GRAVASTAR SOLUTIONS WITH CONTINUOUS PRESSURES AND EQUATION OF STATE

A. DeBenedictis ∗ † D. Horvat ‡ † S. Ilijić ‡ S. Kloster § K. S. Viswanathan †

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Abstract

We study the gravitational vacuum star (gravastar) configuration as proposed by [1] in a model where the interior de Sitter spacetime segment is continuously extended to the exterior Schwarzschild spacetime. The multilayered structure of [2] - [4] is replaced by a continuous stress-energy tensor at the price of introducing anisotropy in the (fluid) model of the gravastar. Either with an ansatz for the equation of state connecting the radial $p_r$ and tangential $p_t$ pressure or with a calculated equation of state with non-homogeneous energy/fluid density, solutions are obtained which in all aspects satisfy the conditions expected for an anisotropic gravastar [1]. Certain energy conditions have been shown to be obeyed and a polytropic equation of state has been derived. Stability of the solution with respect to possible axial perturbation is shown to hold.

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1 INTRODUCTION

Recently it has been proposed by several authors that objects other than black holes could be formed by gravitational collapse of a massive star. Black hole horizons introduce number of theoretical problems and a consensus of solutions of those problems has not yet been reached [2], [3].

∗Pacific Institute for the Mathematical Sciences, Simon Fraser University site, Burnaby, British Columbia, V5A 1S6, Canada (adebened@sfu.ca)
†Department of Physics, Simon Fraser University, Burnaby, British Columbia, V5A 1S6, Canada
‡Department of Physics, Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, HR 10000 Zagreb, Croatia
§Center for Experimental and Constructive Mathematics, Simon Fraser University, Burnaby, British Columbia, V5A 1S6, Canada
One proposal, which was initiated recently by Mazur and Mottola (M-M) [2]-[4], is the so-called “gravastar”. In this scenario, quantum vacuum fluctuations are expected to play a non-trivial role in the collapse dynamics. A phase transition is believed to occur yielding a repulsive de Sitter core which helps balance the collapsing object thus preventing horizon (and singularity) formation [6], [5]. It is expected that this transition occurs very close to the limit $2m(r)/r = 1$ so that, to an outside observer, it would be very difficult to distinguish the gravastar from a true black hole. Since this proposal, different versions of the original gravastar model have appeared with all variety of ingredients. For example, it has been shown how a gravastar structure can form from a Born-Infeld scalar field such as that predicted by low energy string theory [7]. In the multi-layer structure of the M-M model with de Sitter core and asymptotical Schwarzschild outside region, additional features have been added. In [8] gravastar solutions have been studied in a generalized (Reissner-Nordström) exterior and solutions of the model stemming from the original Mazur-Mottola model have been analyzed. Gravastar type solutions in the context of solutions to Einstein’s equations with tube-like cores have also been recently considered [9]. Pioneering in-depth studies of spherically symmetric systems with deSitter asymptotics may be found in [10] - [13].

It has recently been shown [1] that the gravastar configuration (see the next section for the elaborated physical model of it) has to have anisotropic pressures which in addition should obey some of the energy conditions of General Relativity. In this context one would ideally construct such a spherically symmetric model of an anisotropic fluid with a corresponding equation of state. The model, by the definition of the gravastar, should not possess a horizon, it should to be stable and its (anisotropic) pressures and density ideally would not violate energy conditions. Certainly this last requirement is somewhat relaxed in the sense that configurations constructed in this way will violate some of the (usual) energy conditions from its very initial definition.

In this paper we present solutions for the gravastar as proposed by Cattoen, Faber and Visser (CFV) [1]. In their paper an attractive sketch (see Fig.1 drawn here for the convenience of the reader) is given as a guide to all the future gravastar model builders. In the next section a general description of gravastars as anisotropic fluid spheres is presented.

