Axisymmetric stability criteria for a composite system of stellar and magnetized gaseous singular isothermal discs

Yu-Qing Lou\(^1\), Yue Zou\(^1\)

\(^1\)Physics Department and Tsinghua Center for Astrophysics (THCA), Tsinghua University, Beijing 100084, China;\n\(^2\)Centre de Physique des Particules de Marseille (CPPM) /Centre National de la Recherche Scientifique (CNRS) /Institut National de Physique Nucléaire et de Physique des Particules (IN2P3) et Université de la Méditerranée Aix-Marseille II, 163, Avenue de Luminy Case 902 F-13288 Marseille, Cedex 09, France;\n\(^3\)Department of Astronomy and Astrophysics, The University of Chicago, 5640 Ellis Ave, Chicago, IL 60637, USA;\n\(^4\)National Astronomical Observatories, Chinese Academy of Science, A20, Datun Road, Beijing 100012, China.

ABSTRACT
Using the fluid-magnetofluid formalism, we obtain axisymmetric stability criteria for a composite disc system consisting of stellar and gaseous magnetized singular isothermal discs (MSIDs). Both (M)SIDs are presumed to be razor-thin and are gravitationally coupled in a self-consistent axisymmetric background equilibrium with power-law surface mass densities and flat rotation curves. The gaseous MSID is embedded with a non-force-free coplanar azimuthal magnetic field \(B_\theta(r)\) of radial scaling \(r^{-1/2}\). Lou & Zou recently constructed exact global stationary configurations for both axisymmetric and nonaxisymmetric coplanar magnetohydrodynamic (MHD) perturbations in such a composite MSID system and proposed the MHD \(D_s\)–criteria for axisymmetric stability by the hydrodynamic analogy. In a different perspective, we derive and analyze here the time-dependent WKBJ dispersion relation in the low-frequency and tight-winding regime to examine axisymmetric stability properties. By introducing a rotational Mach number \(D_s\) for the ratio of the stellar rotation speed \(V_s\) to the stellar velocity dispersion \(a_s\), one readily determines the stable range of \(D_s^2\) numerically to establish the \(D_s\)–criteria for axisymmetric MSID stability. Those MSID systems rotating either too fast (ring fragmentation) or too slow (Jeans collapse) are unstable. The stable range of \(D_s^2\) depends on three dimensionless parameters: the ratio \(\lambda\) for the Alfvén speed to the sound speed in the gaseous MSID, the ratio \(\beta\) for the square of the stellar velocity dispersion to the gas sound speed and the ratio \(\delta\) for the surface mass densities of the two (M)SIDs. Our WKBJ results of (M)SID instability provide physically compelling explanations for the stationary analysis of Lou & Zou. We further introduce an effective MHD \(Q\) parameter for a composite MSID system and compare with the earlier work of Elmegreen, Jog and Shen & Lou. As expected, an axisymmetric dark matter halo enhances the stability against axisymmetric disturbances in a composite partial MSID system. In terms of the global star formation rate in a disc galaxy system, it would appear physically more sensible to examine the MHD \(Q_M\) stability criterion against galactic observations. Relevance to large-scale structures in disc galaxies are also discussed.

Key words: MHD waves — ISM: magnetic fields — galaxies: kinematics and dynamics — galaxies: spiral — star: formation — galaxies: structure.

1 INTRODUCTION

In contexts of galactic structures, disc stabilities and global star formation rates in spiral galaxies, we derive here instability criteria for axisymmetric coplanar magnetohydrodynamic (MHD) perturbations in a composite disc system with an azimuthal magnetic field in the gas disc component and establish a generalized definition of an effective MHD \(Q_M\) parameter appropriate to such a magnetized gravitational system. Formulated as such, this is an idealized and limited theoretical MHD disc problem yet with several key conceptual elements included. The simple physical rationale...
is that zonal regions of higher gas density and magnetic field are vulnerable to active formation of massive stars with various scales involved differing by many orders of magnitudes. On the same ground, non-axisymmetric stability criteria are equally important but are more challenging to establish (see e.g. Shu et al. 2000 for relevant issues). Over four decades, important development have been made for instability criteria relevant to galactic disc dynamics (see Lin 1987, Binney & Tremaine 1987 and Bertin & Lin 1996 and extensive references therein). The original studies of axisymmetric instabilities were conducted by Safronov (1960) and Toomre (1964) who introduced the dimensionless $Q$ parameter for the local stability (i.e. $Q > 1$) against axisymmetric ring-like disturbances. For disc galaxies, it would be more realistic and sensible to investigate a composite disc system consisting of a stellar disc, a magnetized gas disc and a massive dark matter halo. There have been extensive theoretical studies on this type of composite two-component disc systems of various sub-combinations (Lin & Shu 1966, 1968; Kato 1972; Jog & Solomon 1984a; Bertin & Romeo 1988; Romeo 1992; Elmegreen 1995; Jog 1996; Lou & Fan 1998b, 2000a, b; Lou & Shen 2003; Shen & Lou 2003, 2004a, b; Lou & Zou 2004; Lou & Wu 2005). In particular, there have been several studies trying to define a proper effective $Q$ parameter for local axisymmetric instability criterion in a composite disc system (Elmegreen 1995; Jog 1996; Lou & Fan 1998b, 2000a, b; Shen & Lou 2003, 2004a, b). From different perspectives, these analyses offer insights for instability properties of a composite disc system and provide a theoretical basis for understanding the large-scale dynamics in such a system (e.g. Lou & Fan 2000a, b; Lou & Shen 2003; Shen & Lou 2003, 2004a, b; Lou & Zou 2004; Lou & Wu 2005).

The main motivation here is to explore basic properties the MSID model in a composite system and obtain conceptual insights for astrophysical applications in magnetized spiral galaxies and in estimating global star formation rates in disc galaxies (e.g., Lou & Bian 2005). In theoretical studies of modeling disc galaxies, the class of SID models has a distinguished history since the pioneering work of Mestel (1963) (Zang 1976; Toomre 1977; Lemos, Kalnajs & Lynden-Bell 1991; Lynden-Bell & Lemos 1993; Syer & Tremaine 1996; Goodman & Evans 1999; Chakrabarti, Laughlin & Shu 2003) and has gained considerable attention and interests recently by considering a composite disc system and by incorporating effects of magnetic field (Shu et al. 2000; Lou 2002; Lou & Fan 2002; Lou & Shen 2003; Shen & Lou 2003, 2004a, b; Shen, Liu & Lou 2004; Lou & Zou 2004; Lou & Wu 2005). Specifically, Shu et al. (2000) see also Galli et al. (2001) studied global stationary (i.e., zero pattern speed) perturbation configurations in an isopedically magnetized SID without invoking the usual WKBJ or tight-winding approximation. They obtained exact global solutions for both aligned and unaligned axisymmetric and non-axisymmetric logarithmic spiral configurations and interpreted the axisymmetric solution for perturbations with radial propagations as demarcating the boundaries between the stable and unstable regimes. By these axisymmetric instabilities, a SID with a sufficiently slow rotation speed would jeans collapse induced by perturbations of larger radial scales, while a SID with a sufficiently fast rotation speed may suffer the ring fragmentation instability induced by perturbations of smaller radial scales (see fig. 2 of Shu et al. 2000). By introducing a rotational Mach number $D$, defined as the ratio of the SID rotation speed $V$ to the isothermal sound speed $a$, the critical values of the highest and lowest $D$ for an axisymmetric stability can be determined directly from the marginal stability curve. To support their physical interpretations, they invoked the well-known Toomre $Q$ parameter and found that the highest $D$, namely the minimum of the ring fragmentation curve, corresponds to a $Q$ value very close to unity, thus heuristically suggesting the correspondence between the $D$—criterion and the $Q$—criterion.

