Background independent duals of the harmonic oscillator

Viqar Husain

Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo ON, Canada N2L 2Y5
Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada E3B 5A3
(Dated: June 7, 2006)

We show that a class of topological field theories are quantum duals of the harmonic oscillator. This is demonstrated by establishing a correspondence between the creation and annihilation operators and non-local gauge invariant observables of the topological field theory. The example is used to discuss some issues concerning background independence and the relation of vacuum energy to the problem of time in quantum gravity.

PACS numbers: 04.60.Ds

Two general themes have been a part of the dialogue in quantum gravity in the past decade. These are the ideas associated with the terms "background independence" and "duality."

The first of these stems from the intuition that a quantum theory of gravity should be background independent. Exactly what constitutes a background is a part of the debate in quantum gravity [1, 2], since there are levels of structure, from a set of points to a manifold with a metric, that may be fixed when formulating a classical or quantum theory. What is clear is the semi-classical requirement that in weak gravity an effective metric should emerge for some quantum states, and that in strong gravity no metric should be preferred. For example, the formal statement of the partition function for quantum gravity as a functional integral over all Lorentzian 4-manifold to be background structure includes fixing the possible fixed structure at boundaries. In the canonical framework, the background structure includes fixing the 4-manifold to be $R \times \Sigma$, where space $\Sigma$ also has fixed topology. The same structure would exist in the path integral framework in a fixed time gauge.

An operating definition of background independence may be taken to be metric independence. Classical general relativity is background independent in the sense that the metric is varied in the action, and therefore does not constitute a fixed structure. In contrast the definition of graviton is background dependent since it is a perturbation on a fixed metric. It is widely accepted that a non-perturbative theory of quantum gravity should not be one that prefers a fixed background metric at the fundamental level.

Background independence is not well understood for several reasons. The most prominent among these is that no concrete examples of background independent quantum field theories are known in which a metric is emergent in the semi-classical limit, where one expects to obtain the approximation of quantum fields on a fixed background. The issue is compounded by the fact that no background independent formulation is known even for the free scalar field that can make contact with conventional quantum field theory dynamics where the propagator is fundamentally metric dependent.

Duality on the other hand appears to be better understood, at least in some contexts. In one of its incarnations, it is the idea that two apparently distinct theories, not necessarily in the same spacetime dimension, are equivalent at the quantum level. "Equivalent" means that there is a precise correspondence between operators and quantum states in the dual theories, a relation between their coupling constants, and a matching of the spectra of operators, at least in some limits.

The earliest example of a duality dates to the seventies, when the first equivalence between field theories in 2-dimensions was established. This is the duality of the massive Thirring and sine-Gordon theories [3]. More recent examples of dualities are the series of AdS/CFT correspondences, and mirror symmetry. The former is a proposed duality between certain supersymmetric conformal field theories on Minkowski spacetime and string theory on asymptotically anti-deSitter spaces [4]. The mirror symmetries are equivalences between the quantum theories of certain 2-dimensional supersymmetric sigma models whose the target spaces are Calabi-Yau manifolds.

Duality and background independence become related if one of a dual pair of theories is a theory of geometry, and the other is a conventional theory describing field dynamics on a fixed metric background. Dualities of this type could perhaps allow a background independent question, such as a quantum gravity transition amplitude, or questions about black hole physics, to be reformulated and answered in a conventional fixed-metric setting. The AdS/CFT correspondences are often cited as providing an example of this sort. So in principle at least, they provide a scenario where quantum gravity questions are addressable in the CFT, and vice versa. This has motivated discussion on whether any CFT has a geometric, or background independent dual theory.

It is therefore of interest to seek other examples of dualities of this type and to see what can be learned from them. It is also important to distinguish the cases where there is imposed asymptotic background structure, as in the asymptotically flat or anti-deSitter cases, from the...
cases where space has no boundary. This is because a gravity theory with no classically fixed time has a Hamiltonian constraint rather than a usual Hamiltonian, so any comparison of its operator spectra with those of a proposed dual theory with a fixed time concept requires interpretive care.

