Bonabeau hierarchy models revisited

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Abstract

What basic processes generate hierarchy in a collective? The Bonabeau model provides us a simple mechanism based on randomness which develops self-organization through both winner/looser effects and relaxation process. A phase transition between egalitarian and hierarchic states has been found both analytically and numerically in previous works. In this paper we present a different approach: by means of a discrete scheme we develop a mean field approximation that not only reproduces the phase transition but also allows us to characterize the complexity of hierarchic phase. In the same philosophy, we study a new version of the Bonabeau model, developed by Stauffer et al. Several previous works described numerically the presence of a similar phase transition in this later version. We find surprising results in this model that can be interpreted properly as the non-existence of phase transition in this version of Bonabeau model, but a changing in fixed point structure.

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1 Introduction

It is usual, in sociological works, to describe how global behavior appears, in many levels of social activities [1]. Before that, it is more fundamental to understand in which way citizens gather [2]: since in a little collective every single seems to play the same status, in big societies diversity appears [3]. Hierarchic dominance and hierarchic stratification has been studied with several different approaches [2,4,5]. As long as these matters can be considered as many-body dynamical systems, they have attracted the attention of physicists in the latest years. The emergent area of Sociophysics involves those social complex systems, dealing with many different social situations with a statistical physics approach [8,1,3].

In this terms, a simple and fruitful model of diversity generation has been proposed by Bonabeau et al. [7,6]. Related to this model, some modified versions have been proposed, as the Stauffer et al. [10] version, or the one from Ben-Naim and Redner [16] (this later one has been solved analytically).

The purpose of this paper is double: first, by means of a discrete mean field approximation, we reproduce analytically the numerical results found by Bonabeau. We discover a non trivial complex structure in the hierarchy generation path. After this, we apply the same scheme to Stauffer version [10,9], widely used as a model of hierarchy generation [10,9,13,14,11,15], in order to obtain analytical evidences of its numerical behavior.

2 The Bonabeau model

The Bonabeau model [7,6] has been proposed as a simple model showing self-organization to explain hierarchic dominance in Ethology. With subtle modifications it has been reallocated in Sociophysics area as a model of social stratification [9,10,11,16]. It starts from a community composed of $N$ agents, randomly distributed over a regular lattice $L \times L$, that is to say, with a population density $\rho = \frac{N}{L^2}$. Each agent $i = 1, 2, ..., N$ is characterized by a time dependent variable $h_i(t)$, the agent fitness, that from now on we will name status. Initially all agents share the same status $h_i(t = 0) = 0$: the so-called egalitarian situation. System dynamics are:

(1) *Competition with feedback*: an agent $i$ chosen randomly moves in a four nearest neighbor regular lattice (Newmann neighborhood). If the target site is empty, the agent takes the place. If it is already occupied by an agent $j$, a fight occurs. The attacker agent $i$ will defeat agent $j$ with some probability:

$$ P_{ij}(t) = \frac{1}{1 + \exp(\eta(h_j(t) - h_i(t)))}. $$

(1)
Where $\eta > 0$ is a free parameter. If $i$ wins, he exchanges positions with $j$. Otherwise, positions are maintained. After each combat, status $h_i(t)$ are updated, increasing by 1 the winner’s status and decreasing by $F$ the looser’s status. Note that $F$ is a parameter of the system that weights the defeats such that $F \geq 1$. The case $F = 1$ will be the symmetric case from now on. In the asymmetric case, $F > 1$, the fact of loosing will be more significative for the individual status than the fact of winning [9].

(2) Relaxation: a natural (Monte Carlo) time step is defined as $N$ movement processes (with or without combat). After each time step all agents update their status by a relaxation factor $(1 - \mu)$, such that $0 < \mu < 1$; this effect is interpreted as a fading memory of agents.