In section 3 we describe the evolution of the anisotropic gravastar model and one of the solutions (model 1) which reproduces the CFV sketch of the anisotropic gravastar. The value of the surface redshift is calculated while a more elaborate analysis of this important quantity is relegated to a future work. Section 4 is devoted to the solution (model 2) which starts from the construction of an equation of state and consistently solving the remaining generalized Tolman-Oppenheimer-Volkoff equation. Stability of this solution with respect to axial perturbations is calculated. Finally, we conclude with comments and possible extensions of the anisotropic gravastar solutions.
FROM ISOTROPY TO ANISOTROPY

Einstein’s field equations, being exceedingly complicated because of their nonlinear character, have one most important closed form solution outside a spherical star of total mass $M$, namely the Schwarzschild metric. Its line element in curvature coordinates is

$$ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2,$$

(1)

where $d\Omega^2 := d\theta^2 + \sin^2 \theta d\phi^2$.

For the interior of the star one has to choose a “physically reasonable” stress-energy tensor. One attractive possibility is to use the perfect fluid model of matter where

$$T_{\mu \nu} = (\rho + p) u^\mu u_\nu + p \delta^\mu_\nu,$$

(2)

with $\rho$ and $p$ the energy density and pressure respectively in the co-moving frame of the fluid, and $u^\mu$ being the fluid four-velocity. In the static case, which is studied here, the interior metric may be written as

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2.$$

(3)
The Einstein equations, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, give a system of equations:

$$e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = 8\pi p,$$  \hspace{1cm} (4a)

$$e^{-\lambda} \left( \frac{1}{2} \nu'' - \frac{1}{4} \lambda' \nu' + \frac{1}{4} (\nu')^2 + \frac{\nu' - \lambda'}{2r} \right) = 8\pi p,$$  \hspace{1cm} (4b)

$$e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 8\pi \rho.$$  \hspace{1cm} (4c)

These are supplemented with the conservation law $T_{\nu\mu} = 0$, which in this case yields only one non-trivial equation:

$$T'_{11} + \frac{2}{r} \left[ 1 + \frac{r}{4} \nu' \right] T^1_1 - \frac{1}{2} \lambda' T^0_0 - \frac{2}{r} T^2_2 = 0.$$  \hspace{1cm} (5)

Elimination of the function $\nu(r)$ from the above under-determined system leads to a convenient form of the conservation equation i.e. the Tolman-Oppenheimer-Volkov (TOV) \[14], \[15] equation:

$$\frac{dp}{dr} = - \left[ \frac{\rho(r) + p(r)}{r^2} \right] \frac{[m(r) + 4\pi p(r)r^3]}{[1 - 2m(r)/r]}.$$  \hspace{1cm} (6)

with

$$m(r) := -4\pi \int_0^r T^0_0(\tilde{r}) \tilde{r}^2 d\tilde{r}.$$  \hspace{1cm} (7)

One may also specify an equation of state which relates pressure and density. If, for simplicity, we adopt at this stage a constant density profile function (with built-in boundary conditions)

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r < R \\ 0 & \text{for } R < r \end{cases} \hspace{1cm} (8)$$

then it turns out to be an oversimplification which leads to analytic integration of (6) and the corresponding field equations but pressure and density do not obey the energy conditions which are analogous to the requirement of mass positivity in the Newtonian mechanics.

With the isotropic fluid and the above homogeneous energy density static solutions are allowed for objects with a restricted total mass $M$ to radius $R$ ratio i.e. $2M/R \leq 8/9$ \[16].

2.1 ANISOTROPY

The idea of anisotropy in the spherically symmetric geometry was perhaps first introduced by G. Lemaître \[17] and suggested by Einstein (as quoted in Ref. \[17]). The limiting case of $p_r \to 0$ is mentioned there and the remaining transversal pressure was
said to be enough to support a (stable) sphere (see also [18]). Further development has brought different refinements of the original anisotropy notion (see the papers [19] - [30] for studies and further references).

The perfect fluid requires that the pressure in the interior of a star be isotropic, leading to calculations of isotropic polytropes for descriptions of objects like white dwarfs or neutron stars. Another option giving more freedom to the equation of state within the spherical symmetry is the introduction of the stress-energy tensor which is anisotropic in its principal pressures. The anisotropy is sometimes (spontaneously) produced by extending the notion of a (perfect) fluid to phenomenological models including eg. scalar fields, or strongly interacting matter, although it is not known how large this anisotropy may be in realistic models.