Motivated in part by this debate, we point out in this note an exact duality between a topological (and therefore background independent) field theory in $n$ dimensions $n > 2$, and the simple harmonic oscillator in $0+1$ dimensions. This is done by finding the fully gauge invariant observables of the topological theory, and showing that certain functions of these satisfy the algebra of the creation and annihilation operators. Our aim is to clarify using this example the extent to which quantum gravity questions may be addressable via a dual background dependent theory.

The topological field theories of interest here are the so called BF models on an $n$-dimensional manifold $M \subset \mathbb{R}^n$. The action is

$$S = k \int_M \text{Tr}[B \wedge F(A)].$$

where $A$ is a 1-form, $F(A)$ is its curvature, and $B$ is an $n-2$ form. The trace is in the Lie algebra in which the fields are valued.

Our main point concerning duality is illustrated by the Abelian theory in four dimensions on a manifold $M \sim \Sigma \times \mathbb{R}$, where $\Sigma$ is a 3-manifold without boundary. Since the action is first order, it is easy to put into canonical form:

$$S = k \int_{\Sigma \times \mathbb{R}} 2 \epsilon^{abc} [B_{ab} \partial_c A_c + B_{0a} F_{bc} - B_{ab} \partial_c A_0],$$

where $a, b, c$ are indices in $\Sigma$. The canonical phase space coordinates are therefore $\{A_a, E^a\}$, where $E^a = \epsilon^{abc} B_{bc}$, which satisfy the Poisson bracket relations

$$\{A_a(x, t), E^b(y, t)\} = \frac{1}{k} \delta_a^b \delta^3(x, y)$$

The Hamiltonian is a linear combination of the constraints

$$F_{ab} = 0, \quad \partial_a E^a = 0,$$

obtained by varying the action with respect to $B_{0a}$ and $A_0$.

Since the constraints generate gauge transformations of the canonical variables via Poisson brackets, the gauge invariant observables $O(E, A)$ are defined by the Poisson bracket conditions

$$\{O(E, A), C(E, A)\} = 0,$$

where $C$ denotes the two constraints.

In the present case observables satisfying this condition are the non-local functionals

$$O_1(A, \gamma) = \int \tau \hat{\gamma}^a A_a,$$

$$O_2(E, S) = \int d^2 \sigma n_a E^a.$$  

These are parametrized by embedded loops $\gamma$ and surfaces $S$ in $\Sigma$, and $n_a$ is a one form field defining the surface $S$ ($\epsilon^{abc}n_c$ is the area 2-form and $\hat{\gamma}^a$ is tangent vector to the loop $\gamma$). These observables satisfy the Poisson algebra

$$\{O_1(A, \gamma), O_1(A, \beta)\} = 0,$$

$$\{O_2(E, S), O_2(E, S')\} = 0,$$

$$\{O_1(A, \gamma), O_2(E, S)\} = \frac{1}{k} c(\gamma, S) C_1(A, \gamma)$$

where

$$c(\gamma, S) = \int ds \int d^2 \sigma \hat{\gamma}^a n_a \delta^3(\gamma(s) - S(\sigma)),$$

counts the intersections of the loop with the surface. The last Poisson bracket vanishes if $\hat{\gamma}^a n_a = 0$, or if the loop and surface have no points of intersection.

As is well known, first class constraints have two properties. They generate gauge transformations, and they restrict the dynamics to the surface in the phase space defined by their strong imposition. Off the constraint surface there are an uncountable infinity of observables, because of the number of possible loops and surfaces in $\Sigma$. On the constraint surface however, most of these vanish because of the flat connection constraint; only a finite number that depend on the non-contractible loops and surfaces in $\Sigma$ remain. These capture topological information about $\Sigma$. (From a covariant point of view this is seen by the fact that the equations of motion $db = 0$ and $F = dA = 0$ have a finite dimensional solution space given by the dimensions of the cohomology groups of $M$.)

