Further Evidence for a Gravitational Fixed Point

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There is growing evidence that a generalized version of General Relativity, possibly including terms of higher order in curvature, may be asymptotically safe in the sense of [1]. This property hinges on the existence of a nontrivial Fixed Point (FP) of the Renormalization Group (RG) flow of the gravitational couplings. Using Exact Renormalization Group Equation (ERGE) methods, it has been shown that a FP exists in 4 dimensions within the Einstein–Hilbert truncation (where only the cosmological constant and Newton’s constant are retained) and there are various arguments suggesting that this FP is not a mere artifact of the truncation [2,3]. Since in the real world gravity does not exist in isolation, but rather interacts with many matter fields, it is important to establish that matter interactions do not spoil the FP. (In fact, for applications to physics it is not logically required that pure gravity exists as a quantum theory: it is only necessary that a theory of gravity and realistic matter exists).

In this direction, positive evidence has come from the analysis of arbitrary matter fields minimally coupled to gravity in the Einstein–Hilbert truncation [4], and from the study of the coupled scalar-gravity theory with arbitrary couplings of the form $\phi^{2n}$ and $\phi^{2n}R$ with $n \geq 0$ (thus including the original Einstein-Hilbert terms) [5].

One of the central issues of this approach is the extension of the results to include higher gravitational couplings. The beta functions of four–derivative gravity have been the subject of several papers in the past [6], also [7] and references therein; for the state of the art on this subject see [8]. Analyzing the structure of divergences in dimensional regularization, it has been shown that the couplings of the terms quadratic in curvature are asymptotically free; however, due to the fact that dimensional regularization is not well–suited to the discussion of mass and threshold effects, the behaviour of the cosmological constant and Newton’s constant has not been thoroughly understood. In [9] a term proportional to $R^2$ was included in the truncation of the ERGE and it was shown that its presence does not spoil the existence of the FP. In fact, the FP values of the cosmological constant and Newton’s constant are only slightly changed relative to the Einstein–Hilbert truncation, and the FP–value of the new term is quite small. It is unlikely that one could ever include many more terms in the truncation.

In this note we shall consider a different approximation scheme, namely the large $N$ expansion, where $N$ is the number of matter fields. This approximation has been first applied to gravity in [10] and was later used in [11] to prove the existence of a FP. I will use the ERGE in conjunction with the large $N$ approximation and the heat kernel expansion to rederive and extend this result. The truncation will consist of assuming that all matter self–interactions can be neglected. Nothing will be assumed a priori about the gravitational action. In the leading order in $N$ the analysis is so simplified, that we will be able to prove the existence of the FP for arbitrary gravitational couplings, and furthermore we shall prove that there exists a scheme (i.e. a choice of cutoff function) such that all the coefficients of terms of cubic and higher order in the curvature vanish.

The ERGE can be written, slightly symbolically,

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left( \frac{\delta^2 \Gamma_k}{\delta \Phi \delta \Phi} + R_k \right)^{-1} \partial_t R_k$$

(1)

where $\Gamma_k$ is the coarse–grained effective action, $\Phi$ are all the fields in the theory, $\text{Tr}$ denotes a trace over all degrees of freedom of the theory (sum over indices and integration over momenta), $R_k$ is a cutoff function that suppresses the propagation of modes with momenta lower than $k$ and $t = \log k$. The RG flow of the theory can be computed by making an Ansatz for the effective action $\Gamma_k$, inserting it in (1) and reading off the beta functions of the couplings.

We will begin by considering gravity coupled to $N$ real scalar fields. In the spirit of an effective field theory, we will consider arbitrary gravitational actions of the form

$$S_{\text{grav}}(g_{\mu\nu}；g_i^{(n)}) = \sum_{n=0}^{\infty} \sum_i g_i^{(n)} O_i^{(n)}$$

(2)

where $O_i^{(n)} = \int d^n x \sqrt{g} M_i^{(n)}$ and $M_i^{(n)}$ are polynomials in the curvature and its derivatives involving $2n$ derivatives of the metric. The index $i$ is used to label different polynomials having the same $n$ (and hence the
same mass dimension $2n$). The first two polynomials are simply $M^{(0)} = 1$ and $M^{(1)} = R$. The corresponding couplings are $g^{(0)} = 2Z_g\lambda$, where $\lambda$ is the cosmological constant and $g^{(1)} = -Z_g = -\frac{1}{16\pi G}$, where $G$ is Newton’s constant. For $n = 2$ I will follow [6-8] and choose as independent polynomials

