Study of Dissipative System of Charged Particles by Wigner’s Functions

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Abstract
Recently, it is shown that the extended phase space formulation of quantum mechanics is a suitable technique for studying the quantum dissipative system. Here, as an application of this formalism, we consider a dissipative system of charged particles interacting with an external time dependent electric field. Such a system has been investigating by Buch and Denman, and two solutions with completely different structure have been obtained for Schrödinger’s equation in two different gauges. We demonstrate how both gauges lead to the same conductivity by generalizing the gauge transformations to the phase space and using the extended phase space technique.

1 Introduction
Buch and Denman investigated a system of charged particles interacting with an external time dependent electric field [1]. They obtained two distinct solutions with completely different physical structures for Schrödinger’s equation in different gauges. It seems that, by applying the Extended Phase Space (EPS) formulation of quantum mechanics, proposed by Sobouti and Nasiri [2,3], the above discrepancy could be removed. In this respect, Khademi and Nasiri solved the same problem using EPS method and obtained an identical result for conductivity as a measurable quantity in two different gauges [4]. Here we consider a system of dissipative charged particles interacting with an external time dependent electric field and study its dynamics using the Wigner function. It is shown that the expression obtained for the conductivity is the same as given by Khademi and Nasiri.

The layout of the paper is as follows: In section 2, a brief review of the EPS formalism is presented. In section 3, by introducing two distinct gauges the conductivity is calculated using the Wigner functions in these gauges. Section 4 is devoted to conclusions.
2 Review of formulation

A direct approach to quantum statistical mechanics is proposed by Sobouti and Nasiri (SN) [2], by extending the conventional phase space and by applying the canonical quantization procedure to the extended quantities in this space. Assuming the phase space coordinates $p$ and $q$ to be independent variables on the virtual trajectories, allows one to define momenta $\pi_p$ and $\pi_q$, conjugate to $p$ and $q$, respectively. This is done by introducing the extended lagrangian

$$\mathcal{L}(p, q, \dot{p}, \dot{q}) = -\dot{p}q - \dot{q}p + \mathcal{L}^p(p, \dot{p}) + \mathcal{L}^q(q, \dot{q}), \quad (1)$$

where $\mathcal{L}^p$ and $\mathcal{L}^q$ are the $p$ and $q$ space lagrangians of the given system. Using Eq. (1) one may define the momenta, conjugate to $p$ and $q$, respectively, as follow

$$\pi_p = \frac{\partial \mathcal{L}}{\partial \dot{p}} = \frac{\partial \mathcal{L}^p}{\partial \dot{p}} - q, \quad (2)$$

$$\pi_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}^q}{\partial \dot{q}} - p. \quad (3)$$

In the EPS defined by the set of variables $\{p, q, \pi_p, \pi_q\}$, one may define the extended hamiltonian

$$H_{SN}(\pi_p, \pi_q, p, q) = \dot{p}\pi_p + \dot{q}\pi_q - \mathcal{L} = H(p + \pi_q, q) - H(p, q + \pi_p)$$

$$= \sum \frac{1}{n!} \left\{ \frac{\partial^n H}{\partial p^n} \pi_q^n - \frac{\partial^n H}{\partial q^n} \pi_p^n \right\}, \quad (4)$$

where $H(p, q)$ is the hamiltonian of the system. Using the canonical quantization rule, the following postulates are outlined: a) Let $p, q, \pi_p,$ and $\pi_q$ be operators in a Hilbert space, $X$, of all square integrable complex functions, satisfying the following commutation relations

$$[\pi_q, q] = -i\hbar, \quad \pi_q = -i\hbar \frac{\partial}{\partial q}, \quad (5)$$

$$[\pi_p, p] = -i\hbar, \quad \pi_p = -i\hbar \frac{\partial}{\partial p}, \quad (6)$$