Different from yet complementary to the analysis of Shu et al. (2000) on a single isopedically magnetized disc, Lou (2002) studied global coplanar MHD perturbations in a single MSID embedded with an azimuthal magnetic field and revealed that the minimum of the MHD ring fragmentation curve in this MSID model is tightly associated with the generalized MHD $Q_M$ parameter originally introduced by Lou & Fan (1998a) in developing the galactic MHD density wave theory (Fan & Lou 1996). For a composite system of two coupled unmagnetized hydrodynamic SIDs, Lou & Shen (2003) constructed stationary global perturbation configurations and Shen & Lou (2003) suggested a straightforward $D$—criterion for the axisymmetric ring fragmentation instability in such a system on the basis of a low-frequency WKBJ analysis; they revealed that the minimum of the ring fragmentation in their composite SID model is again closely related to a proper effective $Q$ parameter (Elmegreen 1995; Jog 1996). Furthermore, for a composite SID system with an isopedically magnetized gaseous SID and a stellar SID in the fluid description, Lou & Wu (2005) have constructed global stationary MHD perturbation structures and examined stability properties to anticipate a similar $D$—criterion in parallel to the case of Shen & Lou (2003). Meanwhile, Shen & Lou (2004b) have further generalized both work of Syer & Tremaine (1996) and Lou & Shen (2003) to the situation of a composite system for two gravitationally coupled scale-free discs; they also studied the axisymmetric stability of such a composite system in terms of the marginal stability curves and proposed a $D_s$—criterion by the analogy of Shen & Lou (2003).

We have recently examined two-dimensional coplanar MHD perturbations in a composite system consisting of a stellar SID and a gaseous MSID. Both SIDs are expediently approximated as razor-thin and the gas disc is embedded with a non-force-free coplanar magnetic field (Lou & Zou 2004). In this fluid—magnetofluid MSID model approximation, we obtained exact global stationary MHD solutions for aligned and unaligned logarithmic spiral perturbation configurations in such a composite MSID system, expressed in terms of the stellar rotational Mach number $D_s$. In reference to the results of a single SID (Shu et al. 2000; Galli et al. 2001; Lou 2002), it would be natural to suggest that...
the stationary axisymmetric solutions with radial propagations give rise to marginal stability curves (see Figure 2 in this paper later and Lou & Zou 2004). In comparison with the single SID case, the stable range of \( D_s^2 \) is reduced as a result of the mutual gravitational coupling between the two SIDs. However, in comparison with the case of a composite unmagnetized SID system (Lou & Shen 2003; Shen & Lou 2003), the stable range of \( D_s^2 \) expands considerably due to the presence of a coplanar magnetic field.

To confirm heuristic arguments for the above analogy and our intuitive physical interpretations, we conduct in this paper a low-frequency time-dependent stability analysis in the WKBJ or tight-winding approximation for the composite SID system (Shen & Lou 2003, 2004a; Lou & Zou 2004). We shall demonstrate unambiguously the validity of demarcating the stable and unstable regimes by the stellar rotational Mach number \( D_s \). To place our analysis in proper contexts, we also discuss specifically how the two effective Q parameters for a composite MSID system and introduce a few pertinent to the ring fragmentation instability in a composite MSID system.

In Section 2, we derive the time-dependent dispersion relation using the WKBJ approximation for perturbations in a composite MSID system and introduce a few key dimensionless parameters. In Section 3, we present the \( D_s \)-criterion and two effective Q parameters for a composite MSID system being stable against arbitrary axisymmetric perturbations with radial propagations. Main results and discussions are summarized in Section 4.

2 A FLUID-MAGNETOFUID FORMALISM

We consider below a composite system consisting of two gravitationally coupled (M)SIDs with one of the SIDs being magnetized and thus referred to as MSID. For physical variables, we use either superscript or subscript \( g \) to indicate an association with the stellar SID and either superscript or subscript \( s \) to indicate an association with the gaseous MSID. The stellar SID and the gaseous MSID can have constant yet different rotational speeds \( V_s \) and \( V_g \) (related to the phenomenon of asymmetric drift in the galactic context); we thus write the background angular rotation speeds \( \Omega_s \) of the stellar SID and \( \Omega_g \) of the gaseous MSID as

\[
\Omega_s = V_s / r = a_s D_s / r
\]

and

\[
\Omega_g = V_g / r = a_g D_g / r
\]

separately, where \( a_s \) and \( a_g \) are the constant velocity dispersion of the stellar SID and the isothermal sound speed of the gaseous MSID, respectively; \( D_s \) and \( D_g \) are the corresponding rotational Mach numbers. The relevant epicyclic frequencies in terms of \( \Omega_s \) and \( \Omega_g \) are given by

\[
\kappa_s^2 \equiv 2 \frac{D_s}{r} \frac{d}{dr} (r^2 \Omega_s) = 2 \Omega_s^2
\]

and

\[
\kappa_g^2 \equiv 2 \frac{D_g}{r} \frac{d}{dr} (r^2 \Omega_g) = 2 \Omega_g^2
\]

respectively. Similar to a single SID, we take the background azimuthal magnetic field to be in the form of

\[
B_\varphi(r) = \mathcal{F} r^{-1/2}
\]

where \( \mathcal{F} \) is a constant (Lou 2002; Lou & Fan 2002) and

\[
B_r = B_\varphi = 0
\]

For a more general power-law radial variation of the azimuthal magnetic field and those of other related background variables, the interested reader is referred to a recent analysis of Shen, Liu & Lou (2005).

In the fluid approximation of a stellar SID, the mass conservation, the radial component of the momentum equation and the azimuthal component of the momentum equation are given below in order, namely

\[
\frac{\partial \Sigma^s}{\partial t} + \frac{1}{r} \frac{\partial (r \Sigma^s u^s)}{\partial r} + \frac{1}{r^2} \frac{\partial (\Sigma^s j^s)}{\partial \theta} = 0
\]

\[
\frac{\partial u^s}{\partial t} + u^s \frac{\partial u^s}{\partial r} + \frac{j^s}{r^2} \frac{\partial u^s}{\partial \theta} - \frac{j^2}{r^3} = - \frac{1}{\Sigma^s} \frac{\partial \Pi^s}{\partial r} - \frac{\partial \varphi}{\partial r}
\]

\[
\frac{\partial j^s}{\partial t} + u^s \frac{\partial j^s}{\partial r} + \frac{j^2}{r^2} \frac{\partial j^s}{\partial \theta} = - \frac{1}{\Sigma^s} \frac{\partial \Pi^s}{\partial \theta} - \frac{\partial \varphi}{\partial \theta}
\]

where \( u^s \) is the radial component of the bulk flow velocity of the stellar SID, \( j^s \equiv rv^s \) is the stellar specific angular momentum along the \( \hat{z} \) direction, \( v^s \) is the azimuthal component of the stellar bulk flow velocity, \( \varphi \) is the total gravitational potential, \( \Pi^s \) is the vertically integrated pressure, and \( \Sigma^s \) is the surface mass density of the stellar SID.

In the magnetofluid approximation for the gaseous MSID, the mass conservation, the radial component of the momentum equation and the azimuthal component of the momentum equation are given below in order, namely

\[
\frac{\partial \Sigma^g}{\partial t} + \frac{1}{r} \frac{\partial (r \Sigma^g u^g)}{\partial r} + \frac{1}{r^2} \frac{\partial (\Sigma^g j^g)}{\partial \theta} = 0
\]

\[
\frac{\partial u^g}{\partial t} + u^g \frac{\partial u^g}{\partial r} + \frac{j^g}{r^2} \frac{\partial u^g}{\partial \theta} - \frac{j^2}{r^3} = - \frac{1}{\Sigma^g} \frac{\partial \Pi^g}{\partial r} - \frac{\partial \varphi}{\partial r}
\]

\[
\frac{\partial j^g}{\partial t} + u^g \frac{\partial j^g}{\partial r} + \frac{j^2}{r^2} \frac{\partial j^g}{\partial \theta} = - \frac{1}{\Sigma^g} \frac{\partial \Pi^g}{\partial \theta} - \frac{\partial \varphi}{\partial \theta}
\]

where \( u^g \) is the radial component of the gas bulk flow velocity, \( j^g \equiv rv^g \) is the gas specific angular momentum in the \( \hat{z} \) direction, \( v^g \) is the azimuthal component of the gas bulk flow velocity, \( \Pi^g \) is the two-dimensional gas pressure, \( \Sigma^g \) is the gas surface mass density and \( B_r \) and \( B_\varphi \) are the radial and azimuthal components of the magnetic field \( \mathbf{B} \). The last two terms on the right-hand sides of equations (11) and (12) are dynamically coupled by the total gravitational potential \( \varphi \) through
the Poisson integral
\[ \varphi(r, \theta, t) = \int d\psi \int_0^\infty \frac{-G(\Sigma^g + \Sigma^i)\zeta d\zeta}{[\zeta^2 + r^2 - 2\zeta r \cos(\psi - \theta)]^{1/2}}. \] (13)