The stress-energy tensor for an anisotropic matter/fluid distribution is given by

\[ T_{\mu \nu} = (\rho + p_t) u^\mu u_\nu + p_t \delta^\mu_\nu + (p_r - p_t) s^\mu s_\nu, \tag{9} \]

where \( p_r \) is the radial pressure in the co-moving frame and (again) due to the spherical symmetry, the angular components are identified and are denoted as transversal pressures, \( p_t \). The vector \( s^\mu \) is orthogonal to the fluid four-velocity (\( s^\mu u_\mu = 0 \)).

In the gravastar model, the core interior is assumed to be given by a de Sitter solution so the appropriate pressure/density ratio value equal to minus one should be implemented as an initial condition for the density profile function, as well as a corrector to an equation of state connecting the pressure and density.

For the moment we assume constant energy density \( \rho_0 \). In addition to the (energy) density in a prescribed form, a relation between the radial \( p_r \) and the tangential \( p_t \) pressures should be given. An ansatz for the anisotropy measure,

\[ \Delta := \frac{p_t - p_r}{\rho}, \tag{10} \]

will be used following the hints given in [11]. Bounds on the anisotropy measure are calculated and are expressed in terms of the “compactness” \( 2m(r)/r \), so our ansatz has the following form:

\[ \Delta = \frac{\alpha^2}{12} \frac{2m(r)}{r}. \tag{11} \]

The constant \( \alpha^2/12 \) will simplify the (numerical) calculation and it is a measure of anisotropy for this version of the gravastar model. The TOV equation now assumes the following form:

\[ r^2 p_r'(r) = \frac{4\pi r^3}{9} \left[ \alpha^2 \frac{\rho_0^2}{12} - \frac{3[\rho_0 + p_r(r)][\rho_0 + 3p_r(r)]}{1 - 8r^2\pi\rho_0/3} \right]. \tag{12} \]

with the initial condition \( p_r(0) = p_0 \), requiring also that \( p_r(0) = p_t(0) \).

The radial pressure \( p_r(r) \) is given by

\[ p_r(r) = \frac{\rho_0}{3} \left[ -1 + \alpha \sqrt{1 - \mu(r)} \left( -1 + \frac{2(\alpha - 2)}{\alpha - 2 + (\alpha + 2) \exp[\alpha(-1 + \sqrt{1 - \mu(r)})]} \right) \right]. \tag{13} \]
where the “compactness” is \( \mu(r) = 2m(r)/r = 8\pi r^3 \rho_0/3 \), while the transversal pressure can be obtained through (11), and is given by

\[
p_t(r) = \frac{\rho_0}{12} \left\{ 4 \left[ -1 + \alpha \sqrt{1 - \mu(r)} \right] \right. \\
\left. \cdot \left[ -1 + \frac{2(\alpha - 2)}{\alpha - 2 + (\alpha + 2) \exp[\alpha(-1 + \sqrt{1 - \mu(r)})]} \right] \right. \right. \\
\left. \left. + \mu \alpha^2 \right\} . \quad (14)
\]

For all \( \alpha > 0 \), both pressures start at \( -\rho_0 \) in the centre of the gravastar. At \( \mu = 1 \) (if one would allow such a density profile) the values are \( p_r = -\rho_0/3 \), \( p_t = \frac{\rho_0}{3} (\alpha^2 - 1) \).

Only for values \( \alpha > 4.11172 \), does the radial pressure reach positive values before returning back into the negative pressure region. It is also worth noting that this simple model does not lead to an equation of state, since the energy density does not change within the star.

As previously mentioned, this oversimplified model with the constant (energy) density function as given above cannot reproduce the sketched Figure 1 i.e. it cannot provide a gravastar in the proposed model without the surface layer structure as required by junction conditions. The most obvious extension is to use perhaps the same ansatz for the equation of state connecting the behaviour of radial and tangential pressures as given in (11). For the density one takes some non-homogeneous distribution which will give a more complex dependence of \( p_r(r) \) and lead to a possible solution (see the model 1 below).