$$[p, q] = [\pi_p, \pi_q] = 0. \quad (7)$$

By virtue of Eqs. (4), the extended hamiltonian, $H$, will also be an operator in $X$. b) A state function $\chi(p, q, t) \in X$ is assumed to satisfy the following dynamical equation

$$i\hbar \frac{\partial \chi}{\partial t} = H_{SN}\chi = \left[ H(p - i\hbar \frac{\partial}{\partial q}, q) - H(p, q - i\hbar \frac{\partial}{\partial p}) \right]\chi$$

$$= \sum \frac{(-i\hbar)^n}{n!} \left\{ \frac{\partial^n H}{\partial p^n} \frac{\partial^n}{\partial q^n} - \frac{\partial^n H}{\partial q^n} \frac{\partial^n}{\partial p^n} \right\} \chi. \quad (8)$$
c) The averaging rule for an observable $O(p, q)$, a $c$-number operator in this formalism, is given as

$$< O(p, q) > = \int O(p, q) \chi^*(p, q, t) dp dq.$$  \hspace{1cm} (9)

For details of selection procedure of the admissible state functions, see Sobouti and Nasiri [2].

3 Dynamics of Dissipative Charged Particle via Wigner’s Functions

In classical level, the Wigner extended hamiltonian is related to the extended hamiltonian (8) by an extended canonical transformation. Thus, in quantum level, the Wigner hamiltonian operator can be obtained by corresponding unitary transformation as follows [2]

$$\mathcal{H}_w = U \mathcal{H}_{SN} U^\dagger = H(p + \frac{1}{2}\pi_q, q - \frac{1}{2}\pi_p) - H(p - \frac{1}{2}\pi_q, q + \frac{1}{2}\pi_p),$$  \hspace{1cm} (10)

where

$$U = \exp\left(-\frac{i}{\hbar}\frac{\pi q \pi p}{\pi}\right).$$  \hspace{1cm} (11)

The evolution of Wigner’s distribution function $w(p, q)$, is given by Wigner’s equation,

$$i\hbar \frac{\partial w}{\partial t} = \mathcal{H}_w w.$$  \hspace{1cm} (12)

The Kanai hamiltonian [5] for a dissipative particle in a medium with damping constant $\alpha$ is

$$H = \frac{1}{2m}[p - \frac{e}{c} A(q, t)]^2 e^{-\alpha t} + e\phi(q, t) e^{\alpha t}.$$  \hspace{1cm} (13)

In Eq. (13), $\phi$ and $A$ are electromagnetic scalar and vector potentials, respectively.

To study the above problem we choose two distinct gauges, i.e., $A$- and $\phi$-gauges in EPS. The generalization of the gauge transformations to the EPS is done by Khademi and Nasiri [4]. They have shown that, the gauge function in EPS becomes a function of ordinary phase space coordinates $p$ and $q$ which is the sum of the gauge functions in $q$ and $p$ spaces. The corresponding extended unitary transformation which gives the extended hamiltonian operator in different gauges is the product of the unitary transformations in $q$ and $p$ spaces.

3.1 $A$-gauge

A time dependent uniform electric field may be generated by setting

$$A(t) = -c \int_0^t \exp(\alpha \lambda) E(\lambda) d\lambda.$$  \hspace{1cm} (14)
Note that \( A(t) \) depends only on time. In this gauge Wigner's extended hamiltonian assumes the following form

\[
H^A_w = \frac{1}{m} p \exp(-\alpha t) \pi_q - \frac{e}{mc} A(t) \exp(-\alpha t) \pi_q. \tag{15}
\]

Using this hamiltonian in Eq. (12), one gets

\[
i\hbar \frac{\partial w}{\partial t} = H^A_w w = -i \frac{\hbar}{m} \exp(-\alpha t) p \frac{\partial w}{\partial q} + i \frac{\hbar e}{m c} \exp(-\alpha t) A(t) \frac{\partial w}{\partial q}. \tag{16}
\]

