Weyl Symmetry and the Liouville Theory∗†‡

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Abstract

Flat-space conformal invariance and curved-space Weyl invariance are simply related in dimensions greater than two. In two dimensions the Liouville theory presents an exceptional situation, which we here examine.

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1 Conformally and Weyl Invariant Scalar Field Dynamics in $d > 2$

Let us begin by recording the $d$-dimensional Lagrange density for a scalar field $\phi$ with a scale and conformally invariant self interaction.

$$L_0 = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \lambda \phi^{2d-2}$$  \hspace{1cm} (1.1)

Evidently the expression makes sense only for $d \neq 2$, and we take $d > 2$. The theory is invariant against

$$\delta \phi = f^\alpha \partial_\alpha \phi + \frac{d-2}{2d} \partial_\alpha f^\alpha \phi,$$  \hspace{1cm} (1.2)

where $f^\alpha$ is a (flat-space) conformal Killing vector. The usual, canonical energy momentum tensor

$$\theta^\text{canonical}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \left( \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \lambda \phi^{2d-2} \right)$$  \hspace{1cm} (1.3)

is conserved and symmetric, as it should be in a Poincaré invariant theory. But it is not traceless: $\eta_{\mu\nu} \theta^\text{canonical}_{\mu\nu} \neq 0$. Nevertheless, because of the conformal invariance (1.2), $\theta^\text{canonical}_{\mu\nu}$ can be improved by the addition of a further conserved and symmetric expression, so that the new tensor is traceless [1].

$$\theta_{\mu\nu} = \theta^\text{canonical}_{\mu\nu} + \frac{d-2}{4(d-1)} \left( \eta_{\mu\nu} \Box - \partial_\mu \partial_\nu \right) \phi^2,$$  \hspace{1cm} (1.4)

A variational derivation of the canonical tensor (1.3) becomes possible after the theory (1.1) is minimally coupled to a metric tensor $g_{\mu\nu}$, and its action integral is varied with respect to $g_{\mu\nu}$.

$$\theta^\text{canonical}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \left( \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \lambda \phi^{2d-2} \right)$$  \hspace{1cm} (1.5)

Here $G_{\mu\nu}$ is the Einstein tensor, $R$ the Ricci scalar $R = \frac{2}{d-2} g^{\mu\nu} G_{\mu\nu}$, and $D_\mu$ the covariant derivative. In the limit $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ the non-minimal term in $\mathcal{L}$ vanishes, but it survives in the $g^{\mu\nu}$ variation.

$$T_{\mu\nu} \rightarrow \eta_{\mu\nu} \theta_{\mu\nu}$$  \hspace{1cm} (1.6)

Note that $g^{\mu\nu} T_{\mu\nu} = 0$, with the help of the field equation for $\phi$.

$$D^2 \phi + \lambda \frac{2d}{d-2} \phi^{d-1} - \frac{d-2}{4(d-1)} R \phi = 0$$  \hspace{1cm} (1.7)
This ensures the vanishing of \( \eta^{\mu\nu} \theta_{\mu\nu} \).

The precise form of the non-minimal coupling results in the invariance of the curved space action against Weyl transformations, involving an arbitrary function \( \sigma \) [2].

\[
\begin{align*}
g_{\mu\nu} & \xrightarrow{\text{Weyl}} e^{2\sigma} g_{\mu\nu} \\
\phi & \xrightarrow{\text{Weyl}} e^{\frac{2-d}{2} \sigma} \phi
\end{align*}
\] (1.9a)

(1.9b)

The self coupling is separately invariant against (1.9). The kinetic term and the non-minimal coupling term are not, but their non-trivial response to the Weyl transformation cancels in their sum. Also it is the Weyl invariance of the action that results in the tracelessness of its \( g^{\mu\nu} \)-variation i.e. of \( T_{\mu\nu} \), just as its diffeomorphism invariance ensures symmetry and covariant conservation of \( T_{\mu\nu} \).