Certainly the ansatz could be replaced by a calculated equation of state and introduce again an inhomogeneous density distribution (see the model 2 below). On these two possibilities the new results of this paper are based.

### 2.2 ENERGY CONDITIONS

In order to have a reasonably realistic radial pressure component we demand the (most natural) weak energy condition (WEC) be satisfied everywhere:

\[
\rho(r) \geq 0 \quad \text{along with} \quad \rho(r) + p_r(r) \geq 0 \quad \text{and} \quad \rho(r) + p_t(r) \geq 0 . \quad (15)
\]

Also, we require

\[
\rho(r = 0) > 0 \quad (16)
\]

and, since stability of the fluid/gravastar requires that \( \rho(r) \) must not increase outwards,

\[
\frac{d\rho(r)}{dr} \leq 0 . \quad (17)
\]

The above WEC obviously implies:

\[
\rho(r) + p_r(r) \geq 0 \quad \text{and} \quad \rho(r) + p_t(r) \geq 0 , \quad (18)
\]
the so called null energy condition (NEC). There is also a dominant energy condition (DEC) which requires that

$$\rho(r) \geq 0 \quad \text{and} \quad p_r(r) \in [-\rho(r), +\rho(r)] \quad \text{and} \quad p_t(r) \in [-\rho(r), +\rho(r)],$$

and which implies the other two energy conditions.

Another commonly studied energy condition is the strong energy condition (SEC) which states that, for our static system,

$$\rho + p_r \geq 0, \quad \rho + p_t \geq 0 \quad \text{and} \quad \rho + p_r + 2p_t \geq 0.$$  

Since gravastars possess a deSitter core, it is not possible to satisfy this energy condition.

Within the normal range of the gravastar it could be reasonable to propose that the speed of sound shall not exceed 1 (speed of light). This requirement is reasonable to apply in the region where one could expect that the unusual physics govern most of the physical processes (i.e. where the matter possesses the least exotic behaviour). From the sketch in Figure 1 this is expected to be in the gravastar atmosphere, so

$$v_s^2 = \left. \frac{dp}{d\rho} \right|_{\text{atm.}} \leq 1.$$  

In this region one should be able to derive a polytropic equation of state which should be expressed by parameters which are in accordance to the requirement involving the speed of sound (see next Section).

3 MODEL 1: THE ANSATZ AND AN INHOMOGENEOUS DENSITY DISTRIBUTION

In this section the solution of the gravastar non-layered model (i.e. the model sketched in Figure 1) is sought by prescribing an equation of state relating the pressures, i.e. an equation which connects the radial $p_r(r)$ and tangential $p_t(r)$ pressure. An improved ansatz of the form

$$\tilde{\Delta} = \frac{\rho}{\rho_0}$$

is assumed, where $\Delta$ is as given in (11). The (inhomogeneous) density function profiles will be chosen to be either of the exponential form or a polynomial form.

The gravastar is a static configuration with a prescribed behaviour of the (radial and tangential) pressure. As sketched in Figure 1 the radial (and tangential) pressure originates at $r = 0$ with the initial value conforming with the de Sitter interior definition ($p_r(r = 0) = p_t(r = 0) = -\rho(r = 0)$). Investigations concerning a possible analytic form of the (energy) density have produced stringent bounds on the allowed functional behaviour.
In the first approach to the gravastar model we start from the anisotropic TOV equation. Following the requirement imposed on the density profile we will be using a simply behaving function of the form

\[ \rho(r, r_0) = \rho_0 e^{-(r/r_0)^n}, \]

where \( r_0 \) has an appropriate dimension and \( n \) is chosen to be a positive integer. Motivations for the exponential form of \( \rho \) may be found in [13], [31], where they chose \( n = 3 \), which simplifies some calculations. The values of the central density, \( \rho_0 \), and of the fall-off constant, \( r_0 \), may not be set arbitrarily, since one must make sure that the compactness does not exceed unity. Alternatively, one may require a certain total mass of the configuration, and set the upper limit to the compactness within the star. Then the values for the parameters \( \rho_0 \) and \( r_0 \) would follow. In this (and the following) model we will restrict our (numerical) procedures to a general total mass \( M \) and \( r_0 \) by keeping the corresponding quantities as ratios.