The gravitational effect of a massive dark matter halo is not included for the moment. The divergence-free condition for magnetic field \( \mathbf{B} = (B_r, B_\theta, 0) \) is simply
\[ \frac{\partial (r B_r)}{\partial r} + \frac{\partial B_\theta}{\partial \theta} = 0, \] (14)
and the radial and azimuthal components of the magnetic induction equation for gas motions are
\[ \frac{\partial B_r}{\partial t} = \frac{1}{r} \frac{\partial}{\partial \theta} \left( r^2 B_\theta - \psi ^ g B_r \right), \] (15)
\[ \frac{\partial B_\theta}{\partial t} = -\frac{\partial}{\partial r} \left( r^2 B_\theta - \psi ^ g B_r \right). \] (16)

Using Poisson integral (13), one readily derives the following expressions for the background surface mass densities
\[ \Sigma^g_0 = \frac{a^2_2 (1 + D^2_s)}{2\pi G r^2 (1 + \delta)} \] (17)
and
\[ \Sigma^i_0 = \frac{[a^2_2 (1 + D^2_s) - C^2_A/2]\delta}{2\pi G r^2 (1 + \delta)}, \] (18)
where \( \delta \equiv \Sigma^i_0/\Sigma^g_0 \) is the surface mass density ratio of the two dynamically coupled background (MSID) and \( C_A \) is the constant Alfvén wave speed in the MSID defined by
\[ C^2_A \equiv \int dz B^2_\theta / [4\pi \Sigma^g_0] \.] (19)

Apparently, equations (17) and (18) requires
\[ a^2_2 (1 + D^2_s) = a^2_2 (1 + D^2_s) - C^2_A/2 \] (20)
We now introduce two more useful dimensionless parameters here. The first parameter \( \beta \equiv a^2_2/a^2_o \) stands for the square of the ratio of the stellar velocity dispersion to the isothermal sound speed in the MSID and the second parameter \( \lambda^2 \equiv C^2_A/a^2_o \) stands for the square of the ratio of the Alfvén wave speed to the isothermal sound speed in the MSID. In late-type disc galaxies, the stellar velocity dispersion \( a_o \) is usually higher than the gas sound speed \( a_g \), we thus focus on the case of \( \beta \geq 1 \) (Jog & Solomon 1984a; b; Bertin & Romeo 1988; Jog 1996; Elmegreen 1995; Lou & Fan 1998b; Lou & Shen 2003; Shen & Lou 2003, 2004a, b; Lou & Wu 2005). Before going further, we note that a typical disc galaxy system involves a massive dark matter halo, a stellar disc and a gaseous disc of interstellar medium (ISM) on large scales, where the ISM disc is magnetized with the magnetic energy density \( \sim 1\text{eV/cm}^3 \) being comparable to the energy densities of thermal ISM and of relativistic cosmic-ray gas (e.g. Lou & Fan 2003). To comprehensively understand multi-wavelength observations of large-scale spiral structures of disc galaxies and to develop potentially powerful observational diagnostics (Lou & Fan 2000a, b), it would be more realistic and necessary to take into account of magnetic field effects in a composite magnetized disc-halo model.\textsuperscript{*} While there are exceptions, galactic magnetic fields typically tend to be coplanar with the disc plane of a spiral galaxy on large scales. On smaller scales, regions of closed and open magnetic fields are most likely intermingled by the solar analogy (e.g. Lou & Wu 2005). As a first step, Lou (2002) carried out a coplanar MHD perturbation analysis for stationary aligned and unaligned logarithmic spiral structures in a single MSID embedded with an azimuthal magnetic field and demonstrated that the minimum of the ring fragmentation curve in this MSID model is clearly related to the generalized MHD \( \lambda \) parameter (Lou & Fan 1998a). Since the background MHD rotational equilibrium adopted by Lou & Fan (1998a) is not an MSID model, it would be more satisfactory to justify the statement and interpretation of Lou (2002) in a dynamically self-consistent manner. We shall define a \( \lambda \) parameter similar to that of Lou (2002) and show that this \( \lambda \) is equivalent to that of Lou & Fan (1998a).

For a single MSID, we readily derive linear equations (by setting relevant parameters for the stellar SIDs to vanish) for coplanar axisymmetric MHD perturbations with Fourier harmonic dependence \( \exp(ikr + i\omega t) \), where \( k \) is the radial wavenumber and \( \omega \) is the angular frequency. In the usual WKBJ or tight-winding approximation of \( kr \gg 1 \), we obtain the local WKBJ dispersion relation for MHD density waves propagating in an MSID in the form of
\[ \omega^2 = \kappa^2 + k^2 (a^2_o + C^2_A) - 2\pi G |k| \Sigma^g_0, \] (21)
which is the MHD generalization of the WKBJ dispersion relation for coplanar perturbations in an unmagnetized SIDs.

To derive an effective \( Q_M \) parameter for the axisymmetric stability (i.e., \( \omega^2 \geq 0 \)) against MHD perturbations with an arbitrary \( k \), the determinant of the right-hand side of equation (21) should be negative for all \( k \). This requires
\[ Q_M \equiv \frac{\kappa^2 (a^2_o + C^2_A)^{1/2}}{2\pi G \Sigma^g_0} > 1, \] (22)
where \( Q_M \) is an generalized Toomre’s \( Q \) parameter for a single MSID. Inequality (22) is in the form of the effective \( Q \) parameter derived by Lou & Fan (1998a) for fast MHD density waves in a rigidly rotating gas disc with a different formalism but with the same assumption \( kr \gg 1 \). This result establishes the connection found by Lou (2002) that the minimum of the ring fragmentation curve is tightly associated with the \( Q_M \) parameter of Lou & Fan (1998a).

In our low-frequency time-dependent analysis for a composite MSID system, we write coplanar axisymmetric MHD perturbations with the same harmonic dependence of \( \exp(ikr + i\omega t) \). In the WKBJ limit of \( kr \gg 1 \), we obtain the local WKBJ dispersion relation for a composite MSID system in the form of
\[ (\omega^2 - \kappa^2 - k^2 a^2_o + 2\pi G |k| \Sigma^g_0) \times [\omega^2 - \kappa^2 - k^2 (a^2_o + C^2_A) + 2\pi G |k| \Sigma^g_0] \]
\[ = (2\pi G |k| \Sigma^g_0)(2\pi G |k| \Sigma^g_0). \]
(23)

As expected, this dispersion relation explicitly shows the singularity. Relativistic cosmic-ray electrons gyrating around galactic magnetic field give off the observed synchrotron radiation).
mutual gravitational coupling between the two SIDs by the term on the right-hand side. The first factor on the left-hand side is for perturbations in the stellar SID approximated as a fluid, while the second factor is for coplanar MHD perturbations in the gaseous MSID [see expression (19)]. It should be emphasized that while dispersion relation (23) is a local one, the background physical variables are in rotational MHD equilibrium in a consistent manner. In particular, $\Sigma_0$ given by expression (18) contains the information of background magnetic field via $C_A^2$.

As noted by Lou & Shen (2003), the dispersion relation derived here for coplanar MHD perturbations in a composite (M)SID approach is qualitatively similar to those previously obtained by Jog & Solomon (1984), Elmegreen (1995) and Jog (1996) in spirit, but one major distinction is that in the formulation of our composite (M)SID, the rotation speeds of the two SIDs $V_s$ and $V_g$ are different in general. We thus have inequality $\kappa_s \neq \kappa_g$, while in those earlier analyses, $\kappa_s = \kappa_g$ was presumed a priori following the assumption of $V_s = V_g$.

This is conceptually related to the phenomenon of asymmetric drift (e.g. Binney & Tremaine 1987). Physically, stellar velocity dispersions mimic a pressure-like force for the stellar component, while the thermal gas pressure and magnetic pressure forces together act on the magnetized gas component. In the same gravitational potential well determined by the total mass distribution, the difference in the stellar pressure-like force and the sum of the gaseous and magnetic pressure forces would lead to different $V_s$ and $V_g$ and thus the asymmetric drift. The rare situation of $V_s = V_g$ being equal may happen under very special circumstances.