Thus we see that Weyl (and diffeomorphism) invariance in curved space is closely linked to conformal invariance in flat space [2]. But can a conformally invariant, flat space theory always be extended to a Weyl and diffeomorphisms invariant theory in curved space? Evidently, the answer is “Yes” for the self-interacting scalar theories in \( d > 2 \), discussed previously [3]. We now examine what happens in \( d = 2 \).

## 2 Liouville Theory: Conformally Invariant Scalar Field Dynamics in \( d = 2 \)

A 2-dimensional model with non-trivial dynamics that is conformally invariant is the Liouville theory with Lagrange density

\[
L_{\text{Liouville}}^0 = \frac{1}{2} \eta^{\mu\nu} \partial_{\mu} \psi \partial_{\nu} \psi - \frac{m^2}{\beta^2} e^{\beta \psi}.
\] (2.1)

The conformal symmetry transformations act in an affine manner, so that the exponential interaction is left invariant.

\[
\delta \psi = f^\alpha \partial_\alpha \psi + \frac{1}{\beta} \partial_\alpha f^\alpha
\] (2.2)

The canonical energy-momentum tensor

\[
\theta^{\text{canonical}}_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \eta_{\mu\nu} \left( \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi - \frac{m^2}{\beta^2} e^{\beta \psi} \right)
\] (2.3)

again is not traceless: \( \eta_{\mu\nu} \theta^{\text{canonical}}_{\mu\nu} \neq 0 \), but with an improvement it acquires that property.

\[
\theta_{\mu\nu} = \theta^{\text{canonical}}_{\mu\nu} + \frac{2}{\beta} (\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) \psi, \quad \eta^{\mu\nu} \theta_{\mu\nu} = 0
\] (2.4)

Again \( \theta^{\text{canonical}}_{\mu\nu} \) arises variationally when the Liouville Lagrange density is minimally extended by an arbitrary metric tensor. Similarly the improved tensor (2.4) is gotten when
a non-minimal interaction is inserted.

\[ L_{\text{Liouville}} = \frac{1}{\beta} R \psi + \frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - \frac{m^2}{\beta^2} e^{\beta \psi} \]  
\[ T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{|g|} L_{\text{Liouville}} \]
\[ = \partial_\mu \psi \partial_\nu \psi - g_{\mu\nu} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi - \frac{m^2}{\beta^2} e^{\beta \psi} \right) + \frac{2}{\beta} (g_{\mu\nu} D^2 - D_\mu D_\nu) \psi \] (2.5)

However, the curved-space tensor \( T_{\mu\nu} \) is not traceless,

\[ g^{\mu\nu} T_{\mu\nu} = \frac{2}{\beta^2} R \neq 0 \] (2.6)

becoming traceless only in the flat-space limit, when \( R \) vanishes. Correspondingly, the action associated with (2.5) is not invariant against Weyl transformations, which take the following form for the scalar field \( \psi \).

\[ \psi \rightarrow \text{Weyl} \psi - \frac{2}{\beta} \sigma \] (2.7)

This formula is needed so that the interaction density \( \sqrt{|g|} e^{\beta \psi} \) be invariant. However, the kinetic term together with the non-minimal term are not invariant, so that

\[ I_{\text{Liouville}} = \int \sqrt{|g|} L_{\text{Liouville}} \]
\[ \rightarrow \text{Weyl} I_{\text{Liouville}} - \frac{2}{\beta^2} \int \sqrt{|g|} \left( R \sigma + g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right) \] (2.8)

Note that the change in the action — the last term in (2.10) — is \( \psi \) independent. So the field equation

\[ D^2 \psi + \frac{m^2}{\beta} e^{\beta \psi} - \frac{1}{\beta} R = 0 \] (2.9)

enjoys Weyl symmetry, even while the action does not.