Notice that competition rule (1) is a feedback mechanism: status differences $h_j(t) - h_i(t)$ drive the future winning/loosing probabilities of agents $i$ and $j$. If agent $i$ wins/looses it’s winning/loosing probability increases afterwards. This mechanism amplifies agent inhomogeneity. On the other hand, relaxation rule (2) drives the agent status $h_i(t)$ to equalize: status differences are absorbed and toned down. The balance between both mechanisms generates asymptotic stability on $h_i(t)$. Common sense would lead us to expect low fights when the agent’s density is low, so that the relaxation mechanism would overcome and egalitarian situation would prevail ($h_i = h_j = 0, \forall i, j$). But if the system possesses high agent’s density, the rate of fights would increase, and competition mechanism will prevail, leading the system to non neglecting inhomogeneities. The balance of these two mechanisms is crucial at a given density $\rho$. Simulations ran by Bonabeau et al.[7,6] show how this compromise between both effect bring about a phase transition at a critical density, between egalitarian societies for low densities and hierarchical societies for high densities.

A natural measure for the status diversification is the standard deviation of its stationary distribution $\{h_i^*\}_{i=1,\ldots,N}$. However, in [9] another measure is proposed: this is the standard deviation of stationary probability distribution $\{P_{ij}\}_{i=1,\ldots,N}$, defined as:

$$\sigma = \left( \langle P_{ij}^2 \rangle - \langle P_{ij} \rangle^2 \right)^{1/2}. \tag{2}$$

This choice turns out to be more suitable, as long as it is a bounded parameter: $\sigma \in [0, 1]$. It works as an order parameter of the system: for densities lower than the critical, every $h_i$ are equal and therefore all $P_{ij}$ too, then $\sigma = 0$. For densities bigger than the critical, status are different and therefore probabilities are also different: this leads to a non zero value of $\sigma$. 

3
Mean field approximation in the Bonabeau model

In order to tackle the system in a mathematical way, we will obviate spatial correlations, reinterpreting $\rho$ as the probability of two agents combat. In the spatial correlated model this is equivalent to a random mixing of the agent’s positions every time step. Therefore, at each time step, an agent $i$ will possess:

1. Probability $1 - \rho$ of no combat. In that case, the agent will only suffer relaxation.
2. Probability $\rho$ of leading a combat (with probability $1/(N-1)$ the attacked agent will be $j$). In that case: agent $i$ will increase, in average, its status by $P_{ij}(t)(h_i(t) + 1)$, and will decrease by $F$ its status with probability $1 - P_{ij}(t)$. Relaxation will also be applied in this case.

The model is then described by an $N$ equation system ($i = 1, ..., N$) of the following shape:

$$h_i(t + 1) = (1 - \rho)(1 - \mu)h_i(t) + \frac{\rho(1 - \mu)}{N - 1} \sum_{j=1, j \neq i}^{N} \{P_{ij}(t)(h_i(t) + 1) + (1 - P_{ij}(t))(h_i(t) - F)\}.$$  \hfill (3)

In order to analyze the system let’s start with the simplest version $N = 2$. Having in mind that for two agents $P_{12}(t) = 1 - P_{21}(t)$, the system (3) will reduce to:

$$h_1(t + 1) = (1 - \mu)h_1(t) + \rho(1 - \mu)\{P_{12}(t)(1 + F) - F\}$$
$$h_2(t + 1) = (1 - \mu)h_2(t) + \rho(1 - \mu)\{1 - P_{12}(t)(1 + F)\}. \hfill (4)$$

In the egalitarian phase (below the critical density), the fixed point $(h_1^*, h_2^*)$ of the two-automata system 4 will have stationary status of the same value, say $h_1^* = h_2^*$. In order to find the fixed point of the system we can define the mean status of the system as: $\langle h(t) \rangle = (h_1(t) + h_2(t))/2$. The system turns into:

$$\langle h(t + 1) \rangle = (1 - \mu)\langle h(t) \rangle + \frac{\rho(1 - \mu)(1 - F)}{2},$$

with a fixed point:

$$\langle h^* \rangle = \frac{\rho(1 - \mu)(1 - F)}{2\mu}, \hfill (5)$$

always stable (this can be interpreted as the system’s energy, which is conserved).
Notice that, in the egalitarian phase (below critical density), we have $h_1^* = h_2^* = \langle h^* \rangle$. In order to check out stability of $(\langle h^* \rangle, \langle h^* \rangle)$ we compute the Jacobian matrix of the system, evaluated in that fixed point:

$$J = (1 - \mu) \begin{pmatrix} 1 - A & A \\ A & 1 - A \end{pmatrix},$$

where:

$$A = -\frac{\rho \eta (1 + F)}{4}.$$  \hfill (7)

By stability analysis we conclude that egalitarian phase is stable as long as:

$$\rho < \rho_c = \frac{2\mu}{\eta(1 - \mu)(1 + F)}. \hfill (8)$$

From numerical iteration of the equations for two-automata, we present in figure 1 control parameter $\rho$ versus order parameter $\sigma$, at stationary situation, for different values of parameters $F$, $\eta$ and $\mu$. Critical density values $\rho_c$ agree with (8). For the symmetric case ($F = 1$) with $\eta = 1$ and $\mu = 0.1$ (squares) we obtain a critical density $\rho_c \approx 0.11$. We’ll take this particular case as the reference case from now on.