### 3 Axesymmetric Stability Analysis for a Composite MSD System

We describe below coplanar MHD perturbation analysis for the axisymmetric stability of a composite MSID based on a low-frequency time-dependent WKBJ approach. Generalizing the notations of Shen & Lou (2003) yet with the effect of magnetic field included, we here define

$$H_1 \equiv \kappa^2 + k^2 a_s^2 - 2\pi G|k|\Sigma^0$$

$$H_2 \equiv \kappa^2 + k^2(a_s^2 + C_A^2) - 2\pi G|k|\Sigma^0$$

$$G_1 \equiv 2\pi G|k|\Sigma^0$$

$$G_2 \equiv 2\pi G|k|\Sigma^0$$

where $\Sigma_0$ and $\Sigma_g$ are given by equations (14) and (18). In addition to the appearance of $C_A^2$ in equation (25), $\Sigma_g$ given by expression (18) also contains the magnetic field effect. Dispersion relation (23) can be cast into the form of

$$\omega^4 - (H_1 + H_2)\omega^2 + (H_1H_2 - G_1G_2) = 0$$

with the two roots $\omega_+^2$ and $\omega_-^2$ given by

$$\omega_+^2(k) = \frac{a^2}{2\pi^2}(A_2K^2 + A_1K + A_0 - \varphi^{1/2})$$

where

$$A_2 \equiv 1 + \beta + \lambda^2/\beta$$

$$A_1 \equiv -(1 + y)$$

$$A_0 \equiv 2 + 4y + (\lambda^2 - 2)/\beta$$

$$\varphi \equiv B_4K^4 + B_3K^3 + B_2K^2 + B_1K + B_0$$

$$B_4 \equiv [1 - (1 + \lambda^2)/\beta]^2$$

$$B_3 \equiv 2(1 + y)(1 - 1/\beta - 1)(1 + \lambda^2/\beta)/(1 - \delta)$$

$$B_2 \equiv [y^2 + 2y^3 + 3(8\beta - 4 - 2\lambda^2 + 2\lambda^2 + 2\beta\lambda^2)/\beta^2]$$

$$B_1 \equiv 4(1 + y)(1 - \delta)(1 - 1/\beta + \lambda^2/(2\beta))/(1 + \delta)$$

$$B_0 \equiv 4[1 - 1/\beta + \lambda^2/(2\beta)]^2$$

where $\gamma \equiv D_0^2$. The analysis here parallels that of Shen & Lou (2003); the novel magnetic field effect to be explored is contained in the dimensionless parameter $\lambda^2$.

As a result of the one-to-one correspondence between $D_s^2$ and $D_g^2$ dictated by expression (20), it is straightforward to derive an equivalent form of expression (30) in terms of $D_g^2$ instead of $D_s^2$. Mathematical solutions of $D_s^2$ and $D_g^2$ become unphysical for either $D_s^2 < 0$ or $D_g^2 < 0$ or both being negative. One can readily show from condition (20) that $D_s^2 < D_g^2$ for $\beta \geq 1$ (Lou & Zou 2004). Therefore, it suffices to require $D_s^2 > 0$. In the subsequent analysis, we mainly use $D_g^2$ – the square of the rotational Mach number – to examine the axisymmetric stability property in a composite MSID system.

By setting $\lambda^2 = 0$ for zero magnetic field in expressions (30)–(39), they all reduce to the corresponding expressions for a composite system of two coupled unmagnetized SIDs analyzed by Shen & Lou (2003). For scale-free discs more general than SIDs, the reader is referred to the work of Syer...
& Tremaine (1996), Shen & Lou (2004a, b) and Shen, Liu & Lou (2005). To derive an effective MHD $Q_M$ parameter for a composite disc system of one SID and one gaseous MSID, we must determine the value of $K_{\text{min}}$ at which $\omega^2_\alpha$ reaches the minimum value.

### 3.1 The $D_2^s$—Criterion in the WKBJ Regime

Before defining an effective $Q_M$ parameter, we first show unambiguously the $D_2$—criterion for axisymmetric stability and confirm our earlier interpretations for the marginal stability curves in a composite MSID system (Lou & Zou 2004).

According to solution (39), $\omega^2_\alpha$ is a function of $K$ and $D_2^s$. By setting $\omega^2_\alpha = 0$ and assigning values of parameters $\delta$, $\beta$ and $\lambda^2$ in solution (39), we end up with an equation for $D_2^s$ and $K$. Contours of $\omega^2_\alpha$ in $D_2^s$ and $K$ with various combinations of $\delta$, $\beta$ and $\lambda^2$ are given below to compare with our marginal stability curves obtained earlier (Lou & Zou 2004; see also Shen & Lou 2003, 2004a, b, Shen, Liu & Lou 2005 and Lou & Wu 2005).

As an example of illustration, we set $|m| = 0$, $\delta = 0.2$, $\beta = 1.5$ and $\lambda^2 = 1$ and determine numerically contour curves of $\omega^2_\alpha$ in terms of $D_2^s$ and $K$ as displayed in Fig. 1. Physically, the two regions labelled $\omega^2_\alpha < 0$ in the lower-left and upper-right corners are unstable, while the region labelled by $\omega^2_\alpha > 0$ is stable against axisymmetric coplanar MHD perturbations. For comparison, we show the global marginal stability curve in a composite MSID system with the same parameter values in Fig. 2 (figure 11 of Lou & Zou 2004), where $\alpha$ is a dimensionless effective radial wavenumber (see Shu et al. 2000; Lou 2002; Lou & Shen 2003; Lou & Zou 2004). In the WKBJ approximation of large $K$ and $\alpha$, the two upper-right solid curves in Figs. 1 and 2 show good mutual correspondence for the ring fragmentation instability. Thus our previous interpretation for the global stationary axisymmetric MHD perturbation configuration as the marginal stability curve is confirmed by the WKBJ analysis here. In comparison, the correspondence in the small $K$ regime is qualitative with apparent deviations; the WKBJ approximation works better for a local analysis, while the results of Fig. 2 are global and exact without the WKBJ approximation. It is clear that this comparison reveals the physical nature of the demarcation curves as the axisymmetric stability boundaries.

![Figure 1. A contour plot of $\omega^2_\alpha$ as a function of $K$ and $D_2^s$ with $|m| = 0$, $\delta = 0.2$, $\beta = 1.5$ and $\lambda^2 = 1$. The two separated regions labelled $\omega^2_\alpha < 0$ in the lower-left and upper-right corners are unstable. The two solid curves mark $\omega^2_\alpha = 0$.](image1)

![Figure 2. The global marginal stability curve of $D_2^s$ versus effective dimensionless radial wavenumber $\alpha$ for $|m| = 0$, $\delta = 0.2$, $\beta = 1.5$ and $\lambda^2 = 1$ [see figure 11 of Lou & Zou (2004)].](image2)

![Figure 3. A contour plot of $\omega^2_\alpha$ as a function of $K$ and $D_2^s$ with $|m| = 0$, $\delta = 0.2$, $\beta = 1$ and $\lambda^2 = 0.09$. The two separated domains labelled by $\omega^2_\alpha < 0$ are unstable. The two solid curves mark $\omega^2_\alpha = 0$.](image3)
The role of coplanar ring magnetic field in stabilizing a composite MSID system can be physically understood as follows. According to dispersion relation (23), we see that, in the MSID factor, the gas and magnetic pressure terms are explicitly associated with the square of the radial wavenumber \(|k|\), while the background surface mass density term is linear in \(|k|\). For the ring fragmentation instability that occurs at relatively large radial wavenumbers, the two pressure terms dominate and the increase of magnetic pressure tends to enhance the axisymmetric stability of a composite MSID system. For the Jeans collapse instability that occurs at relatively small wavenumbers, the background gas surface mass density term in the MSID factor becomes dominant over the two pressure terms. By the rotational MHD radial force balance condition (18), the background gas surface mass density tends to be reduced by the increase of magnetic field strength and thus the Jeans collapse instability tends to be suppressed. In addition, the right-hand side of dispersion relation (23) represents the mutual gravitational coupling in the presence of coplanar MHD perturbations. A reduction of background gas surface mass density will weaken this coupling and thus increase the axisymmetric stability.