When requirements of anisotropy are being imposed we use the anisotropic fluid stress-energy tensor \( T^\mu_\nu = \text{diag}(\rho, p_r, p_t, p_t) \). As before, we have from the equations (4a)-(5)

\[ m(r) = 4\pi \int_0^r \rho(\tilde{r}) \tilde{r}^2 d\tilde{r} \]  

and the generalized TOV equation

\[ \frac{dp_r(r)}{dr} = -\left[\rho(r) + p_r(r)\right] \frac{m(r) + 4\pi p_r(r) r^3}{r^2[1 - 2m(r)/r]} + 2\rho(r) \frac{\Delta(r)}{r} \]

(25)

where we have introduced the anisotropy parameter \( \Delta(r) \) from above.

In Figure 2 the (numerically obtained) solutions of the transversal \( p_t \) and radial \( p_r \) pressures are depicted when the modified ansatz given in (22) is used as an input for the equation of state. Also the density profile \( \rho(r) \) given in (23) (decreasing dashed line) as well as \( \rho_0 \cdot \mu(r) = 2\rho_0 \cdot m(r)/r \) for \( \rho_0 = 1 \) (initially increasing dashed line) is presented.

The radial pressure \( p_r(r) \) as well as energy density satisfy all the energy conditions assumed to be valid for gravastar solutions.

Instead of insisting on numerical details we want here to show that the proposed gravastar structure could be found also with other energy density profile functions which also follow general requirements for a gravastar. By choosing another independent profile we will show that solutions found in this paper are not just artefacts of a very specifically chosen energy density profile but a more general feature of prescribed initial conditions within the framework of general relativity.
Figure 2: The gravastar model with the energy density profile \( \rho = \rho_0 \exp[-(r/r_0)^n] \) and anisotropy \((p_t - p_r)/\rho = \alpha^2 (\rho/\rho_0) \mu/12\): radial (lower solid line) and transversal (upper solid line) pressures, energy density (dashed line) and the compactness (dotted line). \( M \) is the total mass of the configuration. In this example the parameters are \( n = 3 \), maximal compactness within the gravastar \( \mu_{\text{max}} = 0.822 \), and \( \alpha = 6.69 \). \( M = 1 \) here so, for a body of mass 10 km, distances are measured in units of 10 km.

3.1 MODEL 1b

As the next example of the gravastar model construction we will use the density profile function of the form
\[
\rho(r) = \rho_0[1 - (r/R)^n]; \quad n \geq 2,
\]
and with the ansatz given in (11). Again the values of the central density, \( \rho_0 \), and of \( R \) are adjusted to make sure that the compactness does not exceed unity. As an example we used \( n = 4 \) in (26) to construct a gravastar of total mass \( M = 1 \) with maximal compactness within the gravastar \( \mu_{\text{max}} = 0.93 \). For these parameters we obtained \( \rho_0 = 0.037 \) and \( R = 2.31 \), and the compactness at the surface \( \mu(R) = 0.864 \). If \( M \) is measured in decametres, the central density in this model is approximately \( 4 \times 10^{24} \) kg/Dm\(^3\), approximately 1000 times neutron star density. The gravitational repulsion of the near deSitter type core region is supporting such densities.

By adjusting the value of \( \alpha \) we could obtain a solution in which both pressures vanish at \( r = R \), thus obtaining a gravastar. The appropriate value was \( \alpha = 6.10 \), while for lower/upper values the pressures end up at negative/positive values at the surface of the gravastar, such solutions were not considered any further (although the tangential pressure and energy density need not vanish at the surface). The pressures, the density profile and the compactness of this model are shown in Fig. 3.
Figure 3: The gravastar model with the energy density profile \( \rho = \rho_0 (1 - (r/R)^n) \) and anisotropy \( (p_t - p_r)/\rho = \alpha^2 \rho/\rho_0 \mu/12 \): radial (lower solid line) and transversal (upper solid line) pressures, energy density (dashed line) and the compactness (dotted line). \( M \) is the total mass of the configuration. In this example the parameters are \( n = 4 \), maximal compactness within the gravastar \( \mu_{\text{max}} = 0.889 \), and \( \alpha = 6.10 \). Surface compactness is \( \mu(R) = 0.864 \).

