AMPLITUDE AND PHASE FLUCTUATIONS FOR GRAVITATIONAL WAVES PROPAGATING THROUGH INHOMOGENEOUS MASS DISTRIBUTION IN THE UNIVERSE

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ABSTRACT

When a gravitational wave (GW) from a distant source propagates through the universe, its amplitude and phase change due to gravitational lensing by the inhomogeneous mass distribution. We derive the amplitude and phase fluctuations, and calculate these variances in the limit of a weak gravitational field of density perturbation. If the scale of the perturbation is smaller than the Fresnel scale \( \sim 100pc(f/\text{mHz})^{-1/2} \) (\( f \) is the GW frequency), the GW is not magnified due to the diffraction effect. The rms amplitude fluctuation is \( 1 - 10\% \) for \( f > 10^{-10} \) Hz, but it is reduced less than \( 5\% \) for a very low frequency of \( f < 10^{-12} \) Hz. The rms phase fluctuation in the chirp signal is \( \sim 10^{-3} \) radian at LISA frequency band \( (10^{-5} - 10^{-1} \) Hz). Measurements of these fluctuations will provide information about the matter power spectrum on the Fresnel scale \( \sim 100 \) pc.

Subject headings: gravitational lensing – gravitational waves – large-scale structure of universe

1. INTRODUCTION

The inspiral and merger of SMBHs (Super Massive Black Holes : \( 10^4 - 10^7M_\odot \)) is one of the most promising candidates for LISA (Laser Interferometer Space Antenna), which will be launched around 2014. This detector is sensitive in a frequency range of \( 10^{-5} - 10^{-1} \) Hz and can measure SMBH mergers at cosmological distances with a high signal-to-noise ratio. The binary SMBH systems have recently been called “cosmological standard sirens” (Holz & Hughes 2005). This is because the distance to the source \( r_s \) can be directly measured by using the relation \( A \propto f/(f^3r_s) \), where \( f \) is the GW frequency, \( f \) is its time derivative and \( A \) is the amplitude at the detector (Schutz 1986). The distance can be determined with less than 1% accuracy if the direction is determined by identifying its electromagnetic counterpart (e.g. Cutler 1998; Seto 2002; Hughes 2002; Vecchio 2004).

But in practice, the distance cannot be determined with such high accuracy because of the gravitational lensing caused by inhomogeneous mass distribution in the Universe. Recently, Holz & Hughes (2005) and Kocsis et. al. (2005) discussed the effects of lensing magnification (or demagnification) on determining the distance to SMBH mergers. They concluded that lensing errors are \( 5 - 10\% \), which is greater than the intrinsic distance error.

In weak gravitational lensing, the magnification is \( \mu \approx 1 + 2\kappa \), where \( \kappa \) is the convergence. The rms convergence fluctuation was derived by Blandford et al. (1991), Miralda-Escude (1991) and Kaiser (1992) on the basis of linear perturbation theory. In this previous work, the geometrical optics was assumed.

But for the lensing of GWs, since the wavelength is much longer than that of light, the geometrical optics is not valid in some cases. Recently, suggested by Macquart (2004) and Takahashi, Suyama & Michikoshi (2005), if the scale of the density perturbation \( k^{-1} \) is smaller than the Fresnel scale \( r_F \sim (\lambda r_s)^{1/2} \) (\( \lambda \) is the wavelength), the wave effects become important. This condition is rewritten as \( k^{-1} < 100pc(f/\text{mHz})^{-1/2}(r_s/10Gpc)^{1/2} \). In such a case, the incident wave does not experience perturbation and its amplitude is not magnified.

In this paper, we consider a situation in which GWs propagate through the density perturbation of CDM (cold dark matter) and baryon. In the wave optics, the lensing affects not only the amplitude but also the phase, hence we discuss the lensing effects on both of them. We use units of \( c = G = 1 \).

