E}_{Q_1}[\bar{q}_t^*|\mathcal{X}_t^0] = \bar{q}_t^*, \quad \mathbb{E}_{Q_1}[\bar{q}_t|\mathcal{X}_t^*] = \bar{q}_t^*.
\end{align*}
\]

(12)

Moreover, there exists a unique probability \(P_1\) on \((X, \mathcal{X})\) such that \(P_1(dx)|_\mathcal{X}_t^0 = \bar{q}_t^*(x)Q_1(dx)\) for all \(t \geq 0\). Also \(P_1(dx)|_\mathcal{X}_t^* = \bar{q}_t^*(x)Q_1(dx)\) holds.
2.4 The general setup

It is useful to unify the initial distribution and the distribution of the measurement results in a unique filtered probability space. Let us set:

\[ \Omega := A \times X, \quad \omega := (\alpha, x), \quad \pi_0(\omega) := \alpha, \quad \pi_1(\omega) := x, \] (13a)

\[ \sigma_0 := \sigma_1 \circ \pi_0, \quad q_0 := q_1 \circ \pi_0 = \|\sigma_0\|_1, \quad \rho_0 := \rho_1 \circ \pi_0 = \frac{\sigma_0}{\|\sigma_0\|_1}, \] (13b)

\[ \mathcal{F} := A \otimes \mathcal{X}, \quad Q := Q_0 \otimes Q_1, \] (13c)

\[ \mathcal{F}_0 := \{ B \times X : B \in A \}, \quad \mathcal{F}_1 := \{ A \times Y : Y \in \mathcal{X}_1 \}, \] (13d)

\[ \mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{F}_1 = \sigma\{ B \times Y : B \in A, Y \in \mathcal{X}_1 \}. \] (13e)

By defining \( \Lambda_s^t := \mathbb{1} \otimes \Lambda_s^t \), we extend \( \Lambda_s^t \) to \( L^1(\Omega, \mathcal{F}_s, Q; T(\mathcal{H})) \cong L^1(A, \mathcal{A}, Q_0) \otimes L^1(X, \mathcal{X}_s^0, Q_1; T(\mathcal{H})) \). Similarly, we extend \( U(t, s) \) to \( L^1(\Omega, \mathcal{F}_s, Q; T(\mathcal{H})) \cong L^1(\Omega, \mathcal{F}_s, Q) \otimes T(\mathcal{H}) \). Let us also set:

\[ \sigma_t := \Lambda_s^t[\sigma_0], \quad \sigma_t^* := \sigma_t \circ \pi_1 = \Lambda_s^t[\eta_s], \quad q_t := \|\sigma_t\|_1, \] (14a)

\[ q_t^* := q_t^\ast \circ \pi_1 = \|\sigma_t^*\|_1, \quad \rho_t := \frac{\sigma_t}{\|\sigma_t\|_1}, \quad q_t^* := q_t^\ast \circ \pi_1 = \frac{\sigma_t^*}{\|\sigma_t^*\|_1}. \] (14b)

In the computations of the following sections we shall need various properties of the quantities we have just introduced; here we summarize such properties. Let \( r, s, t \) be three ordered times: \( 0 \leq r \leq s \leq t \). Then, \( \sigma_t \) and \( \sigma_t^* \) are states on \( L^\infty(\Omega, \mathcal{F}_t, Q; L(\mathcal{H})) \) and

\[ \mathbb{E}_Q[q_t | \mathcal{F}_s] = q_s, \quad \mathbb{E}_Q[q_t^* | \mathcal{F}_s] = \mathbb{E}_Q[q_t^* | \mathcal{F}_s] = q_t^*, \] (15a)

\[ \mathbb{E}_Q[q_t | \mathcal{F}_s] = q_t, \quad \mathbb{E}_Q[q_t^* | \mathcal{F}_s] = \mathbb{E}_Q[q_t^* | \mathcal{F}_s] = \sigma_t, \] (15b)

\[ \mathbb{E}_Q[\sigma_t | \mathcal{F}_s] = U(t, s)[\sigma_s], \quad \mathbb{E}_Q[\sigma_t^* | \mathcal{F}_s] = U(t, s)[\sigma_s^*], \] (15c)

\[ \mathbb{E}_Q[\sigma_t^* | \mathcal{F}_s] = \eta_t, \quad \eta_t = \mathbb{E}_Q[\sigma_t], \quad \sigma_t = \sigma_t^*[\sigma_s], \] (15d)

\[ \sigma_t^* = \Lambda_s^t[\sigma_t^*], \quad \frac{\Lambda_s^t[\rho_s]}{\|\Lambda_s^t[\rho_s]\|_1} = \rho_t, \quad \frac{\Lambda_s^t[\rho_s^*]}{\|\Lambda_s^t[\rho_s^*]\|_1} = q_t^*. \] (15e)