For a further comparison, we reproduce the global marginal stability results of figure 13 in Lou & Zou (2004) here as Fig. 6, which has the same set of parameters as Fig. 5 for the local WKBJ solution results. We note again that for the unstable region (upper right) of large radial wavenumber, labelled as the ring fragmentation in Lou & Zou (2004), Figs. 5 and 7 show very good correspondence as expected. As reference, Table 1 contains several lists for the overall stable range of \(D_2^2\) with \(|m| = 0\) and different sets of parameters \(\delta\), \(\beta\) and \(\lambda^2\), including the results both from here using the WKBJ approximation and from those of Lou & Zou (2004) for exact global MHD perturbation calculations in a composite MSID model.

As shown above, axisymmetric stability properties of a composite MSID system can be qualitatively understood using the WKBJ analysis. Nevertheless, the WKBJ approxi-
mation gradually becomes inaccurate in dealing with smaller radial wavenumbers of the Jeans collapse regime. Once the physical interpretation has been firmly established, the exact global marginal perturbation solution procedure of Lou & Zou (2004) should be adopted to identify the relevant stable range of $D_2$ in a composite (M)SID system.

### 3.2 MHD Generalizations of Effective $Q$ Parameters in a Composite MSID System

For the axisymmetric stability of a single disc, the familiar $Q$ parameter (Safronov 1960; Toomre 1964) provides a simple local criterion $Q > 1$. For a single magnetized disc with a coplanar magnetic field, the MHD generalization of this local criterion for the axisymmetric stability becomes $Q_{AM} > 1$ (Fan & Lou 1996; Lou & Fan 1998a). For a composite system of two gravitationally coupled discs without magnetic field, it would be natural to seek an extension of such a local criterion for the axisymmetric stability. Such a local criterion does exist but to find a relatively simple analytical expression is not as easy (Elmegreen 1995; Jog 1996), the local background variables are prescribed a priori so that the two disc rotation speeds and thus the two epicyclic frequencies are the same. In contrast, in the analyses of Lou & Shen (2003) and Shen & Lou (2003), the equilibrium background variables are determined in a dynamically consistent manner so that the two SID rotation speeds and thus the two epicyclic frequencies are allowed to be different in general. In reference to the local criteria of Elmegreen (1995) and Jog (1996), the global $D_2$ criterion of Lou & Shen (2003) for the axisymmetric stability in a composite SID system is more straightforward and precise (Shen & Lou 2003). The main thrust of this section is to discuss and establish the global $D_2$ criterion for the axisymmetric stability in a composite MSID system with a coplanar magnetic field through comparisons (Lou & Zou 2004).

#### 3.2.1 MHD Extension of $Q_E$ Parameter of Elmegreen

Paralleling the procedures of Elmegreen (1995) and of Shen & Lou (2003), we may define an effective $Q_E$ parameter as the MHD extension of the $Q_{E'}$ parameter of Elmegreen for a composite system of MSIDs. From equation (30), the minimum of $\omega_2^2$ is given by

$$\omega_{2\text{-min}}^2 = \frac{a^2_2}{2} \left[ A_2 K_{min}^2 + A_1 K_{min} + A_0 - \varphi^{1/2} \right]$$

$$= \frac{a^2_2}{2} \left( \varphi^{1/2} - A_2 K_{min}^2 - A_1 K_{min} \right) (Q_E^2 - 1), \tag{40}$$

where

$$Q_E^2 \equiv \frac{A_0}{(\varphi^{1/2} - A_2 K_{min}^2 - A_1 K_{min})}, \tag{41}$$

and $\varphi$ defined by equation (41) takes on the value at $K_{min}$. Although the forms of these mathematical expressions are strikingly similar to those of Elmegreen (1995), all relevant coefficients and variables contain the effect of magnetic field through $\lambda^2$. By definition (41) of $Q_E^2$ above, it is clear that for $Q_E^2 > 1$, the minimum of $\omega_2^2 > 0$ and thus the composite MSID system would be stable against axisymmetric coplanar MHD perturbations of arbitrary radial wavelengths. This generalized MHD parameter $Q_E^2$ corresponds to the stable range of $D_2$ where $D_2$ is the rotational Mach number of the stellar SID modelled as a fluid.

The formidable appearance of $\omega_2^2$ given by equation (40) would prevent us from deriving a straightforward analytical expression of $K_{min}$. Nonetheless, instead of minimizing $\omega_2^2$ directly, it is much simpler to determine the critical value $K_c$ for $K$ corresponding to the minimum of variable $W \equiv \omega_2^2 \omega_4^2$. By equation (25), we have

$$W \equiv \omega_2^2 \omega_4^2 = H_1 H_2 - G_1 G_2. \tag{42}$$

For possible extrema of $W$, the relevant cubic equation that $K_c$ should satisfy is

$$dW \overline{dK} = d(H_1 H_2 - G_1 G_2) \overline{dK} = 0, \tag{43}$$

or equivalently

$$dK^3 + aK^2 + bK + c = 0, \tag{44}$$

with the four coefficients explicitly defined by

$$d \equiv 4 \left( 1 + \lambda^2 \right) (\delta + 1), \tag{45}$$

$$a \equiv -3 \left( \beta \delta + \lambda^2 + 1 \right) (y + 1), \tag{46}$$

$$b \equiv 2 (\delta + 1) \left( 2 \beta y - 2 + \lambda^2 + 2 \beta + 2 y \lambda^2 + 2 y \right), \tag{47}$$

$$c \equiv -(y + 1) \left( 2 \beta y + 2 \beta y \delta - 2 + \lambda^2 + 2 \beta \right). \tag{48}$$

For most parameter regimes under consideration, there is only one real solution for the cubic equation (45). This real solution $K_c$ takes the lengthy but straightforward form of

$$K_c = (x - q/2)^{1/3} + (-x - q/2)^{1/3} - a/(3d), \tag{49}$$

where

$$x \equiv (q^2/4 + p^2/27)^{1/2}, \quad p \equiv (b/d) - (a/d)^2/3 \quad \text{and} \quad q \equiv 2(a/d)^3/27 - ab/(3d^2) + c/d. \quad \text{We then use this } K_c \text{ to estimate } K_{min} \text{ and to determine the effective MHD } Q_E \text{ parameter as the generalization of } Q_{E'}. \text{ Relevant curves of MHD } Q_E^2 \text{ versus } D_2^2 \text{ corresponding to different values of } \delta, \beta \text{ and } \lambda^2 \text{ are displayed in Figures 3-11.}$$
By varying parameters $\delta$, $\beta$, and $\lambda^2$, we observe several trends of variation in the profile of MHD $Q_E$ versus $D_s^2$ as displayed in Figs. 8 and 9. When $\beta$ increases with fixed $\delta$ and $\lambda^2$ values, or $\delta$ increases with fixed $\beta$ and $\lambda^2$ values, the stable range of $D_s^2$ shrinks in general, while the increase of $\lambda^2$ tends to expand the stable range of $D_s^2$. Similar variation trends have been noticed earlier for the $D_s$—criterion and the exact global perturbation solutions in our composite MSID model (Lou & Zou 2004).

By definition (11) for the MHD $Q_E$ parameter, one determines the stable range of $D_s^2$ in the WKBJ approximation as shown in Figures 8 and 9. For example, given $\delta = 0.2$, $\beta = 1.5$ and $\lambda^2 = 1$ in Fig. 8 we have $Q_E^2 > 1$ for $0.12 < D_s^2 < 6.34$; the composite MSID system is thus stable against axisymmetric coplanar MHD perturbations within this range of $D_s^2$. More numerical results for the stable ranges of $D_s^2$ for different parameters are summarized in Table 2.

Table 2. Approximate stable ranges of $D_s^2$ determined by the criterion $Q^2 \geq 1$ in a composite system of (M)SIDs. The $D_s^2$ values outside the parentheses are given by the MHD $Q_E$ parameter generalizing that of Elmegreen (1995), while the $D_s^2$ values inside the parentheses are derived from the MHD $Q_J$ parameter generalizing that of Jog (1996). These two sets of $D_s^2$ values corresponding to MHD $Q_E$ and MHD $Q_J$ parameters are nearly the same and the stable $D_s^2$ ranges are very close to the values given by our exact global $D_s$—criterion using the MHD perturbation procedure of Lou & Zou (2004; see also Tab. 1 here).