2. GRAVITATIONALLY LENSED WAVEFORM

The background metric is the Friedmann-Robertson-Walker (FRW) model with a gravitational potential of \( U(\ll 1) \). The perturbed FRW metric \( g_{\mu\nu} \) for a flat universe is given by

\[
dx^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)g^B_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) \left[ - (1 + 2U) dr^2 + (1 - 2U) d\mathbf{x}^2 \right],
\]

where \( \eta \) is the conformal time, the scale factor is normalized such that \( a = 1 \) at present, and \( g^B_{\mu\nu} \) is the perturbed Minkowski metric. The line element is \( dx^2 = dr^2 + d\mathbf{x}^2 \), where \( r \) is a radial coordinate (a comoving distance) from the observer, \( r(z) = \int_0^z dz' / H(z') \), while \( \mathbf{x} \) is a two-dimensional vector perpendicular to the line-of-sight. We show the lens geometry for the observer and the source in Fig.1. The observer is in the origin of the coordinate axes, while the source position is \( \mathbf{x}_s = (r_s, \mathbf{x}_s^\perp) \) with \( |\mathbf{x}_s^\perp| \ll r_s \).

Since the propagation equation of GW is (1) conformally invariant if the wavelength is much smaller than the Hubble radius (see Appendix) and (2) the same as the scalar field wave equation (Peters 1974), we use the scalar field \( \phi \) propagating under the Minkowski background \( g^B_{\mu\nu} \). The basic equation is \( \partial_\mu \left( \sqrt{-g} g^{B\mu\nu} \partial_\nu \phi \right) = 0 \) or

\[
(\nabla^2 + \omega^2) \phi = 4\omega^2 U \phi,
\]

where \( \omega \) is the frequency at the observer\(^1\) and \( \phi(\omega, \mathbf{x}) \) is the Fourier transform of \( \phi(\eta, \mathbf{x}) \).

\(^1\) \( \omega \) is the same as the comoving frequency \( \omega_c \), since \( \omega_c = a\omega \) and \( a = 1 \) at present.
We take $\tilde{\phi}^0$ as the incident wave emitted from the source, which is the solution of Eq. (2) in the unlensed case $U = 0$. We use the spherical wave as $\tilde{\phi}^0(\omega, x) \propto e^{i\omega|x-x_s|/|x-x_s|}$. Including the effect of $U$ on the first order (Born approximation), the gravitationally lensed wave at the observer is given by (Takahashi et al. 2003)

$$\tilde{\phi}^0_{\text{obs}}(\omega) = \tilde{\phi}^0(\omega, x) + \tilde{\phi}^1_{\text{obs}}(\omega),$$

with

$$\tilde{\phi}^1_{\text{obs}}(\omega) = -\frac{\omega^2}{r} \int d^3x \frac{e^{i\omega|x|}}{|x|} U(x) \tilde{\phi}^0(\omega, x),$$

where $\tilde{\phi}^0$ represents the effect of lensing ($|\tilde{\phi}^0| \ll |\tilde{\phi}^1|$). The incident wave $\tilde{\phi}^0$ is gravitationally lensed at $x = (r, \mathbf{x}_\perp)$ and changed into the lensed wave $\tilde{\phi}^0 + \tilde{\phi}^1$ (see Fig.1). Let us define $K$ and $S$ as (Ishimaru 1978, Ch.17),

$$K(\omega) = \text{Re} \left[ \frac{\tilde{\phi}^0_{\text{obs}}(\omega)}{\tilde{\phi}^0_{\text{obs}}(\omega)} \right];
S(\omega) = \text{Im} \left[ \frac{\tilde{\phi}^0_{\text{obs}}(\omega)}{\tilde{\phi}^0_{\text{obs}}(\omega)} \right].$$

Then we have

$$\tilde{\phi}^0_{\text{obs}}(\omega) = [1 + K(\omega)] \tilde{\phi}^0(\omega)e^{iS(\omega)},$$

from Eq. (5). Hence $K$ means the magnification ($K > 0$) or demagnification ($K < 0$) of the wave amplitude, while $S$ means the phase shift due to the lensing. Hereafter, we call $K$ amplitude fluctuation and $S$ phase fluctuation, respectively.