In order to understand the dynamics of the fixed points of the two-automata system, we apply the following change of variables: $h^* = h_2^* - h_1^*$. The fixed points of the system become now the solutions $h^*$ of:

$$h^* = \frac{\rho(1 - \mu)(1 + F)}{\mu} \left(1 - \frac{2}{1 + \exp(\eta h^*)}\right).$$

As to the case of reference, notice that for each value of $\rho$ the system, as a fact of symmetry, has two fixed points: $(h_1^*, -h_1^*), (h_2^*, -h_1^*)$. In figure (2) we represent a case below transition ($\rho < \rho_c$), with a singular solution, and a case above transition ($\rho > \rho_c$) with three solutions: this results in the solution bifurcation of $h_1^*$ and $h_2^*$ in figure 3. In the egalitarian phase we have: $h_1^* = h_2^* = \langle h^* \rangle = 0$. At $\rho = \rho_c$ a bifurcation occurs, from which we have $h_1^* = -h_2^* \neq 0$.

In the successive figures we can observe what effects produce a variation of parameters $F$, $\eta$ and $\mu$ on status stratification, as for the reference case (figure 3): (1) Increasing the asymmetry $F$ (figure 4) decreases $\rho_c$ and grows up inequality. (2) Increasing the relaxation $\mu$ (figure 5) increases $\rho_c$ and diminish inequality. (3) A decrease of $\eta$ (figure 6) increases $\rho h_o_c$ and has no effect on
We now apply the same philosophy to the general system (3). We may define the mean status of the system as:

\[ \langle h(t) \rangle = \frac{1}{N} \sum_{i=1}^{N} h_i(t). \]

Now due to the fact that \( P_{ij} + P_{ji} = 1 \) and then \( \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} P_{ij} = \frac{N(N-1)}{2} \), we can sum and normalize the \( N \) equations, obtaining:

\[ \langle h(t+1) \rangle = \langle h(t) \rangle - \frac{\rho(1-\mu)(F-1)}{2}, \]

whose fixed point is independent of the number of automata and agrees with the very first result given at (5) in the case of two-automata (the system’s energy). Again we get that \( (h_1^*, h_2^*, \ldots, h_N^*) \) with \( h_1^* = h_2^* = \ldots = h_N^* = \langle h^* \rangle \) is a fixed point of the system whose stability determines the transition. The Jacobian matrix of the linearized system, evaluated at the fixed point, is:

\[
J = (1-\mu) \begin{pmatrix}
1-A & \frac{A}{N-1} & \ldots & \frac{A}{N-1} \\
\frac{A}{N-1} & 1-A & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\frac{A}{N-1} & \ldots & \ldots & 1-A
\end{pmatrix},
\]

a circulating matrix [12] where \( A = -\frac{\rho \eta (1+F)}{4} \) and its eigenvalues being: \( \lambda = (1-\mu)((1-A) - \frac{A}{N-1}) \) with multiplicity \( N-1 \), and \( \lambda = (1-\mu) \) with multiplicity 1. The egalitarian phase is therefore determined by:

\[
\rho < \rho_c = \frac{4\mu(N-1)}{\eta(1-\mu)N(1+F)}.
\]

This result is on agreement with the particular case of two-automata \( (N=2) \), and if \( N \gg 1 \) we have:

\[
\rho_c = \frac{4\mu}{\eta(1-\mu)(1+F)}. \tag{9}
\]

Notice that as long as \( 0 \leq \rho_c \leq 1 \), the phase transition will only occur under:

\[
\mu < \frac{\eta(1+F)}{4 + \eta(1+F)}. \tag{10}
\]