### 3 Obtaining the \( d = 2 \) Liouville theory from the \( d > 2 \) Weyl invariant theories

We see that the 2-dimensional situation is markedly different from what is found for \( d > 2 \):
for the latter theories there exists a Weyl-invariant precursor, with a traceless energy-momentum tensor in curved space, which leads to a traceless energy-momentum tensor in flat space. For \( d = 2 \) the precursor is not Weyl invariant and the energy-momentum tensor becomes traceless only in the flat-space limit.

To get a better understanding of the 2-dimensional behavior, we now construct a limiting procedure that takes the Weyl invariant models at \( d > 2 \), (1.5), to two dimensions. Thereby we expose the steps at which Weyl invariance is lost.
In order to derive the $d = 2$ Liouville theory (2.5) from the $d > 2$, Weyl invariant models with polynomial interaction (1.5), we set

$$\varphi = \frac{2d}{\beta(d-2)} e^{\frac{d}{2\beta} (d-2) \psi},$$

and take the limit $d \to 2$, from above. We examine each of the three terms in (1.5) separately.

For the self interaction, we have

$$\lambda \varphi^2 \frac{d}{d-2} = \lambda \left( \frac{2d}{\beta(d-2)} \right)^{\frac{2d}{d-2}} e^{\beta \psi} \frac{d}{d-2} \frac{m^2}{\beta^2} e^{\beta \psi}.$$

In the last step, to absorb the singular factor we renormalize the constant $\lambda$ by defining $m^2 / \beta^2$. For the kinetic term, the limit is immediate.

$$\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \frac{d}{d-2} \frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi.$$

But the non-minimal term has no limit, so we expand the exponential.

$$\frac{d-2}{8(d-1)} R \varphi^2 = \frac{d^2}{2\beta^2(d-1)(d-2)} R e^{\frac{d}{2\beta} (d-2) \psi}$$

$$= \frac{d^2}{2\beta^2(d-1)(d-2)} R + \frac{d}{2\beta(d-1)} R \psi + \cdots$$

In the $d = 2$ limit, (3.2) and (3.3) and the last term in (3.4) lead to the curved space Liouville Lagrange density (2.5). The first term in (3.4) gives an indeterminate result in the action.

$$\int \sqrt{|g|} \mathcal{L}_{d>2} \frac{d}{d-2} \int \sqrt{|g|} \mathcal{L}^{\text{Liouville}} + \frac{2}{\beta^2} \frac{\int \sqrt{|g|} R}{d-2}$$

The indeterminacy arises from the fact that in two dimensions $\sqrt{|g|} R$ is the Euler density and its integral is just a surface term – effectively vanishing as far as bulk properties are concerned. So the last term in (3.5) gives 0/0 at $d = 2$. Evidently, the Liouville model is regained when 0/0 is interpreted as 0, but this leads to a loss of Weyl invariance. To maintain Weyl invariance on the limit $d \downarrow 2$, we must carefully evaluate the $\psi$-independent $\int \sqrt{|g|} R/(d-2)$ quantity – we need a kind of L’Hospital’s rule for dimensional reduction.

It turns out that a precise evaluation of $\int \sqrt{|g|} R/(d-2)$ in the limit $d \downarrow 2$ can be found, by reference to Weyl’s original ideas.

Before describing this, let us remark that the conformal and Weyl transformation rules for $\psi$, (2.2) and (2.9), are correctly obtained by substituting (3.1) into the corresponding rules for $\varphi$, (1.2) (1.9b), and passing to limit $d \downarrow 2$. The same connection exists between the equations of motion (2.11) (1.8). However, the reduction of the $\varphi$ energy-momentum tensor (1.6) produces the $\psi$ tensor (2.6) plus the term $\frac{1}{d-2} G_{\mu\nu}/(d-2)$, which is indeterminate at $d = 2$, since both the numerator and denominator vanish. Notice that taking the trace of this quantity, before passing to $d \downarrow 2$, leaves $\frac{1}{d-2} (1-d/2)R/(d-2) = -\frac{2}{d^2} R$, which cancels the non-vanishing trace of Liouville energy-momentum tensor. This again identifies the indeterminacy as the source of Weyl non-invariance.
4 Weyl’s Weyl Invariance