$$
M_1^{(2)} = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2
$$

$$
M_2^{(2)} = G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2
$$

$$
M_3^{(2)} = R^2; \quad M_4^{(2)} = \nabla^2 R,
$$

where $C$ is the Weyl tensor and $G$ is $32\pi^2$ times the integrand of the Gauss–Bonnet topological invariant. The coefficients $a_k^{(2)}$ and $a_i^{(2)}$ are sometimes denoted $1/\lambda$ and $-\omega/3\lambda$ [7]. The second and fourth terms are total derivatives and do not contribute to local physical processes. We will assume that the scalar fields are all massless and minimally coupled to gravity, with inverse propagator $z = -\nabla^2$. For the running effective action $\Gamma_k$ we assume that it is the sum of the scalar action and (2), where all $k$-dependent. This defines our truncation. We will discuss later the stability of this truncation against quantum fluctuations.

The $1/N$ expansion consists in assuming that the number of matter fields $N \to \infty$. Here we shall restrict ourselves to the leading order approximation, where one simply neglects the gravitational contribution with respect to the matter contribution. This eliminates most of the complications that arise in the gravitational ERGE calculations and provides the simplest way of arriving at a gravitational FP. Taking into account the cutoff function, the modified inverse propagator is $P_k(z) = z + R_k(z)$; we can then write the ERGE

$$
\partial_t \Gamma_k = \frac{N}{2} \text{Tr} \left( \partial_t P_k \frac{P_k}{P_k} \right).
$$

The trace of a function $f$ of the Laplacian can be written as

$$
\text{Tr} f \equiv \sum_i f(\lambda_i) = \int_0^\infty dt \text{Tr} K(t) \tilde{f}(t)
$$

where $\lambda_i$ are the eigenvalues of the Laplacian, $\tilde{f}$ is the Laplace transform of $f$ and $\text{Tr} K(t) = \sum_n e^{-i\lambda_n t}$ is the trace of the heat kernel of $-\nabla^2$. (We assume that there are no negative and zero eigenvalues; if present, these will have to be dealt with separately.)

One can then use the known properties of the heat kernel to calculate the trace in various circumstances. If we are interested in the local behaviour of the theory (i.e. the behaviour at scales $k$ much smaller than the typical curvature) we can use the asymptotic expansion

$$
\text{Tr} K(t) \approx b_0 t^{-2} + b_2 t^{-1} + b_4 + b_6 t + \ldots
$$

where $b_{2k} = \int d^4x \sqrt{g} b_{2k}$, $b_{2k}$ being well-polynomials in the curvature tensor and its covariant derivatives. Then we get

$$
\text{Tr} f = b_0 Q_2(f) + b_2 Q_1(f) + b_4 Q_0(f) + b_6 Q_{-1}(f) + \ldots
$$

where

$$
Q_n(f) = \int_0^\infty dt t^{-n} \tilde{f}(t)
$$

We can rewrite these integrals in terms of the original function $f$: for $n \geq 1$ it is a Mellin transform

$$
Q_n(f) = \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} f(z)
$$

whereas for $n \leq 0$

$$
Q_n(f) = (-1)^n f^{(n)}(0)
$$

where $f^{(n)}$ denotes the $n$-th derivative of $f$.

Using (7) we obtain

$$
\partial_t \Gamma_k = \frac{N}{2} \left( \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left[ Q_2 \left( \frac{\partial_t P_k}{P_k} \right) + \frac{1}{6} R Q_1 \left( \frac{\partial_t P_k}{P_k} \right) \right] + \frac{1}{180} \left( \frac{3}{2} \lambda^2 - \frac{1}{2} G + \frac{5}{2} R^2 + 6\nabla^2 R \right) Q_0 \left( \frac{\partial_t P_k}{P_k} \right) + \ldots \right)
$$

Define the dimensionful beta functions $\partial_t g_i^{(n)} = \beta_i^{(n)} = k^{4-2n} g_i^{(n)}$. The coefficient of each term in (11) is the beta function of the corresponding coupling. The beta functions of the dimensionless couplings $\tilde{g}_i = g_i^{(n)} k^{2n-4}$ are given by

$$
\partial_t \tilde{g}_i = \tilde{\beta}_i = (2n-4)\tilde{g}_i + a_i
$$

where the numbers $a_i^{(n)}$ can be read off (11). For all
n ≠ 2 this leads to a fixed point
\[ \hat{g}^{(n)}_{\ast} = \frac{1}{4 - 2n} a^{(n)}_i \] (13)

For \( n = 2 \) one gets instead the following solution:
\[ g^{(2)}_{\ast}(k) = g^{(2)}_{\ast}(k_0) + a^{(2)}_i \ln(k/k_0) \] (14)

This logarithmic running seems to contradict the hypothesis of the existence of a FP. However, one has to remember that the correct variables describing the coupling of curvature squared terms are the inverses of the variables \( g^{(2)}_i \). The couplings \( (g^{(2)}_i)^{-1} \) are asymptotically free, approaching zero from above or from below depending on the sign of \( a^{(0)}_i \).