We have that \( \{ q_t, t \geq 0 \} \) is a non-negative, mean one, \( Q \)-martingale. Then, there exists a unique probability \( P \) on \( (\Omega, \mathcal{F}) \) such that \( \forall t \geq 0 \)

\[ P(d\omega)|_{\mathcal{F}_t} = q_t(\omega)Q(d\omega). \] (16)

Moreover,

\[ P(d\alpha \times X) = P_1(d\alpha), \quad P(A \times dx) = P_1(dx), \] (17)

\[ P(d\omega)|_{\mathcal{F}_t} = q_t^*(\omega)Q(d\omega), \quad \eta_t = \mathbb{E}_P[\rho_t] = U(t, s)[\eta_s]. \] (18)

3 Mutual entropies and informational bounds

Here and in the following we shall have always \( 0 \leq u \leq r \leq s \leq t \).
3.1 The state $q_t$ and the classical information

Let us consider the state $q_t$ and its marginals $E_Q[q_t|F_s] = q_r$, $E_Q[q_t|F^r_s] = q^r_t$. Then, we can introduce the classical mutual entropy:

$$S_c(q_t|q_r q^r_t) = \int_{\Omega} P(d\omega) \log \frac{q_t(\omega)}{q_r(\omega)q^r_t(\omega)} =: I_c(r, t). \quad (19a)$$

Note that $I_c(t, t) = 0$. For $r = 0$ we have the input/output classical information gain:

$$I_c(0, t) = S_c(q_t|q_t) = \int_{A \times X} P(da \times dx) \log \frac{q_t(\alpha, x)}{q_t(\alpha)q_t(x)}.$$ \hspace{1cm} (19b)

By applying the monotonicity theorem and the channel $E_Q[\bullet|F_s]$ to the couple of states $q_t$ and $q_r q^r_t$, we get

$$S_c(q_t|q_r, q^r_t) \geq S_c(E_Q[q_t|F_s]|E_Q[q_r q^r_t|F_s]) = S_c(q_s|q_r q^r_t), \quad (20)$$

which becomes

$$I_c(r, t) \geq I_c(r, s). \quad (21)$$

The function $t \mapsto I_c(s, t)$ is non-decreasing.

3.2 The state $\sigma_s$ and the main bound

A useful quantity, with the meaning of a measure of the “quantum information” left in the a posteriori states, is the mean $\chi$-quantity

$$\overline{\chi}(s, t) := \int_{\Omega} P(d\omega) S_q(\rho_t(\omega)|q_t^r(\omega)) = \mathbb{E}_P[S_q(\rho_t|q^r_t)]. \quad (22)$$

The interpretation as a mean $\chi$-quantity is due to the fact that $\overline{\chi}(s, t) = \mathbb{E}_P[S_q(\rho_t|q^r_t)|F^r_s]$. But by Eq. (19) and $E_P[\rho_t|F_r] = q^r_t$, $E_P[q_t^r|F^r_s] = q^r_t$, we have that $E_P[S_q(\rho_t|q^r_t)|F^r_s]$ is a random $\chi$-quantity. Note that

$$\overline{\chi}(t, t) = \int_{\Omega} P(d\omega) S_q(\rho_t(\omega)|\eta_t) =: \chi\{P, \rho_t\}. \quad (23)$$

Let us consider the state $\sigma_s$ and its marginals $E_Q[\text{Tr}\{\sigma_s\}|F_s] = q_r$, $E_Q[\sigma_s|F^r_s] = \sigma^r_s$. Then, we have the mutual entropy

$$S(\sigma_s|q_r, \sigma^r_s) = I_c(r, s) + \overline{\chi}(r, s). \quad (24)$$

For $r = s$ and for $r = 0$ this equation reduces to

$$S(\sigma_s|q_r, \eta_s) = \chi\{P, \rho_s\}, \quad S(\sigma_0|q_0, \rho_0) = \chi\{P, \rho_0\} = \chi\{P, \rho_t\}. \quad (25)$$

By applying the monotonicity theorem and the channel $\Lambda^r_t$ to the couple of states $\sigma_s$ and $q_r, \sigma^r_s$, we get

$$S(\sigma_s|q_r, \sigma^r_s) \geq S(\Lambda^r_t|\sigma_s|\Lambda^r_s|q_r, \sigma^r_s) = S(\sigma_t|q_r, \sigma^r_t), \quad (26)$$

which becomes

$$\overline{\chi}(r, s) - \overline{\chi}(r, t) \geq I_c(r, t) - I_c(r, s) \geq 0. \quad (27)$$
Therefore, the function $t \mapsto \chi(s, t)$ is non-increasing.