To proceed further we must fix the topology of $\Sigma$. This determines the number of independent observables, and hence the number of degrees of freedom. Perhaps the simplest example is provided by the case $\Sigma \sim S^3 \times S^2$ for which there is one non-contractible loop and surface, with $c(\gamma, S) = 1$. Thus there are exactly two degrees of freedom, which satisfy the Poisson algebra

$$\{C_1, C_2\} = \frac{1}{k}.$$

A quantization for this spatial topology is obtained by realizing this as a commutator on a Hilbert space. An occupation number representation is obtained by defining the operators

$$\hat{a}^\pm = \sqrt{\frac{k}{2\hbar}} \left( \hat{C}_1 \pm i\hat{C}_2 \right)$$

where $\hat{C}_1$ and $\hat{C}_2$ are the quantum versions of $C_1$ and $C_2$, respectively.
with their usual action. This establishes a duality with the usual algebraic quantization of the harmonic oscillator, up to the issue of Hamiltonian. This is discussed below.

Other spatial topologies in the abelian theory give more observables in the reduced theory. For example the case $\Sigma \sim T^3$ has three pairs of surface and loop observables, which is equivalent to three uncoupled oscillators.

A similar correspondence with the harmonic oscillator exists for the non-abelian theory. The (unreduced) phase space of the theory is that of Yang-Mills theory, but with additional constraint functions. These are

$$D_a E^{ai} = 0, \quad F^i_{ab} = 0,$$  \hspace{1cm} (14)

where $i$ is a Lie algebra index. The observables for the theory are a bit more involved than for the Abelian theory \[9\]. As for any theory with a Gauss law, these are made from the holonomy of the connection $A'_a$ around loops $\alpha$

$$U_{\alpha}(A) = P \exp \int_\alpha ds \, A_a(\alpha(s)) \, \dot{\alpha}^a(s).$$ \hspace{1cm} (15)

The first type of observable is the trace of holonomy

$$T^0(A; \alpha) = \text{Tr} \, [U_\gamma(A)].$$ \hspace{1cm} (16)

The second type depends on a loop $\alpha$ and a closed 2-surface $S$,

$$T^1(A, E; \alpha, S) = \int_S d^2 \sigma \, n_{\sigma} \text{Tr} \left[ E^a(\sigma) U_{\alpha}(\sigma) \right].$$ \hspace{1cm} (17)

The integrand in the latter is a function of the holonomy of loops $\alpha$ whose base point lies in the surface $S$. The surface integral is over all locations of the base point in $S$. Both observables may be constructed in a fixed representation of the group, which we take to be the fundamental one. It is readily verified that $T^1$ Poisson commutes with the constraints, and so is a fully gauge invariant observable. $T^0$ satisfies this trivially. $(T^1)$ is an integrated version of one of a series of partially gauge invariant observables for quantum gravity introduced in Ref. [11].

The Poisson algebra of the observables for the $SU(2)$ theory has a rather nice structure:

$$\{ T^0(\alpha), T^0(\beta) \} = 0,$$ \hspace{1cm} (18)

$$\{ T^0(\alpha), T^1(S, \beta) \} = i c(\alpha, S) \times \left[ T^0(\alpha \circ \beta) - T^0(\alpha \circ \beta^{-1}) \right].$$ \hspace{1cm} (19)

$$\{ T^1(\alpha, S), T^1(\beta, S') \} =$$

$$ic(\alpha, S') \left[ T^1(\alpha \circ \beta, S) - T^1(\alpha \circ \beta^{-1}) \right]$$

$$-ic(\beta, S) \left[ T^1(\beta \circ \alpha, S) - T^1(\beta \circ \alpha^{-1}) \right],$$ \hspace{1cm} (20)

where $\alpha \circ \beta$ etc. denote a product of holonomies for the respective loops.