Taking the transformation

\[
\xi = \frac{-e E(t)}{imw(\alpha + iw)} + q, \tag{17}
\]

\[
\eta = p, \tag{18}
\]

\[
\tau = t, \tag{19}
\]

and using them in Eq. (16), one gets

\[
- \frac{\eta}{m} e^{-\alpha \tau} \frac{\partial w}{\partial \xi} = \frac{\partial w}{\partial \tau} \tag{20}
\]

Equation (16) can be solved and yields

\[
w = c \exp(-\frac{k}{\alpha} e^{-\alpha t}) \exp(-\frac{km}{\alpha} \xi) \delta(\eta - a), \tag{21}
\]

where \( c \) and \( k \) are normalization and separation constants. The conductivity for a system of \( N \) particles is given by

\[
\sigma = \frac{Ne < \dot{q} >}{E(t)}, \tag{22}
\]

where

\[
\dot{q} = \frac{\partial H^A_w}{\partial \pi_q} = \frac{\eta}{m} \exp(-\alpha \tau) + \frac{e E(\tau)}{m(\alpha + iw)}. \tag{23}
\]

Then

\[
\sigma = \frac{Ne^2}{m(\alpha + iw)} \tag{24}
\]

Note that the first term in Eq. (23) is a transient term and vanishes as \( t \to \infty \).
3.2 \(\phi\)-gauge

In \(\phi\)-gauge, \(\phi(q, t)\) is defined as

\[
\phi(q, t) = -qE(t).
\] (25)

The Wigner extended Hamiltonian becomes

\[
H_w^\phi = \frac{1}{m} \exp(-\alpha t) p\phi_q + eE(t) \exp(\alpha t) \phi_p.
\] (26)

The evolution equation for \(w\) gives

\[
i\hbar \frac{\partial w}{\partial t} = H_w^\phi \frac{\partial w}{\partial q} - i\hbar E(t) \exp(\alpha t) \frac{\partial w}{\partial p}.
\] (27)

Taking the transformation

\[
\xi' = q - \frac{eE(t)}{m\hbar(w(\alpha + iw))},
\] (28)

\[
\eta' = p - \frac{eE(t) \exp(\alpha t)}{\alpha + iw},
\] (29)

\[
\tau' = t.
\] (30)

one has

\[
-\eta' \frac{\exp(-\alpha t)}{m} \frac{\partial w}{\partial \xi'} = \frac{\partial w}{\partial \tau'}
\] (31)

which has the same form as Eq. (20). Thus the Wigner function has the same mathematical structure in both gauges. To calculate the conductivity one may uses

\[
\dot{q} = \frac{\partial H_w^\phi}{\partial \pi_q} = \eta' \frac{\exp(-\alpha t)}{m} + \frac{eE(t'')}{m(\alpha + iw)}.
\] (32)

One may easily show that \(\dot{q}(\xi, \eta, \tau)\) and \(\dot{q}(\xi', \eta', \tau')\) have exactly the same functional forms. Therefore, the solution of Eqs. (16) and (27) will be identical, giving the same physical results for conductivity in \(\Lambda\) and \(\phi\) gauges.

4 Conclusions

In an earlier work, Khademi and Nasiri [4], used the EPS formulation proposed by Sobouti and Nasiri [2], to study the quantum behavior of a system of dissipative charged particles in a time dependent electric field and, in contrast to the Bush and Dennman, they showed that the conductivity for the system is gauge independent. In this paper, using the Wigner functions, we studied the same problem and obtained the same expression for the conductivity in two different \(\Lambda\) and \(\phi\) gauges. The results are the same as those obtained by Khademi and
Nasiri. This is because, the Wigner hamiltonian can be obtained by an extended canonical transformation on the SN hamiltonian [2]. Then, one expects that in quantum level the corresponding quantum distribution functions to be related by a unitary transformation. Therefore, the physical quantities, i.e., the conductivity, must be the same in two approaches.

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References