To obtain a definite value for the behavior of $\int \sqrt{|g|} R/(d - 2)$ in the limit of $d \downarrow 2$, we examine once again the Weyl transformation properties of the kinetic term for a scalar field theory in $d$ dimensions. (The self-interaction is Weyl invariant, and needs no further discussion.) As already remarked, the kinetic term is not Weyl invariant, and this is compensated by the non-minimal interaction, to produce the Weyl invariant kinetic action.

$$I_{\text{kinetic}} = \int \sqrt{|g|} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{d - 2}{8(d - 1)} R \phi^2 \right)$$  \hspace{1cm} (4.1)

However, Weyl proposed a different mechanism for the construction of a Weyl invariant kinetic term: Rather than using a non-minimal interaction, he introduced a “gauge potential” $W_\mu$ to absorb the non-variance \[3\]. One verifies that

$$I_{\text{Weyl}} = \int \sqrt{|g|} \left( \frac{1}{2} g^{\mu\nu} \left[ \partial_\mu \phi + (d - 2) W_\mu \phi \right] \left[ \partial_\nu \phi + (d - 2) W_\nu \phi \right] \right)$$  \hspace{1cm} (4.2)

is invariant against (1.9), provided $W_\mu$ transforms as

$$W_\mu \rightarrow \text{Weyl} W_\mu - \frac{1}{2} \partial_\mu \sigma.$$  \hspace{1cm} (4.3)

We now demand that $I_{\text{kinetic}}$ in (4.1) coincideds with $I_{\text{Weyl}}$ in (4.2). This is achieved when the following holds.

$$\frac{R}{4(d - 1)} = D^\mu W_\mu + (d - 2) g^{\mu\nu} W_\mu W_\nu$$  \hspace{1cm} (4.4)

this curious Riccati-type equation is familiar in $d = 2$, where it states that $\sqrt{|g|} R$ is a total derivative; a condition that is generalized by (4.4) to arbitrary $d > 2$.

With the help of (4.4) we evaluate, before passing to $d \downarrow 2$, the ambiguous contribution to the action – the last term in (3.5). We have from (4.4)

$$\frac{\int \sqrt{|g|} R}{4(d - 1)(d - 2)} = \frac{1}{d - 2} \int \partial_\mu (\sqrt{|g|} W^\mu) + \int \sqrt{|g|} g^{\mu\nu} W_\mu W_\nu.$$  \hspace{1cm} (4.5)

The first term does not contribute, even when $d \neq 2$, because the integrand is a total derivative for all $d$, while the remainder leaves

$$\lim_{d \downarrow 2} \frac{\int \sqrt{|g|} R}{d - 2} = 4 \int \sqrt{|g|} g^{\mu\nu} w_\mu w_\nu$$  \hspace{1cm} (4.6)

where $w_\mu \equiv W_\mu |_{d=2}$ satisfies, according to (3.4),

$$4D^\mu w_\mu = R \quad \text{at } d = 2.$$  \hspace{1cm} (4.7)

[Note that (4.3) and (4.7) are consistent with the Weyl transformation formula for $R$ at $d = 2$: $R^{\text{Weyl}} e^{-2\sigma} (R - 2D^2 \sigma)$.]
Thus to achieve Weyl invariance, the action should be supplemented by the metric-dependent, but $\psi$-independent term.

$$\Delta I = \frac{8}{\beta^2} \int \sqrt{|g|} \ g^{\mu\nu} w_\mu w_\nu$$

(4.8)

According to (4.3) and (4.7), the Weyl variation of $\Delta I$ is

$$\Delta I \overset{\text{Weyl}}{\longrightarrow} \Delta I + \frac{2}{\beta^2} \int \sqrt{|g|} \left( R + g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right).$$

(4.9)

This cancels the Weyl non-invariant response of $I_{\text{Liouville}}$; see (2.10).