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\beta$</th>
<th>$\lambda^2$</th>
<th>lower limit of $D_s^2$</th>
<th>upper limit of $D_s^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1</td>
<td>1</td>
<td>0.12 (0.12)</td>
<td>6.34 (6.34)</td>
</tr>
<tr>
<td>0.2</td>
<td>1</td>
<td>0.09</td>
<td>0.16 (0.16)</td>
<td>5.96 (5.96)</td>
</tr>
<tr>
<td>0.2</td>
<td>1</td>
<td>3.61</td>
<td>0.11 (0.11)</td>
<td>7.95 (7.95)</td>
</tr>
<tr>
<td>0.2</td>
<td>10</td>
<td>1</td>
<td>0.12 (0.12)</td>
<td>4.55 (4.55)</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>1</td>
<td>0.04 (0.04)</td>
<td>2.28 (2.28)</td>
</tr>
</tbody>
</table>

By inspection and comparison, the stable $D_s^2$ ranges are almost the same as those derived from our $D_s$—criterion as expected (see Table 1). In comparison with the stable $D_s^2$ ranges derived from the exact global axisymmetric perturbation solutions of Lou & Zou (2004), we see again that the results obtained by the two procedures have very good correspondence in the larger wavenumber regime, while in the Jeans collapse regime of smaller radial wavenumbers, the WKBJ approximation leads to apparent deviations. The generalized instability criterion characterized by the MHD $Q$ parameter is physically relevant to the MHD ring fragmentation instability in a composite disc system of gravitationally coupled SID and MSID.

Alternatively, we may follow a similar procedure of Lou & Zou (2004) to show the tight correspondence between the marginal stability curve and the effective MHD $Q$ parameter. For specific values of parameters $\delta$, $\beta$ and $\lambda^2$, we calculate the minimum of the ring fragmentation curve for $D_s^2$ using the procedure of Lou & Zou (2004). Inserting the re-
resulting \( D_s^2 \) into expression (41), we readily obtain the value of MHD \( Q^2_D \) parameter corresponding to the minimum of the ring fragmentation curve. For example, given \( \delta = 0.2 \), \( \beta = 1.5 \) and \( \lambda^2 = 1 \) in figure 11 of Lou & Zou (2004) (or Fig. 2 here), the minimum of \( D_s^2 \) in the ring fragmentation curve is about 6.1554. Using definition (11), we obtain the corresponding MHD \( Q^2_D = 1.0216 \). More numerical results are summarized in Table 2.

In all these cases, the relevant values of MHD \( Q^2_D \) are fairly close to unity. Therefore the effective MHD \( Q \) parameter \( Q_D \) is indeed closely relevant to the MHD ring fragmentation instability for axisymmetric coplanar MHD perturbations in a composite MSID system.

3.2.2 MHD Extension of \( Q_J \) Parameter of Jog

We have seen that the MHD \( Q_E \) parameter generalizing the \( Q_E \) of Elmegreen (1995) is pertinent to the MHD ring fragmentation perturbation in a composite MSID system. However, the determination of the MHD \( Q_E \) parameter may occasionally become cumbersome when equation (37) has three real roots of \( K \). It would be tedious to identify the absolute minimum of \( \omega^2 \). Alternative to this procedure, Jog (1996) adopted a seminumerical approach to define another effective \( Q \) parameter for an unmagnetized composite disc system, referred to as \( Q_J \), here. For comparison, Shen & Lou (2003) have used a generalized procedure to derive the \( Q_J \) parameter for a composite SID system with a self-consistent background equilibrium allowing two different SID rotation speeds but in the absence of magnetic field. We shall introduce below the MHD \( Q_J \) parameter for a composite (M)SID system with a coplanar azimuthal magnetic field and compare these results with those of the MHD \( Q_E \) parameter.

According to the minus-sign solution (29), \( \omega^2 \) becomes positive and negative depending on whether \( H_1H_2 - G_1G_2 > 0 \) and \( < 0 \), respectively. The critical condition for the axisymmetric stability is thus characterized by \( H_1H_2 - G_1G_2 = 0 \) which can be rearranged into the form of

\[
\frac{2\pi G k \Sigma_0^0}{|\kappa_x^2 + k^2(\alpha_x^2 + C_A^2)|} + \frac{2\pi G k \Sigma_0^0}{|\kappa_z^2 + k^2\alpha_z^2|} = 1. \tag{50}
\]

Note that the form of the expression here is strikingly similar to that of Jog (1996) with the magnetic terms being both explicit in the expression and implicit in \( \Sigma_0^0 \) through \( C_A^2 \),

![Figure 11. Several curves of the MHD \( Q^2_J \) versus \( D_s^2 \) for different \( \beta \) values. Other relevant parameters \( |m| = 0 \), \( \delta = 0.2 \) and \( \lambda^2 = 1 \). For each MHD \( Q^2_J \) curve, the two intersection points at \( Q_J^2 = 1 \) bracket the stable range of \( D_s^2 \). This stable \( D_s^2 \) range shrinks as \( \beta \) increases.](image)

With magnetic field, we define a new variable \( \mathcal{F} \) by

\[
\mathcal{F} \equiv \frac{2\pi G k \Sigma_0^0}{|\kappa_x^2 + k^2(\alpha_x^2 + C_A^2)|} + \frac{2\pi G k \Sigma_0^0}{|\kappa_z^2 + k^2\alpha_z^2|} = \frac{2(\beta + 1 + y) - 1 + \lambda^2/2 + K^2(1 + \lambda^2)}{(2y + K^2)} + \frac{K(1 + y)/(1 + \delta)}{(2y + K^2)}, \tag{51}
\]

where \( y \equiv D_s^2 \) and \( K \equiv |k|r \). We then numerically search for \( K_{min} \) at which \( \omega^2 \) reaches the minimum value. We now define an effective MHD \( Q_J \) parameter such that

\[
\frac{2}{(1 + Q_J^2)} = \frac{2}{(2y + K_{min})} + \frac{K_{min} \beta(1 + y)/(1 + \delta)}{(2y + K_{min})}, \tag{52}
\]

[see equations (5) and (6) of Jog (1996) and equation (30) of Shen & Lou (2003)]. It follows immediately that \( Q_J^2 > 1 \) and \( Q_J^2 < 1 \) correspond to axisymmetric MHD stability and instability, respectively. Given \( Q_J^2 > 1 \), it indeed follows that \( H_1H_2 - G_1G_2 > 0 \) for the \( \omega^2 \) corresponding to \( K_{min} \). That is, the minimum of \( \omega^2 \) is positive for arbitrary \( K \) and consequently, the composite (M)SID system is stable against axisymmetric coplanar MHD disturbances. The procedure of obtaining the MHD \( Q_J \) parameter can be summarized as follows. For a given set of \( \delta \), \( \beta \), \( \lambda^2 \) and \( D_s^2 \), one first determines the value of \( K_{min} \) numerically using equation (52). By inserting this \( K_{min} \) into equation (29) one then obtains the numerical value of MHD \( Q_J^2 \) in a composite MSID system for the given set of \( \delta \), \( \beta \), \( \lambda^2 \) and \( D_s^2 \).

We have explored relevant parameter regimes of interest and revealed several qualitative variation trends of MHD \( Q_J^2 \) parameter as shown in Figures 11 and 12. Similar to the MHD \( Q_E \) parameter, the range of \( D_s^2 \) for MHD \( Q_J^2 > 1 \) corresponds to the axisymmetric stability of a composite MSID system. Corresponding to MHD \( Q_J^2 > 1 \), we have calculated stable ranges of \( D_s^2 \) given several sets of \( \delta \), \( \beta \) and \( \lambda^2 \) and the detailed results are summarized in Table 2 along...
Figure 13. Several curves of MHD $Q^2_J$ versus $D_s^2$ for different $\delta$ values. Other relevant parameters $|m| = 0$, $\beta = 10$ and $\lambda^2 = 1$ are fixed. For each MHD $Q^2_J$ curve, the two intersection points at $Q^2_J = 1$ bracket the stable range of $D_s^2$. This stable $D_s^2$ range expands as $\delta$ increases.