For $r = s$ we get

$$S(\sigma_s | q_s \eta_s) \geq S(\sigma_t | q_s \sigma_t^*). \quad (28)$$

which gives the upper bound for $I_c$:

$$0 \leq I_c(s, t) \leq \chi\{ P, \rho_s \} - \chi(s, t). \quad (29)$$

For $s = r = 0$, it reduces to

$$0 \leq I_c(0, t) \leq \chi\{ P_i, \rho_i \} - \int_{A \times X} P(d\alpha \times dx) S_q(\rho_t(\alpha, x) | \eta_r^0(x)). \quad (30)$$

The bound (30) is the translation in terms of continual measurements of the bound of Section 3.3.4 of [7], which in turn is a generalization of a bound by Schumacher, Westmoreland and Wootters [5]. Equation (30) is a strengthening of the Holevo bound [3] $I_c(0, t) \leq \chi\{ P_i, \rho_i \}$.

### 3.3 Quantum information gain

Let us consider now the quantum information gain defined by the quantum entropy of the pre-measurement state minus the mean entropy of the a posteriori states [1,2,4]. It is a measure of the gain in purity (or loss, if negative) in passing from the pre-measurement state to the post-measurement a posteriori states. In the continual case, we can consider the quantum information gain in the time interval $(s, t)$ when the system is prepared in the ensemble $\{ P_i, \rho_i \}$ at time 0 or when it is prepared in the state $\eta_r$ at time $r$:

$$I_q(s, t) := \int_{\Omega} P(d\omega) \left[ S_q(\rho_s(\omega)) - S_q(\rho_t(\omega)) \right], \quad (31a)$$

$$I_q(r; s, t) := \int_{\Omega} P(d\omega) \left[ S_q(\tilde{\rho}_s(\omega)) - S_q(\tilde{\rho}_t(\omega)) \right]. \quad (31b)$$

By this definition we have immediately

$$I_q(r, t) = I_q(r, s) + I_q(s, t), \quad I_q(u; r, t) = I_q(u; r, s) + I_q(u; s, t). \quad (32)$$

It has been proved [4] that the quantum information gain is positive for all initial states if and only if the measurement sends pure initial states into pure a posteriori states.

As in the single time case [6–8], inequality (27) can be easily transformed into an inequality involving $I_q$:

$$I_q(r; s, t) - I_q(s, t) \geq I_c(r, t) - I_c(r, s) \geq 0. \quad (33)$$

Let us take an initial ensemble made up of pure states: $\rho_i(\alpha)^2 = \rho_i(\alpha)$, $\forall \alpha \in A$. Let us assume that the continual measurement preserve pure states: the states $\rho_i(\alpha, x)$ are pure for all choices of $t, \alpha, x$. Then, the von Neumann entropy of $\rho_i(\omega)$ vanishes and we have $I_q(s, t) = 0$ for all choices of $s$ and $t$. From the second of Eqs. (32) and Eq. (33) we get

$$I_q(u; r, t) - I_q(u; r, s) = I_q(u; s, t) \geq I_c(u, t) - I_c(u, s) \geq 0, \quad (34)$$

8
i.e. the function $t \mapsto I_q(u; r; t)$ is non decreasing for “pure” continual measurements.

In particular, by taking $u = r = 0$ we have

$$I_q(0; 0, t) = S_q(\eta_0) - \int_X P_1(dx)S_q(\tilde{g}^0_0(x)).$$

(35)

For a continual measurement sending every pure initial state into pure a posteriori states, $\forall \eta_0 \in \mathcal{S}(\mathcal{H})$ the quantum information gain $I_q(0; 0, t)$ is non negative, non decreasing in time and with $I_q(0; 0, 0) = 0$.

**Acknowledgments**

Work supported by the European Community’s Human Potential Programme under contract HPRN-CT-2002-00279, QP-Applications, and by Istituto Nazionale di Fisica Nucleare.

**References**