### 3.2 THE EQUATION OF STATE AND SURFACE REDSHIFT

Apart from the energy condition relations among the components of \( T_{\mu \nu} \), an equation of state for the matter described by \( T_{\mu \nu} \) should exist. An equation of state could be either given in the sense that it is prescribed as an initial condition under which the Einstein field equations should be solved or derived as a posteriori calculated law which should be in accordance with the expected characteristics of a physical system described by the \( T_{\mu \nu} \) content.

The dependence of the (radial) pressure on the energy density is usually, in the perfect fluid regime, expressed in a mathematical form

\[
p = \kappa \rho^{1+1/n_p} = \kappa \rho^{\gamma}
\]

which represents the polytrope, and \( n_p \) is a polytropic index \([32]\).

According to the graph of the radial pressure vs. density (in the mirrored presentation - see Figure 4) it is clear that the equation of polytrope could/should be calculated only in the “atmosphere” of the gravastar (see Figure 1), where radial pressure and density follow the usual behavior of a more standard fluid and the equation of state is assumed to relate energy density to radial pressure, as is commonly employed when dealing with anisotropic fluids. In the polytropic regime the speed of sound respects the bound \( dp/d\rho < 1 \). The specific models constructed here are designed to illustrate the maximum compactness allowed by the speed of sound causality condition \([21]\). The parameters can easily be adjusted to produce models where the speed of sound is much less than unity.
One method of determining a star’s properties is the measurement of its surface redshift, $Z$, of spectral lines produced in the star’s photosphere. It is defined by the fractional change ($\Delta \lambda$) between observed ($\lambda_0$) and emitted ($\lambda_e$) wavelength compared to emitted wavelength:

$$Z := \frac{\Delta \lambda}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1,$$

and, according to our notation, its surface value becomes

$$Z_R = e^{-\nu(R)/2} - 1.$$

In the context of the presented model calculations it is important to recall that anisotropy affects the (surface) redshift so that for the static perfect fluid sphere the surface redshift is not larger then $Z_s = 2$ \cite{10, 23} whereas in \cite{23, 26}, and references therein, the maximal surface redshift for anisotropic sphere is found to be 3.842. The model 1a (see Fig. 2) provides the value $Z_s = 1.230$ (with maximum compactness $\mu_{\text{max}} = 0.822$). The model 1b (Fig. 3), with $\mu_{\text{max}} = 0.889$ and surface compactness $\mu_s = \mu(R) = 0.864$ gives the value $Z_s = 1.712$. Both values are well within expected values for a realistic (neutron star) object.

![Figure 4: The equation of state resulting from the solution shown in Fig. 2: radial pressure (thick solid line) and its derivative with respect to the energy density (thick dashed line). The polytropic fit computed in the range indicated by the marker and extrapolated over the whole range of densities: radial pressure (thin solid line) and its derivative (thin dashed line). The parameters of the polytropic fit are $\kappa = 9.16$ and $\gamma = 1.70$. The atmosphere corresponds to the left region of the graph.](image)

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Figure 5: The radial pressure of Fig. 2 (solid line) and the polytropic fit radial pressure (dashed line) vs. $r$.