![Figure 13](image)

with those corresponding to the MHD $Q^2_E$ parameter. It is apparent that the stable range of MHD $D_s^2$ given by MHD $Q^2_J \geq 1$ is almost the same as that given by MHD $Q^2_E \geq 1$. Meanwhile, in Figures [14] and [15] we reveal the same variation trends as those obtained by MHD $Q^2_E$ and by exact global MHD perturbation procedure of Lou & Zou (2004). When $\beta$ increases with fixed $\delta$ and $\lambda^2$ values (see Fig. [14]), or $\delta$ increases with fixed $\beta$ and $\lambda^2$ values (see Fig. [15]), the stable range of $D_s^2$ tends to shrink, while the increase of $\lambda^2$ tends to expand the stable range of $D_s^2$ (see Fig. [16]).

Once again, we follow the similar procedure of Lou & Zou (2004) to reveal the close relationship between the minimum of the $D_s^2$ marginal stability curve and the effective MHD $Q$ parameter. For specified values of parameters $\delta$, $\beta$ and $\lambda^2$, we compute the minimum of the ring fragmentation $D_s^2$ curve using the exact global MHD perturbation procedure of Lou & Zou (2004), insert the resulting $D_s^2$ into expression (52) and obtain the value of MHD $Q^2_J$ parameter corresponding to the minimum of the ring fragmentation curve. For example, given $\delta = 0.2$, $\beta = 1.5$ and $\lambda^2 = 1$ in figure 11 of Lou & Zou (2004) (or Figure 2 here), the minimum of $D_s^2$ in the ring fragmentation curve is $\sim 6.1554$. Using definition (52), we obtain the corresponding MHD $Q^2_J = 1.02223$. More detailed numerical results are summarized in Table 3 for reference.

By these numerical experiments, we demonstrate that the values of MHD $Q^2_J$ and MHD $Q^2_E$ corresponding to the minima of $D_s^2$ ring fragmentation curves are nearly the same, with the relevant values of MHD $Q^2_J$ being very close to unity. Therefore, the effective MHD $Q^2_J$ parameter is also pertinent to the MHD ring fragmentation instability for axisymmetric coplanar MHD perturbations in a composite MSID system.

In comparison to the determination of MHD $Q^2_E$, the search for the MHD $Q_J$ value requires numerical explorations for each given $D_s^2$. The major advantage is that the definition (52) for MHD $Q^2_J$ remains valid for the entire parameter regime and avoids improper situations of unusual sets of $\delta$, $\beta$ and $\lambda^2$.

3.3 A Composite Partial MSID System

In most disc or spiral galaxies, there are overwhelming observational evidence for the existence of massive dark matter haloes in general. To include the large-scale gravitational effect of a massive dark matter halo, we add a gravitational potential $\Phi$ term associated with the dark matter halo in our basic MHD equations (5), (9), (11) and (12), where $\Phi$ is presumed to be axisymmetric for simplicity and for the lack of information. Based on $N$-body numerical simulations for galaxy formation, typical velocity dispersions of dark matter ‘particles’ are fairly high (more than a few hundred kilometers per second). Hence, another major simplification of our analysis is to ignore perturbation responses of the massive dark matter halo to coplanar MHD perturbations in a composite MSID system (e.g., Syer & Tremaine 1996; Shu et al. 2000; Lou 2002; Lou & Fan 2002; Lou & Shen 2003; Shen & Lou 2004a, b). As before, we introduce a dimensionless ratio $F = \varphi/|\varphi + \Phi|$ for the fraction of the disc potential relative to the total potential all in a background equilibrium state (e.g., Syer & Tremaine 1996; Shu et al. 2000; Lou 2002; Lou & Shen 2003; Lou & Zou 2004). The background rotational MHD equilibrium of a composite MSID is thus strongly modified by this additional $\Phi$ term. As before, we write $\Omega_s = a_sD_s/r$, $\Omega_b = a_bD_b/r$, $\kappa_s = \sqrt{2}\Omega_s$, and $\kappa_b = \sqrt{2}\Omega_b$. It follows from the radial force balance in the (M)SID system that the background surface mass densities now become

$$
\Sigma_0^s = F\frac{a_s^2(1 + D_s^2)}{2\pi Gr(1 + \delta)}, \tag{53}
$$

$$
\Sigma_0^b = F\frac{a_b^2(1 + D_b^2) - C_b^3/2\delta}{2\pi Gr(1 + \delta)}, \tag{54}
$$
where $0 \leq F < 1$ for a partial composite MSID and $F = 1$ for a full composite MSID that has been studied in detail in the previous subsections. Performing the standard MHD perturbation analysis, we linearize dependent physical variables but ignore dynamical feedbacks from the massive dark matter halo to coplanar MHD perturbations in the MSID system. In the WKBJ approximation, it is then straightforward to derive a strikingly similar dispersion relation in the form of equation (23) but with modified background equilibrium properties (53) and (54). Following the same procedure of $D_s$-criterion analysis described in section 4.1, we obtain the $\omega^2$ contour plot as a function of stellar rotational Mach number $D_s^2$ and radial wavenumber $K \equiv |k|r$ with the potential ratio $F$ as an additional parameter. Typical results are displayed in Fig. 14 as an example of illustration.

The example of Fig. 14 with $F = 1$ corresponds to a full composite MSID system where no background dark matter halo is involved, as has been studied in subsection 3.1. As $F$ becomes less than 1 corresponding to an increase of the potential fraction of the massive dark matter halo, the stable range of $D_s^2$ becomes enlarged as clearly shown in Figure 14 for the case of $F = 0.1$. From these $\omega^2 < 0$ contour plots, it is apparent that the introduction of a massive dark matter halo tends to stabilize a composite MSID system as expected (Ostriker & Peebles 1973; Binney & Tremaine 1987; Lou & Shen 2003). For late-type spiral galaxies, one may take $F = 0.1$ or smaller. Such a composite partial MSID system is stable against axisymmetric coplanar MHD disturbances in a wide range of $D_s^2$.

### 3.4 Quantitative Estimates and Galactic Applications

For our numerical examples, the disc mass density ratio parameter $\delta \equiv \Sigma_0^d/\Sigma_0^g$ has been taken to be 0.2 and 0.02 (see Fig. 12). For late-type spiral galaxies, this ratio ranges from 0.05 to 0.15. For relatively young and gas-rich spiral galaxies, this ratio $\delta$ can reach 0.2 and even higher. In nearby spiral galaxies, the strength of magnetic field is typically inferred to be a few to $10 \mu G$; by the equipartition argument, magnetic field strength may reach a few tens of $\mu G$ in circumnuclear regions and within towards the centre (e.g., Lou et al. 2001). We shall take the ratio $\lambda \equiv C\lambda/a_g$ to be of the order of 1. As estimates, we have taken $\beta \equiv (a_s/a_g)^2$ to be 1, 1.5, 10, and 30 in Figure 11. For spiral galaxies, the ratio $a_s/a_g$ can be of the order of or greater than 5 or 6. With these estimates in Figs. 11–12, we see clearly that without a massive dark matter halo, a typical disc galaxy system with a sufficiently fast rotation (e.g., a $V_s \sim 150 - 250 \text{ km s}^{-1}$) would suffer the ring-fragmentation instability. Although the presence of magnetic field offers the stabilizing effect (see Fig. 9 against the ring-fragmentation instability, the development of such instability would be unavoidable for typically inferred galactic magnetic field strengths. By Fig. 13 the inclusion of a massive dark matter halo holds the key to prevent such ring-fragmentation instability.

In the observational studies of Kennicutt (1989) on global star formation rates in spiral galaxies, the theoretical rationale is therefore not sufficiently strong. Firstly, the application of the Toomre stability criterion for a single disc is too simple to be physically sensible. Secondly, for a composite disc system without magnetic field, the generalized criteria (Elmegreen 1995; Jog 1996; Lou & Fan 1998) for the Toomre instability cannot be readily applied to a real galactic disc. Thirdly, even for a composite disc system with a coplanar magnetic field which shows clear stabilizing effects, the MHD ring-fragmentation would occur for typically inferred parameters of a spiral galaxy without a sufficiently massive dark matter halo. Finally, our conclusion is therefore that in relating Toomre-type instabilities with global star formation rates in spiral galaxies, one should conceive new physical rationales by incorporating the dynamical interplay between a massive dark matter halo and a magnetized composite disc system (Wang & Silk 1994; Silk 1997; Lou & Fan 2002a, b).