Using the Fourier transform of the potential, $\tilde{U}(r, k) = \int d^3x \tilde{U}(x)e^{ik\cdot x}$, with $|\mathbf{x}_\perp| \ll r$ and $|\mathbf{x}_\perp| \ll r_s$, the result in Eq. (6) is reduced to

$$\tilde{\phi}^1_{\text{obs}}(\omega) = -\frac{\omega^2}{r} \int dr \frac{d^3k}{(2\pi)^3} \tilde{U}(r, k) \times \exp \left[ -ik_r r - \frac{r}{r_s} k_\perp \cdot \mathbf{x}_\perp^s - \frac{r (r_s - r)}{2\omega r_s} |k_\perp|^2 \right],$$

where $k_r$ and $k_\perp$ are the radial and perpendicular components of $k$. In particular for high frequency limit $\omega \to \infty$, $K$ and $S$ are rewritten as

$$K(\omega) \to \kappa = \int_0^{r_s} dr \frac{r (r_s - r)}{r_s} \nabla^2 U(r, r/r_s \mathbf{x}_\perp^s),$$
$$S(\omega) \to \omega t_d = -2\omega \int_0^{r_s} dr U(r, r/r_s \mathbf{x}_\perp^s).$$

Here, $\kappa$ is the convergence field along the line-of-sight to the source and $t_d$ is the gravitational time-delay. The above results are consistent with that in weak gravitational lensing (Bartelmann & Schneider 2001).

3. AMPLITUDE FLUCTUATION

3.1. Variance in the Amplitude Fluctuation

In this section, we derive the variance in the amplitude fluctuation $K$. The gravitational potential satisfies Poisson’s equation (Peebles 1980)

$$\hat{U}(r, k) = -\frac{3H_0^2 \Omega_m}{2a(r)} k^{-2} \hat{\delta}(r, k),$$

where $\delta$ is the density perturbation. The fluctuation of $\hat{\delta}$ is characterized by the power spectrum $\langle \hat{\delta}(r, k)\delta^*(r', k') \rangle = (2\pi)^3 P_\delta(r, k) \delta^2(k - k') \delta(r - r')$. Then, the correlation in the potentials $\hat{U}(r, k)$ and $\hat{U}(r', k')$ is

$$\langle \hat{U}(r, k)\hat{U}(r', k') \rangle = \left( \frac{3H_0^2 \Omega_m}{2a(r)} \right)^2 k^{-4} \times (2\pi)^3 P_\delta(r, k) \delta^2(k - k') \delta(r - r').$$

To calculate $P_\delta$, we use the linear power spectrum (Eisenstein & Hu 1999) with the nonlinear correction of Peacock & Dodds (1996). We adopt a COBE-normalized $\Lambda$ CDM model in a flat $\Lambda$CDM cosmology with $\Omega_m = 0.04$, $\Omega_r = 0.3$, $\Omega_k = 0.7$, and $H_0 = 70$ km s$^{-1}$Mpc$^{-1}$.

The variance in the amplitude fluctuation is given from Eqs. (5), (7) and (10) as

$$\Delta^2_K(\omega) \equiv \langle K^2(\omega) \rangle = \left( \frac{3H_0^2 \Omega_m}{4\pi} \right)^2 \int_0^{r_s} dr \frac{1}{a^2(r)} \times \frac{r^2 (r_s - r)^2}{r_s^2} \int d^2k_\perp P_\delta(r, k_\perp)F_K(\omega, r, k_\perp)$$

where the filter function $F_K$ is defined as

$$F_K = \left[ \frac{\sin \left( \frac{r^2 k_\perp^2}{2} \right)}{r^2 k_\perp^2} \right]^2; \quad r_\perp^2 = \frac{r (r_s - r)}{\omega r_s}.$$

Here, $r_F$ is called the Fresnel scale (Macquart 2004). This is roughly given by

$$r_F \simeq 120 \text{pc} \left( \frac{f}{\text{mHz}} \right)^{-1/2} \left( \frac{r (r_s - r)/r_s}{10 \text{Gpc}} \right)^{1/2},$$

where $f = \omega/2\pi$. We show the filter function $F_K$ as a function of $r_F k_\perp$ in Fig.2. From this figure, $F_K = 1$ if the scale of the density perturbation $\sim k_\perp$ is larger than $2$ It corresponds to $\sigma_8 = 0.8$ which is the present amplitude of the mass fluctuation at $8h^{-1}$ Mpc.
the Fresnel scale \( r_F \), while \( F_K \) decreases rapidly in proportion to \( (r_F k) F \) if \( k \ll r_F \). Hence, the amplitude fluctuation is affected by a density perturbation of scale larger than the Fresnel scale. In the geometrical optics limit \( F_K \rightarrow 1 \), the result (11) is the same as \( \langle \kappa^2 \rangle^{1/2} \).