Figure 6: The equation of state resulting from the solution shown in Fig. 3: radial pressure (thick solid line) and its derivative with respect to the energy density (thick dashed line). The polytropic fit computed in the range indicated by the marker and extrapolated over the whole range of densities: radial pressure (thin solid line) and its derivative (thin dashed line). The parameters of the polytropic fit are $\kappa = 38.0$ and $\gamma = 2.18$. The atmosphere corresponds to the left region of the graph.
Figure 7: The radial pressure of Fig. 3 (solid line) and the polytropic fit radial pressure (dashed line) vs. $r$. 
Here we select a density profile of the form (23). Specifically, for numerical calculations, we chose
\[ \rho_0 = \frac{1}{16\pi M^2}, \quad r_0 = (12)^{1/3} M, \quad M = 6000 \text{ m}. \]

The energy density is related to the radial pressure via a Mbonye-Kazanas (MK) equation of state [31]:
\[ p_r(\rho) = \left[ s - (s + 1) \left( \frac{\rho(r)}{\rho_0} \right)^m \right] \left( \frac{\rho(r)}{\rho_0} \right)^{1/n} \cdot \rho(r). \] (30)

The MK equation of state possesses several desirable features (with appropriately chosen parameters), namely: the speed of sound is less than one in the atmosphere and the WEC and DEC are satisfied. The model considered here utilizes (30) with the parameters \( m = 2 \) and \( n = 1 \). The tangential pressure will then be given by the anisotropic TOV equation (25).

In Figure 8 we plot \( p_r/\rho \) (solid) and \( p_t/\rho \) (dashed) in order to study the DEC.

![Figure 8: The DEC with radial pressure \( p_r(r)/\rho(r) \) and with tangential pressure \( p_t(r)/\rho(r) \). The parameters are: \( s = 2.2135 \) and total mass \( M = 6 \text{ km} \).](image)

From Figure 8 we see that there exists DEC violation with respect to the tangential pressure. This violation occurs in the crust of the gravastar, where the physics is expected to be "exotic" and generally cannot be avoided as noted in [1]. Note that in a model constructed via this method, the DEC violation is minimal and the DEC can be respected in the outer layer of the star.

The expression for the anisotropy measure \( \Delta \) (which is now calculated and not prescribed as in model 1), although in closed form, is not simple. Instead, for the specific model considered here, we plot it in Figure 9. Notice from this figure that
\( \Delta \) vanishes in several locations. It should be possible at these points to patch the solution to a perfect fluid yielding a regular star-like structure in the outer region.

Following the calculation in the previous model, the specific model constructed here possesses a surface redshift of \( Z_R = 0.4 \).

![Figure 9: The anisotropy parameter, \( \Delta = (p_t(r) - p_r(r))/\rho(r) \), for model 2.](image)

### 4.1 AXIAL STABILITY

Here we analyze stability of this model against axial perturbations. Stability of gravastar models with thin-shells against spherical perturbations has been analyzed in several papers \cite{33, 34}. The model of Lobo \cite{34} considers the interesting scenario where the gravastar like object is formed from the gravitational condensation of dark energy, believed to be responsible for the current acceleration of the universe. The physical motivation for considering dark energy stars may be found in \cite{35}.

We concentrate here on the issue of axial perturbations, which allow for perturbative rotations. This is useful as, in a realistic astrophysical collapse scenario, one would expect the star to possess some amount of angular momentum. For details the interested reader is referred to the book by Chandrasekhar \cite{36} and the paper by Dymnikova and Galaktionov \cite{37}. Before proceeding we shall establish some notation.

\[
g_{00} = -e^{\nu(r)} = -e^{-\lambda(r)}e^{\Gamma(r)}, \tag{31}
\]

where

\[
e^{-\lambda(r)} = 1 - \frac{2m(r)}{r}
\]

and

\[
\Gamma(r) = \int_{\tilde{r}=0}^{\tilde{r}} h(\tilde{r}) \, d\tilde{r},
\]

\[
h(r) = \frac{8\pi r^2 (\rho + p_r)}{r - 2m(r)} \tag{32}
\]
A more general line element may be written as
\[ ds^2 = -e^{\nu} dt^2 + e^\lambda dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta [d\phi - \omega dt - q_2 dr - q_3 d\theta]^2. \] 

(33)

By comparison with (3), the unperturbed metric has \( \omega = q_2 = q_3 = 0 \). Axial perturbations correspond to these becoming non-zero. Our analysis is to linear order in these quantities.

We write
\[ \omega(r, \theta, t) = \tilde{\omega}(r, \theta) e^{i\sigma t}, \] 

(34)

and similarly for \( q_2 \) and \( q_3 \).