Let us again consider the case $\Sigma \sim S^1 \times S^2$. The non-trivial observables on the reduced phase space are again specified by the non-contractible loops and surfaces in $\Sigma$: all the loops and surfaces that wrap once around the circle and the sphere respectively give equivalent non-trivial observables. Thus there is exactly one basic observable of each type, which we denote by $T^0(\alpha)$ and $T^1(\alpha, S)$, where $(\alpha, S)$ denote the circle and the sphere. The observable on the reduced phase space simplifies to

$$\{ T^0(\alpha), T^0(\alpha) \} = 0 = \{ T^1(\alpha, S), T^1(\alpha, S) \},$$

$$\{ T^0(\alpha), T^1(\alpha, S) \} = i \left[ T^0(\alpha^2) - T^0(\alpha \alpha^{-1}) \right].$$ \hspace{1cm} (21)

The last equation may be rewritten using the trace identity $\text{Tr}(A)\text{Tr}(B) = \text{Tr}(AB) + \text{Tr}(AB^{-1})$ for $SU(2)$ matrices $A, B$, and the fact that $T^0(\alpha \alpha^{-1})$ is the trace of the $2 \times 2$ identity matrix (since we are using the fundamental representation of $SU(2)$). This gives

$$\{ T^0(\alpha), T^1(\alpha, S) \} = i [T^0(\alpha)^2 - 4].$$ \hspace{1cm} (22)

A correspondence of this algebra with that of the harmonic oscillator is established by defining the creation and annihilation variables by

$$A^\pm = \frac{1}{\sqrt{2}} \left[ T^0 \pm \frac{T^1}{(T^0)^2 - 4} \right].$$ \hspace{1cm} (23)

It is possible to proceed similarly with other spatial topologies and gauge groups. The common feature in all examples is the expression of one or more copies of the oscillator variables as functions of the basic gauge invariant non-local observables of the topological field theory. The central and non-trivial property that permits this is that the observable algebra of the BF theories we have discussed not only closes, but is also simple enough to reveal the combinations of functions that are canonically conjugate.

Although of mathematical interest, the central question concerning dualities between background independent theories and conventional ones is what can be learned about the physics of one theory from the other. For the cases discussed here, it is fair to ask what one can learn about the harmonic oscillator from the topological field theory, beyond the identification of oscillator variables.

A first observation concerns the problem of time [12]. The topological examples provided here are fully background independent in the sense that there are no fixed asymptotic structures. This is unlike the AdS/CFT correspondences, where from the start the asymptotic isometry group of the spacetimes under consideration are prescribed, and carry with them a notion of time. The important physical difference between topological field theories and the harmonic oscillator is that the latter has
a non-vanishing Hamiltonian which gives non-trivial dynamics with respect to an external (Newtonian) time. On the other hand, the Hamiltonian of any theory with time reparametrisation invariance is a phase space constant so that time evolution is pure gauge, and gauge invariant observables are also constants of the motion. Therefore the Heisenberg evolution equations are trivially satisfied for the topological field theory observables. This means that one cannot "see" the dynamics of the oscillator in the topological theory, although the "number operator" of the latter is the dual of the oscillator Hamiltonian:

$$H_{osc} \leftrightarrow (a^\dagger a)_{tft} \quad (24)$$

In the cases where there is no classically fixed asymptotics, there are two known ways to introduce a time variable in a background independent theory: a classical time gauge fixing, or a relational time where a "slow" phase space variable is chosen as an internal clock, and evolution of other phase space variables are viewed with respect to it.

Without such choices, perhaps the best that can be done is a kinematic correspondence like the one presented here, where an equivalence is established between observable algebras up to time evolution on one side: the oscillator algebra of the operators $\hat{a}(t), \hat{a}^\dagger(t)$ at any value of Newtonian time maps to the timeless algebra of the gauge invariant observables of the topological theory. Equivalently one can say that the correspondence is at the level of the time independent Schrödinger equations of the two theories, where on the topological theory side this equation is $(a^\dagger a)_{tft} |\psi\rangle = E |\psi\rangle$. There is no non-trivial evolution equation for the fully gauge invariant variables, since these are (by definition) also constants of the motion.