It remains to determine $w_\mu$ by solving (4.7). We are of course interested in a local solution, so that the Weyl-invariant Liouville action be local. Such a solution has been found [4]. It is

$$w^\mu = \frac{\varepsilon^{\mu\nu}}{4\sqrt{|g|}} \left( \frac{\varepsilon^{\alpha\beta}}{\sqrt{|g|}} \partial_\alpha g_{\beta\nu} + (\cosh \omega - 1) \partial_\nu \gamma \right).$$

(4.10)

The second term in the parenthesis is the canonical SL (2, R) 1-form, with

$$\cosh \omega = \frac{g^{++}}{\sqrt{|g|}} \quad \text{and} \quad e^\gamma = \frac{\sqrt{g^{++}}}{\sqrt{g^{--}}}.$$

(4.11)

[(+,-) refer to light-cone components $\frac{1}{\sqrt{2}}(x^0 \pm x^1)$.] This portion of $w^\mu$ is Weyl invariant, while the rest verifies the transformation law (4.3). The solution (4.10) is not unique. One may add to (4.10) any Weyl-invariant term of the form $\frac{\varepsilon^{\mu\nu}}{\sqrt{|g|}} \partial_\nu X$, since this will not contribute to (4.7).

Remarkably $w^\mu$ in (4.10) is not a contravariant vector, even though $D_\mu w^\mu$ is the scalar $R/4$. Consequently our Weyl invariant Liouville action $I_{\text{Liouville}} + \Delta I$ is not diffeomorphism invariant. Its $g^{\mu\nu}$ variation defines a traceless energy-momentum tensor, which however is not (covariantly) conserved.

We do not know what to make of this. Perhaps the above mentioned ambiguity can be used to remedy the diffeomorphism non-invariance, but we have not been able to do so. It would seem therefore that a local, curved-space Liouville action can be either diffeomorphism invariant or Weyl invariant, but not both.

If this conjecture is true, we are facing an “anomalous” situation in a classical field theory, which has previously been seen only in a quantized field theory. It is know that in two dimensions, the diffeomorphism invariant Lagrange density $\frac{1}{2} \sqrt{|g|} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$ is also invariant against Weyl transformations that transform the metric tensor, but not the scalar field $\varphi$ [i.e. Eq. (1.9) at $d = 2$]. However, the effective quantum action that is obtained by performing the functional integral over $\varphi$, yields a metric expression which is either diffeomorphism invariant or Weyl invariant, but not both [5].

If locality is abandoned, one may readily construct a covariant solution for $w_\mu$ in the form $\partial_\mu w$,

$$\omega_\mu = \partial_\mu \omega,$$

(4.12)

with $w$ transforming under a Weyl transformation as [compare (4.3)]

$$w \overset{\text{Weyl}}{\longrightarrow} w - \frac{\sigma}{2}.$$  

(4.13)
Evidently

\[ D^2 w = R/4, \]

\[ w(x) = \frac{1}{4} \int d^2 y \sqrt{|g(y)|} \frac{1}{D^2(x, y)} R(y), \]

(4.14)

where the Green’s function is defined by

\[ D_x^2 \frac{1}{D^2(x, y)} = \frac{1}{\sqrt{|g|}} \delta^2(x - y). \]

(4.15)

Eq. (4.13) is verified by (4.14), and the addition to the Liouville action is just the Polyakov action [5].

\[ \Delta I = \frac{1}{23^2} \int \partial^2 x d^2 y \sqrt{|g(x)|} R(x) \frac{1}{D^2(x, y)} \sqrt{|g(y)|} R(y) \]

(4.16)

This then provides a diffeomorphism and Weyl invariant action for the Liouville theory, which however is non-local. Whether locality can be also achieved remains an open question.

References