Choosing a cutoff function \( R_k \) determines the numbers \( a^{(n)}_i \), and hence affects the position of the FP. This choice does not affect physically measurable quantities, however. One particularly convenient choice of cutoff function is the so-called optimized cutoff \( R_k(z) = (k^2 - z)\theta(k^2 - z) \), where \( \theta \) is the Heaviside step function [12]. With this choice we get \( Q_2 \left( \frac{\partial P_{\lambda}}{\partial k} \right) = k^4 \), \( Q_1 \left( \frac{\partial P_{\lambda}}{\partial k} \right) = 2k^2 \), \( Q_0 \left( \frac{\partial P_{\lambda}}{\partial k} \right) = 2 \) and \( Q_{-1} \left( \frac{\partial P_{\lambda}}{\partial k} \right) = 0 \) for \( n ≤ -1 \). In terms of the cosmological constant and Newton’s constant, the FP occurs for
\[ \hat{\Lambda} = \Lambda k^{-2} = -\frac{3}{4} ; \quad \hat{G} = Gk^2 = -\frac{12π}{N} \] (15)

Perhaps the biggest surprise is the fact that in this scheme the FP–value of all the couplings with \( n ≥ 3 \) is zero. This is due to the flatness of the function \( \frac{\partial P_{\lambda}}{\partial k} \) for small \( z \). With the optimized cutoff, it would therefore be consistent to neglect all the terms with more than four derivatives of the metric.

The dimension of the critical surface is determined by the linearized flow, which is defined by the matrix \( M_{ij} = \frac{\partial a^{(n)}_i}{\partial P_{\lambda}} \). Since the \( a^{(n)}_i \) in [12] are independent of the couplings, the eigenvalues of \( M \) are equal to the canonical dimensions. Therefore, the cosmological constant and Newton’s constant are UV relevant (attractive), the \( g^{(2)}_i \) couplings are marginal and all the higher terms are irrelevant.

With the same techniques, one can explore also other regimes. For large \( t \), the heat kernel on a noncompact manifold has the asymptotic behaviour \( K(t) ≈ \frac{1}{(4πt)^{d-1}} S \) where
\[ S = \int d^4x \sqrt{g} \left[ \frac{1}{6} R + \frac{1}{12} R \frac{1}{\sqrt{2}} R - \frac{2}{3} R_{\mu\nu} \frac{1}{\sqrt{2}} R^{\mu\nu} + \ldots \right] \]

Since all the terms have the same dimension, they also have the same overall running. The beta functions of the operators appearing in \( S \) are equal to \( \frac{1}{(4πt)^{d-1}} Q_1 \left( \frac{\partial P_{\lambda}}{\partial k} \right) \), times the coefficient of the operator as it appears in \( S \). Consequently, there is a FP where the FP–action is proportional to \( S \), the exact coefficient being scheme–dependent. With the optimized cutoff, the FP–action is \( \Gamma_\ast = \frac{N}{6\pi^2} k^2 S \). We note that nonlocal actions of this form may have some relevance to cosmology [15].

Let us now consider the effect of other matter fields. Following [13], we can calculate the contribution to the ERGE of \( n_S \) scalar fields, \( n_D \) Dirac fields, \( n_M \) Maxwell fields, all massless and minimally coupled. Unlike in [13], however, we shall not make any assumption about the background metric, so as to be able to discriminate the coefficients of various contractions of two curvatures. For each type of field we choose the cutoff function in such a way that the modified propagator has the form \( P_k(\mathcal{O}) \), where \( \mathcal{O}^{(S)} = -\nabla^2 \) on scalars, \( \mathcal{O}^{(D)} = -\nabla^2 + \frac{4}{3} \) on Dirac fields and \( \mathcal{O}^{(V)} = -\nabla^2 \delta_\nu^\nu + R^\nu\nu \) on Maxwell fields in the gauge \( \lambda = 1 \). We can then use the heat kernel coefficients given e.g. in [16]. In the case of the Maxwell fields, the calculation of the effective action with nonminimal operators (corresponding to other gauges) was given in [17], showing that the results are gauge–independent.