In figure (7) we represent, for \( N \gg 1 \), the parameter space, where we distinguish the zone where the egalitarian-hierarchical transition is allowed.
4 Additive relaxation

Bonabeau et al. in their seminal paper [6], proposed an additive relaxation as an alternative to the multiplicative relaxation developed above. That additive relaxation updates the status as it follows:

\[ h_i \rightarrow h_i - \mu \tanh(h_i). \]  

(11)

They developed then a mean field approximation, based on stochastic differential equations, where they found the egalitarian-hierarchical phase transition. We can of course apply, in the discrete model that we propose, this additive mechanism of relaxation. In the two-automata system, mean field equations reduce to:

\begin{align*}
    h_1(t+1) &= h_1(t) + \rho (P_{12}(t)(1 + F) - F) - \mu \tanh(h_1) \\
    h_2(t+1) &= h_2(t) + \rho (1 - P_{12}(t)(1 + F)) - \mu \tanh(h_2).
\end{align*}

Following our previous steps, it’s quite easy to deduce that the fixed point \( h_1^* = h_2^* \) is stable as long as \( \rho < \rho_c = \frac{2\mu}{\eta(1+F)} \). This result is on agreement with those of the continuum model proposed by Bonabeau et al. [6].

A simplified model has been recently proposed [16] by Ben-Naim and Redner, in order to obtain analytical evidences of the phase transition which is observed in the Bonabeau model. In their version (which is highly inspired on the Bonabeau model) the relaxation process is additive (though it is not a function of the status but simply a fixed constant \( h_i \rightarrow h_i - 1 \)), and the competition mechanism is not stochastic but deterministic (except for the situation where both agents have the same status). A phase transition between both regimes (equality/hierarchy) is then found analytically.

Comparing additive to multiplicative relaxation in the Bonabeau model, we must say that additive works worse than multiplicative: status differences grow excessively and then, computing limitations are exceeded (due to the exponential explosion).

Anyway, if we introduce a new parameter \( Q \geq 0 \) (instead of increasing by one the winner status, we increase it by \( Q \), so that we can tune both winning and losing effects \( Q, F \)), the system doesn’t explode numerically with a good tuning of the parameters. In figure 8 with \( N = 10 \), we took \( F = 0.7, Q = 0.7, \mu = 0.0001 \) and \( \eta = 0.001 \). As we see, status reach values of five order of magnitude, what we think is not realistic.

The hierarchy scheme in the multiplicative relaxation model (figures 9-13) develops much more complexity than the additive one. An increase of density, above critical one, stills generates hierarchy, as common sense would have
dictated us. This fact is traduced by a periodic fixed point coordinates $h_i^*$ splitting, even after the phase transition. This hierarchy growing is not trivial; something that has for sure been unnoticed for the moment. In the additive relaxation model the hierarchical structure is simple, it doesn’t change with $\rho$ at hierarchical phase. Instead of that, there is a fixed point coordinates $h_i^*$ splitting at $\rho_c$, and dynamical evolution in hierarchical phase is poor.

5 Mean field in the Stauffer version

The model developped by Stauffer et al. comes from Bonabeau’s. It was firstly introduced to carry out the supposed lack of transition of the previous model, discussed numerically in \[10,9,11\]. In this version, the free parameter $\eta$ is now exchanged with the order parameter $\sigma$, such that

$$P_{ij}(t) = \frac{1}{1 + \exp (\sigma(t)(h_j(t) - h_i(t)))}.$$  \hspace{1cm} (12)

This modification somehow introduces a dynamical feedback to the system: probability of winning/loosing is directly related to the global inequality of the system, therefore, depending on the climate’s aggressiveness of the system, agents will behave more or less aggressive themselves.

Just as in the case of Bonabeau’s model, by introducing a mean field we expect to find and reproduce analytically the phase transition that is proclaimed in the literature.

In the case of two-automata system, we develop the mean field equations having in mind that:

(1) Each automaton updates at each time step with the same dynamics that the Bonabeau model. The probability calculation is different though ($\eta \rightarrow \sigma$).

(2) At each time step, variable $\sigma$ is updated: the order parameter is now a dynamical parameter of the system and therefore evolves with it.