The second observation concerns the cosmological constant, and arises from the issue of the matching of spectra. With the correspondence of Eqn. (24), the eigenvalues differ by a constant shift – the ground state energy of the oscillator. The number operator of the topological theory does not have an "energy" interpretation beyond that suggested by the correspondence. However it does raise an issue which is ultimately connected with the cosmological constant (or vacuum energy) problem.

The conventional vacuum energy problem arises in the context of quantum fields on a fixed background spacetime [12]. The association of matter vacuum energy with the cosmological constant is made using the semiclassical equation

$$G_{ab} + \Lambda^{(f)} g_{ab} = 8\pi G \langle \hat{T}_{ab} \rangle \quad (25)$$

where $\Lambda^{(f)}$ is a fundamental (or bare) cosmological constant. From this equation the predicted cosmological constant is given by

$$\Lambda^{(theory)} = \Lambda^{(f)} - 2\pi G \langle \hat{T}_{ab} \rangle g^{ab}, \quad (26)$$

where the expectation value is taken in some "vacuum" state. If the background metric has a timelike Killing vector field, there is a preferred matter Hamiltonian, and hence a vacuum. For dynamical metrics on the other hand, the additional assumption of a time gauge choice is needed to identify a Hamiltonian and its vacuum. Thus in general it is evident that $\Lambda^{(theory)}$ is dependent on the choice of time. With an assumed Planck scale cutoff and $\Lambda^{(f)} = 0$, the statement of the cosmological constant problem is the oft quoted discrepancy $13^{15}$ $\Lambda^{(theory)}/\Lambda^{(obs)} = 10^{120}$. If $\Lambda^{(f)} \neq 0$, this translates to a "fine tuning" problem.

To reformulate all this at a more fundamental level one needs a notion of time, and its associated non-vanishing quantum gravity Hamiltonian density $\hat{H}(\hat{q}, \pi; \phi, P_\phi; \Lambda^{(f)}, g_i; t)$. This is a function of the gravity $(g_{ab}, \pi^{ab})$ and matter $(\phi, P_\phi)$ operators, the cosmological constant, and other coupling constants $g_i$. Furthermore, it must have explicit dependence on a time variable $t$, however it arises from a fundamental background independent theory. (This is evident for example in the reduced Hamiltonians obtained by imposing various time gauge fixings in cosmological models.)

The task is to find ground state(s) $|g, \phi\rangle_0$, of this Hamiltonian and compute the vacuum energy. It is at this stage that there may be an emergent "cosmological constant problem" if the energy of the relevant state of $\hat{H}$ does not match the observed one, i.e. if it turns out that

$$0 \langle g, \phi | \hat{H} | g, \phi \rangle_0 \equiv \rho_0(\Lambda^{(f)}, g_i; t) \sim \rho^{(obs)} \quad (27)$$

requires fine tuning of $\Lambda^{(f)}$ and $g_i$ when the present value of time is inserted on the left hand side of this equation. Furthermore, since the expectation value has explicit time dependence, it is evident that to agree with observations, the observed value of vacuum energy density must not be a fixed constant.

What is apparent from these observations is that if one starts from a background independent gravity-matter theory, the problem of time must be solved before one can even ask if there is a cosmological constant problem.

In summary, what we have learned from the duality example given here is that, although it may be possible to establish exact dualities between a background independent theory and a background dependent one by presenting an operator dictionary, the challenge remains to establish a dynamical correspondence between physical processes. Both this, and a more fundamental quantum gravity based statement of the cosmological constant problem, require a solution of the problem of time in quantum gravity.

This work was supported in part by the Natural Science and Engineering Research Council of Canada. I would like to thank Laurent Freidel, Stefan Hofmann, Lee Smolin and Oliver Winkler for discussions.
[6] For some recent ideas on this topic see Laurent Freidel, Etera R. Livine "Ponzano-Regge model revisited III: Feynman diagrams and Effective field theory" (hep-th/0502106), and C. Rovelli, "Graviton propagator from background-independent quantum gravity," [gr-qc/0508124].