It can be shown that the system governing the perturbations can be reduced to a single second-order differential equation, which can be solved by separating the variables \( r \) and \( \theta \) [37]. In brief, the perturbed field equations give a relation between \( \omega \) and \( q_2 \) and \( q_3 \), allowing the elimination of \( \omega \). The quantity \( Q := e^\nu r^2 \sin^3 \theta (\partial_\theta q_2 - \partial_r q_3) \) is written as \( Q = R(r)\Theta(\theta) \) and the resulting radial equation is
\[ r^2 e^\nu \frac{d}{dr} \left[ \frac{e^\nu dR}{r^2 \frac{d}{dr}} \right] - \mu_l^2 \frac{e^\nu R}{r^2} + \sigma^2 R = 0, \]

(35)

where \( \mu_l^2 \) is the eigenvalue of the angular equation which can take on the values \( \mu_l^2 = (l + 2)(l - 1) \) for \( l = 2, 3, \ldots \).

We can make a change of coordinates,
\[ r_* = \int e^{-\nu} dr, \]

so that (35) reduces to a Schrödinger type equation:
\[ \left[ \frac{d^2}{dr_*^2} - V_l(r) \right] Z_l = -\sigma_l^2 Z_l \]

(36)

where
\[ Z_l(r) = \frac{R_l(r_*)}{r}, \]

and
\[ V_l(r) = \left( \frac{e^\nu}{r^2} \right) \left( \mu_l^2 + 2e^\nu - e^\nu r^2 \right). \]

(37)

The allowed values of \( \sigma \) are therefore the eigenvalues of (36). If all the eigenvalues are positive, then \( \sigma \) will always be real. Hence the time dependence of the perturbations will oscillate but not grow. These perturbations are then stable.

The eigenvalues, \( \sigma_l \), are determined by the potential \( V_l(r) \). In fact, knowing the asymptotic behaviour of the potential is sufficient as observed in [37]. The eigenvalues will be positive if the asymptotic behaviour near zero and as \( r \to \infty \) both go as \( r^{-2} \).
We consider the limit as $r \to 0$ which, for gravastars with de Sitter centers gives

$$\lim_{r \to 0} \nu(r) = 0 \quad \text{and} \quad \lim_{r \to 0} \nu'(r) = 0.$$ 

These conditions lead to

$$V_l \to \frac{l(l + 1)}{r^2}$$

as $r \to 0$. In the far region, $r \to \infty$, we need to show that $\Gamma(r)$ possesses finite limit. For large enough $r$ it can be seen (from Figure 8) that $r - 2m(r) \gg 1$ and that $p_r(r) < \rho(r)$ hence

$$h(r) \leq 16\pi r^2 \rho(r).$$

Since the integral of (39) has finite limit, $\Gamma(r)$ will also have finite limit and therefore $\nu(r)$ also possesses finite limit. In addition, $rh(r) \to 0$ in the far zone so that $r\nu'(r)$ vanishes. Thus, for very large $r$, $V_l(r)$ has $r^{-2}$ behaviour. This completes the argument for axial stability.

5 CONCLUDING REMARKS

In this paper we have constructed several models of gravastars which do not employ thin shells. These models obey the conditions posed in [1]. The “stellar construction” is based on two methods. One method employs the prescription of the energy density and anisotropy parameter, where the system can be shown to fit a polytropic equation of state in the stellar atmosphere. In the second method, a prescription of the energy density is imposed, and then the radial pressure is defined via a reasonable equation of state. A general model of the latter type has been shown to be stable under axial perturbations. It is expected that similar models, utilizing the former method, would also exhibit such stability. In closing we should mention that the models presented here are proof-of-concept models. That is, they are constructed to show that gravastars, under the requirements set out by previous authors (for example [1]) can be explicitly achieved within the framework of general relativity theory. It would be of interest to study whether the gravitational collapse of a heavenly body, with reasonable initial conditions and undergoing the phase transition proposed in [2]-[5] at late times, will always form an object without an event horizon.

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