We once again redefine $h = h_2 - h_1$. With this change of variable, we pass from a three equation system ($h_1, h_2, \sigma$) to a two equation system ($h, \sigma$), without lack of generality, because $h_1$ and $h_2$ are related through the mean status (system’s energy). The system equation is therefore:
\begin{align*}
  h(t + 1) &= (1 - \mu)h(t) + \rho(1 - \mu)(1 + F) \cdot \left(1 - \frac{2}{1 + \exp(\sigma(t)h(t))}\right), \\
  \sigma(t + 1) &= \left|\frac{1}{1 + \exp(\sigma(t)h(t))} - \frac{1}{2}\right|.
\end{align*}

It is likely to expect the same qualitative results of the stability analysis of this system than those about Bonabeau’s model, that is to say, loss of stability of the egalitarian regime (i.e. the fixed point \(h^*(= h_2^* - h_1^*) = 0, \sigma^* = 0\) would become unstable at some critical density \(\rho_c\)).

The Jacobian matrix of the linearized system is, in general:
\[
J = \begin{pmatrix}
  1 - \mu + AB\sigma^* & 2ABh^* \\
  -\sigma^*A & -h^*A
\end{pmatrix},
\]
where:
\[
A = \exp(\sigma^*h^*)/(1 + \exp(\sigma^*h^*))^2,
\]
\[
B = \rho(1 - \mu)(1 + F).
\]
Evaluating \(J\) in the egalitarian fixed point \((h^* = 0; \sigma^* = 0)\), we find that eigenvalues of \(J\) are:
\[
\lambda_1 = 0 < 1, \lambda_2 = 1 - \mu < 1, \forall \rho
\]
(13)

We find that egalitarian zone is stable for all densities. How come a phase transition can then occur, as is presented in many previous model simulations \([10,9,13,14,11,3,15]\)? How come hierarchical situation can be achieved starting from equality, if the egalitarian zone is always stable? The key of the dilemma is set on the simulation methods that have been applied until now. Figure (14) shows stationary values of order parameter \(\sigma\) vs. control parameter \(\rho\). For each \(\rho\), we take random initial conditions for status and \(\sigma\) (these can be both zero or non-zero). The figure shows stationary value \(\sigma = 0\) for all initial conditions, below a certain density. Above it, we have stationary values of \(\sigma\) being zero in some cases (we remain in the egalitarian zone) and non-zero in others (hierarchical zone).

Numerically we observe that the system has one stable fixed point below the critical density \(\rho_c\) \((h^* = 0; \sigma^* = 0)\), and two stable fixed points above
\( h^* = 0; \sigma^* = 0, h^* \neq 0; \sigma \neq 0 \). The egalitarian zone is therefore always stable \((\forall \rho)\). At \( \rho_c \), a saddle-node bifurcation occurs, and brings about the hierarchic branch. Notice that the stability scheme is totally different from what we founded in the Bonabeau model: while in that model, equality-hierarchy transition was generated across Pitchfork bifurcation, due to loss of stability of the egalitarian regime, in the Stauffer version the egalitarian regime is always stable, but here at \( \rho_c \) a saddle-node bifurcation takes place. The stable branch of this bifurcation is related to the hierarchic regime, and the unstable branch (not drawn in figure 14) plays the role of frontier between the two domains of attraction.

Depending what initial conditions we give to \((h_i, \sigma)\), (i.e. depending in which domain of attraction we start), the system, above \( \rho_c \), will evolve towards the egalitarian domain or the hierarchic one. In figure 15 we can understand how this stability is developed. Below \( \rho_c \) the system has only one fixed point (triangles), indeed stable due to (13): every set of initial conditions \( h, \sigma \) will evolve towards \( h^* = 0, \sigma^* = 0 \) (egalitarian zone). Above \( \rho_c \) the system has three fixed points (circles), moreover, from (13) and figure 14 we know that upper and lower fixed point are stable and characterize both egalitarian and hierarchic zones, this leads to an unstable fixed point between them, performing the frontier. If the initial conditions belong to the egalitarian domain, the system will evolve towards an asymptotic egalitarian state. On the contrary, if the initial conditions belong to the hierarchic domain, the system will evolve to an asymptotic hierarchical state.

If in the model simulations, we set egalitarian initial conditions, the result would be the ”absence of transition”. But if we fix some other initial conditions, this could lead us to interpret the results as ”existence of transition”. In simulations made by Stauffer [3] they say ”for the first ten Monte Carlo steps per site, \( \sigma = 1 \) to allow a buildup of hierarchies”. This fact probably allows initial fluctuations develop so that initial conditions will be in the hierarchic domain.