### 4 SUMMARY AND DISCUSSION

In this paper, we examined the axisymmetric MHD linear stability properties of a composite system consisting of a stellar SID and a gaseous MSID coupled by the mutual gravity and a massive dark matter halo, using the WKBJ approximation. Our main purpose is to confirm the physical interpretation for the global marginal stability curve (Lou & Zu 2004) and to establish the MHD generalization of the $Q$ parameter (Safronov 1960; Toomre 1964) for a composite MSID system in reference to the earlier work (e.g., Elmegreen 1995; Jog 1996; Lou & Fan 1998a, b; Lou 2002; Lou & Shen 2003; Lou & Zu 2004). We now summarize the main theoretical results below.

We have recently constructed exact global solutions for stationary coplanar MHD perturbations in a composite system of a stellar SID and a gaseous MSID for both aligned

![Figure 14. An $\omega^2$ contour plot as a function of $K$ and $D_s^2$ with $m = 0$, $\delta = 0.2$, $\beta = 1.5$, $\lambda^2 = 1$ and $F = 0.1$. The region labelled by $\omega^2 < 0$ is unstable and the Jeans collapse regime disappears completely. In comparison with Fig. 11 it is apparent that the stable range of $D_s^2$ is enlarged as $F$ decreases.](image-url)
and unaligned logarithmic spiral cases (Lou & Zou 2004). Lou & Zou (2004) have extended the analyses of Shu et al. (2000) on an isodispersely magnetized SID, of Lou (2002) on a single coplanarly magnetized SID, and of Lou & Shen (2003) and Shen & Lou (2003) on an unmagnetized composite SID system. In a broader perspective, a composite MSID system is only a special case belonging to a more general class of composite scale-free magnetized disc systems (Syer & Tremaine 1996; Shen & Lou 2004a, b; Shen et al. 2005). In analogy of Shu et al. (2000), Lou & Shen (2003) and Shen & Lou (2003), Lou & Zou (2004) naturally interpreted the marginal curves for stationary axisymmetric coplanar MHD perturbations with radial propagations as the marginal stability curves. Based on the low-frequency time-dependent WKBJ analysis here (Shen & Lou 2003, 2004a), we establish the physical scenario for the presence of the two unstable regimes referred to as the ‘MHD ring fragmentation instability’ and the ‘MHD Jeans collapse’ in a composite MSID system. Consequently, it is intuitively appealing and physically sensible, and the MHD dimensionless parameter applied to our exact neutral stability criterion for a composite MSID system in order to examine its axisymmetric stability and obtain the stable range of $D_s^2$, where $D_s$ is the rotational Mach number of the stellar SID.

In the WKBJ approximation, we relate our MHD $D_s$–criterion to the two effective $Q$ parameters extended to the MHD regime, namely, the MHD $Q_{E}$–criterion generalizing that of Elmegreen (1995) and the MHD $Q_J$–criterion generalizing that of Jog (1996). However, our procedures differ from those of Elmegreen (1995) and of Jog (1996), because our MHD background of rotational equilibrium is dynamically self-consistent with $\nu_s \neq \nu_q$ in general. We show that MHD generalizations of both $Q_{E}$– and $Q_J$–criteria lead to nearly the same stable range for $D_s^2$ extended to the MHD realm. This confirms theclose relation between our MHD $D_s$–criterion and the MHD $Q_{E}$– and $Q_J$–criteriain the exact global MHD perturbation procedure of Lou & Zou (2004) are all close to unity. Our interpretation of the axisymmetric marginal stability curve as the demarcation of stable and unstable regimes is physically sensible, and the MHD $Q_{E}$ and $Q_J$ parameters are associated with the MHD ring fragmentation instability in a composite MSID.

Finally, we have shown the axisymmetric MHD stability property of a composite partial MSID by including the gravitational effect from an axisymmetric massive dark matter halo. It is apparent that the dark matter halo has a strong stabilizing effect for a composite MSID system.

In addition to theoretical interest of disc instabilities for forming large-scale galactic structures (n.b., non-axisymmetric ones are not studied here), there has been a keen desire to somehow relate such instabilities to global star formation rates in disc galaxies and their evolution (e.g., Jog & Solomon 1984a, b; Kennicutt 1989; Wang & Silk 1994; Silk 1997). The overall chain of star formation processes from large-scale disc instabilities, to giant molecular clouds, to cloud collapses, to clusters of stars, to disc accretion onto individual stars and so on is quite complicated and involves many scales of different orders of magnitudes. Conceptually, large-scale axisymmetric ring structures in a disc must be further broken down non-axisymmetrically into smaller pieces in order to initiate this conceive chain of collapses. While various stages of this ‘chain’ have been intensively studied separately, the ultimate relation or connection between the large-scale axisymmetric instabilities and the global star formation rate remains unclear and needs to be established (e.g. Elmegreen 1995; Lou & Bian 2005).

Observationally, Kennicutt (1989) attempted to infer an empirical relation between the $Q$ parameter of the stellar disc alone and the global star formation rate. Wang & Silk (1994) pursued a similar idea with an estimate of $Q$ parameter for a composite disc system of two fluid discs; however, their approximation for $Q$ parameter may be off too much under various relevant situations (Lou & Fan 1998b, 2000a). Should this line of reasoning indeed contain an element of truth for addressing the issue of the global star formation rate, then the $Q$ parameter adopted should really correspond to that of a composite disc system with a magnetized gas disc component and in the presence of a massive dark matter halo. The basic physical reason behind this suggestion is that stars form directly in the magnetized gas disc under the joint gravitational influence of the dark matter halo, the stellar disc and the magnetized gas disc itself. If this line of reasoning does indeed make physical sense, then an interesting possibility arises. That is, the dark matter halo may play an important role of regulating global star formation rates in disc galaxies and thus galactic evolution. For example, if the mass of a dark matter halo is very much greater than the mass of a composite disc system, then star formation activities become weaker. On the other hand, if the dark matter halo is not sufficiently massive, then the disc system rapidly evolves into a bar system. It is also possible that the dark matter halo is only marginal to maintain a stability of a composite disc. In this case, the global star formation activities in the disc system proceed in a regulated manner.

Our MHD model analysis in this paper, highly idealized in many ways, does contain several requisite elements for establishing an MHD generalization of the $Q$ parameter and the corresponding criterion for axisymmetric stability or instability. Observationally, it would be extremely interesting to examine the relation between the MHD $Q_M$ parameter in galactic systems and global star formation rates. This is not expected to be a trivial exercise given various sources of uncertainties.

By presuming that such axisymmetric disc instabilities characterized by either $D_s^2$ or $Q_M$ parameters might somehow hint at or connect to the global star formation rate, there are a few model problems similar to the current one that can be explored further. For example, the models of Shen & Lou (2004b) and Shen, Liu & Lou (2005) can be combined to construct a composite disc system consisting of two scale-free discs with the gaseous one being coplanarly magnetized in the presence of a dark matter halo. Likewise,
the work of Lou & Wu (2005) can be generalized to two coupled scale-free disc with the gaseous one being isopelically magnetized in the presence of dark matter halo (Lou & Wu 2006 in preparation). The real situation is more complicated. We hope these analyses may offer certain hints and insights for different aspects of the problem (Lou & Bai 2005 in preparation).

ACKNOWLEDGEMENTS

This research has been supported in part by the ASCI Center for Astrophysical Thermonuclear Flashes at the University of Chicago under Department of Energy contract B341495, by the Special Funds for Major State Basic Science Research Projects of China, by the Tsinghua Center for Astrophysics, by the Collaborative Research Fund from the National Natural Science Foundation of China (NSFC) for Young Outstanding Overseas Chinese Scholars (NSFC 10028306) at the National Astronomical Observatory, Chinese Academy of Sciences, by NSFC grants 10373009 and 10533020 (QL) at the Tsinghua University, and by the special fund 20050003088 and the Yangtze Endowment from the Ministry of Education through the Tsinghua University. QYL acknowledges supported visits by Theoretical Institute for Advanced Research in Astrophysics (TIARA) of Academia Sinica and National Tsinghua University in Taiwan. The hospitality and support of School of Physics and Astronomy, University of St Andrews, Scotland, U.K., and of Centre de Physique des Particules de Marseille (CPPM/IN2P3/CNRS) et Université de la Méditerranée Aix-Marseille II, France are also gratefully acknowledged. Affiliated institutions of QYL share this contribution.

REFERENCES

Fan Z.H., Lou Y.-Q., 1996, Nat, 383, 800
Kato S., 1972, PASJ, 24, 61