We assume that the numbers \( n_S, n_W \) and \( n_M \) are all of order \( N \), or zero. The ERGE becomes
\[ \partial_t \Gamma_k = \frac{n_S}{2} Tr(S) \left( \frac{\partial P_k}{P_k} \right) - \frac{n_D}{2} Tr(D) \left( \frac{\partial P_k}{P_k} \right) + \frac{n_M}{2} Tr(M) \left( \frac{\partial P_k}{P_k} \right) - n_M Tr(S) \left( \frac{\partial P_k}{P_k} \right) \] (16)

where the argument of each function appearing under the traces is the appropriate operator \( \mathcal{O} \) and the last term is due to the ghosts. With the optimized cutoffs, the beta functions are characterized by the following coefficients:
\[ a^{(0)}_i = \frac{1}{96\pi^2} (n_S - 2n_D + 4n_M) \]
\[ a^{(1)}_i = \frac{1}{96\pi^2} (n_S - 2n_D - 4n_M) \]
\[ a^{(2)}_i = \frac{1}{2880\pi^2} \left( \frac{3}{2} n_S + \frac{25}{4} n_D + 18 n_M \right) \]
\[ a^{(2)}_i = \frac{1}{2880\pi^2} \left( -\frac{1}{2} n_S - \frac{11}{4} n_D - 31 n_M \right) \]
\[ a^{(2)}_i = \frac{5}{2880\pi^2} n_S \]
\[ a^{(2)}_i = \frac{1}{2880\pi^2} (6n_S + 3n_D - 18 n_M) \] (17)

The fixed point occurs at
\[ \hat{\Lambda}_\ast = -\frac{3}{4} n_S - 2n_D + 2n_M \]
\[ \hat{G}_\ast = \frac{12\pi}{n_S - 2n_D + 4n_M} \] (18)

We see that there is a FP for any value of \( n_S, n_W \) and \( n_M \), but Newton’s constant is only positive if the number of scalar field is not too large. The couplings \( (g^{(2)}_i)^{-1} \) are asymptotically free and again \( g^{(n)}_i = 0 \) for \( n ≥ 3 \).
Let us now discuss the consistency of the truncation that has been used in this paper. In general, one could expect processes with intermediate gravitons to generate matter interactions even if they were initially set to zero. However, this will not happen in the UV limit if the interactions are asymptotically free. It has been shown in [5] that a large class of scalar interactions becomes asymptotically free in the presence of gravity. Thus, the truncation we used here for the scalar fields is probably consistent. Gauge fields are asymptotically free already in flat space and it is reasonable to expect that the situation will not change in the presence of dynamical gravity. More troublesome are the Yukawa couplings, which in the standard model have positive beta functions. It is tempting to conjecture that the phenomenon that occurs for scalar fields is more general, and that there exists a “Gaussian–Matter FP” where all the matter interactions are asymptotically free and only the purely gravitational couplings have nonvanishing values at the FP. Earlier calculations reported in [7] suggest that this is the case. Then, the truncation adopted in this paper with regard to the matter fields would be a consistent truncation.

The use of the large $N$ limit has several advantages over previous calculations. First of all, it dramatically simplifies the derivation of the beta functions, making the result very transparent. Furthermore, the calculations presented here are free of the gauge fixing ambiguities that occur when graviton loops are taken into account. This shows that the FP cannot be an artifact of the gauge choice. The most important result derived here is the existence of the FP for all the terms of higher order in curvature. What is perhaps even more striking, all the couplings with $n \geq 3$ can be made to vanish by a simple choice of cutoff scheme. It is known that the position of the FP is scheme dependent, but in the case of the cosmological constant and Newton’s constant no sensible scheme is known were they vanish at the FP.

One of the problems with applications of the ERGE is the absence of systematic ways of estimating the errors due to the truncation. If one can work with a sufficiently general consistent truncation, the large $N$ limit introduces just such an expansion parameter. Calculating the next-to-leading order in $1/N$ requires the evaluation of graviton contributions. This is obviously impossible with the full gravitational action, but the results presented here suggest that in certain schemes it may be consistent to set to zero all terms with $n \geq 3$. This would not be an a priori, uncontrollable truncation: due to the scheme–dependence of the physical results, it would be equivalent to calculating with the full action.

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References