6 Conclusions

The Bonabeau model has been criticized [10,9,11] in the last years. In this paper we revisit Bonabeau model in order to obtain analytical evidences that give clear proof of the phase transition that the system shows. We obtain, in the model with multiplicative relaxation, a high complex structure of the hierarchical regime, a fact that we think deserves an in-depth investigation. The Stauffer version, which is an alternative to Bonabeau model, proposed by Stauffer et al. [10,9,11], is the base of recent works, basically focused on simulations [13,14,15]. In this paper we tackle this version with the same phi-
losophy applied in the Bonabeau model. Surprisingly, this one doesn’t show a phase transition in rigor, as far as there is no sudden growth of hierarchy if we start from equality.

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References

Fig. 1. Equality-hierarchy phase transition with control parameter $\rho$ and order parameter $\sigma$, in the mean field model of two automata ($N = 2$). From left to right we represent the transition derived from iteration of the two automata equation system 4, for different values $(F; \mu; \eta)$: $(2; 0.1; 1)$ diamonds, $(1; 0.1; 1)$ squares, $(1; 0.1; 0.5)$ left triangles, $y (1; 0.3; 1)$ circles. The continuous lines are just guides for eye.

Fig. 2. The two automata system switches from having one fixed point $h^* = h_2^* - h_1^* = 0$ for densities $\rho < \rho_c$, to three fixed points when $\rho > \rho_c$ : $h^* = \{0, +a, -a\}$ (first one unstable and the rest stable), equivalent to the three fixed points of the two automata system, say $\{(0, 0), (+a/2, -a/2), (-a/2, +a/2))\$. 


Fig. 3. Bifurcation of stationary values $h_1^*$ and $h_2^*$ of the reference case (symmetric case $F = 1$ with $\eta = 1$ and $\mu = 0.1$). At the egalitarian zone we have: $h_1^* = h_2^* = \langle h^* \rangle = 0$. At $\rho = \rho_c$ a bifurcation occurs, from where, due to symmetry, $h_1^* = -h_2^* \neq 0$.

Fig. 4. Bifurcation of stationary values of $h_1^*$ and $h_2^*$ for values $F = 2$, $\mu = 0.1$ and $\eta = 1.0$.

Fig. 5. Bifurcation of stationary values of $h_1^*$ and $h_2^*$ for values $F = 1$, $\mu = 0.3$ and $\eta = 1.0$. 
Fig. 6. Bifurcation of stationary values of $h_1^*$ and $h_2^*$ for values $F = 1$, $\mu = 0.1$ and $\eta = 0.5$.

Fig. 7. Parameter space: delimitation, for $N >> 1$, of the regions where transitions can be whether achieved or not achieved.

Fig. 8. Bifurcations of stationary values of fixed point components (system 3) depending on $\rho$, for the symmetric case of reference, in the additive relaxation model, when $N = 10$. 
Fig. 9. Bifurcations of stationary values of fixed point components (system 3) depending on $\rho$, for the symmetric case of reference when $N = 3$.

Fig. 10. Bifurcations of stationary values of fixed point components (system 3) depending on $\rho$, for the symmetric case of reference when $N = 4$.

Fig. 11. Bifurcations of stationary values of fixed point components (system 3) depending on $\rho$, for the symmetric case of reference when $N = 6$. 
Fig. 12. Bifurcations of stationary values of fixed point components (system 3) depending on $\rho$, for the symmetric case of reference when $N = 8$.

Fig. 13. Bifurcations of stationary values of fixed point components (system 3) depending on $\rho$, for the symmetric case of reference when $N = 10$.

Fig. 14. $\sigma$ vs $\rho$ in the 2 automata system. Initial conditions for each $\rho$ are taken randomly. This leads to stationary values of $\sigma$ being zero (egalitarian zone) or non zero (hierarchical zone) depending of those chosen initial conditions.
Fig. 15. Crossover from one stable fixed point of the system at \( \rho < \rho_c \) (triangles) to two stable fixed points -egalitarian/hierarchic- (circles) and an unstable fixed point as the delimiting branch between both domains of attraction.