The reason why a critical scale is \( r_F \) can be explained as follows: For the lensing by a compact object with a mass \( M \), if the wavelength is larger than the Schwarzschild radius, \( \lambda > M \), the diffraction effect becomes important (see Takahashi & Nakamura 2003, and references therein). Inserting this condition to the Einstein radius \( r_F = 4M(r_s - r)/r_s^{1/2} \), we have the Fresnel scale in Eq. (12).

3.2. Results

We show the rms (root-mean-square) amplitude fluctuation \( \Delta K \) in Eq. (11) as a function of the frequency in Fig. 3. The source redshift is \( z_s = 3 \). The solid line is \( \Delta K \) and the dashed line is the geometrical optics limit \( \langle \kappa^2 \rangle^{1/2} \). The difference between these two lines is small for the LISA frequency band (10^{-5} to 10^{-1} Hz). But for very low frequency \( f < 10^{-10} \) Hz, \( \Delta K \) is clearly smaller than \( \langle \kappa^2 \rangle^{1/2} \). Thus the frequency band for the pulsar timing array \( f \approx 10^{-9} \) Hz (e.g. Jenet et al. 2005), \( \Delta K \) should be used instead of \( \langle \kappa^2 \rangle^{1/2} \). The wave amplitude is magnified due to lensing by the density perturbation on a scale larger than the Fresnel scale (see Sec.3.1). The integration of Eq. (11) is mainly contributed on the scale of \( 0.1 - 1 \) Mpc (since \( P_s k^2 \) has a peak there). For \( f < 10^{-10} \) Hz, the Fresnel scale is \( r_F > 0.4 \) Mpc/(10^{-10} Hz)^{-1/2} which is larger than the contributed scale (0.1 – 1 Mpc), and hence the amplitude is not magnified.

Fig. 4 is the same as Fig. 3 but as a function of \( z_s \). The dashed line is the geometrical optics limit \( \langle \kappa^2 \rangle^{1/2} \), the solid (dotted) line is for the frequency of 10^{-10}, 10^{-12} Hz. For \( z_s = 1 - 10 \) the rms amplitude fluctuation is \( 1 - 10\% \) for \( f > 10^{-10} \) Hz, while it decreases less than 5\% for \( f < 10^{-12} \) Hz.

3.3. Diffraction effect in the amplitude fluctuation

To investigate the diffraction effect in the amplitude fluctuation, we expand \( K(\omega) \) in terms of \( 1/\omega \). From Eqs. (5) and (7), we have

\[
K(\omega) = 2\omega \int_0^{\infty} dr \sin \left( \frac{1}{2} r_F^2 k^2 \right) U \left( r, \frac{r}{r_s} x_n \right),
\]

where \( \sin(r_F^2 k^2/2) = \sum_{n=1}^{\infty} (-1)^{n+1} (r_F^2 k^2/2)^{2n-1} / (2n-1)! \). The leading term \( (n = 1) \) is the convergence \( \kappa \) in Eq. (8). The \( n \)-th term \( (n \geq 2) \), being of the order of \( (r_F^2 k^2)^{2(n-1)} k \sim (r_F/k) k^{(n-1)} \), is a correction term due to the diffraction effect. The perturbation on a scale smaller than the Fresnel scale affects the diffraction.

4. PHASE FLUCTUATION
4.1. Effects of the Phase Fluctuation in Chirp Signal

Next, we discuss the effects of the phase fluctuation. We consider the inspiraling BH binaries as the sources. As the binary system loses its energy due to gravitational radiation, the orbital separation decreases and the orbital frequency increases. Hence, the frequency of the gravitational waves increases with time (\(df/dt > 0\)). This is called a chirp signal. We consider the frequency to be swept from \(f_1\) to \(f_2\). For the binary masses \(M_1\) and \(M_2\) at redshift \(z_s\), the frequency of 1 yr before the final merging is

\[
f_1 = 4.1 \times 10^{-5} \left( \frac{M_z}{10^6 M_\odot} \right)^{-5/8} \text{ Hz}
\]

(15)

where \(M_z = (M_1 M_2)^{3/5}(M_1 + M_2)^{-1/5}(1 + z_s)\) is the redshifted chirp mass. The frequency at the final merging is

\[
f_2 = 4.4 \times 10^{-3} \left( \frac{M_z}{10^6 M_\odot} \right)^{-1} \text{ Hz}
\]

(16)

where \(M_z = (M_1 + M_2)(1 + z_s)\) is the redshifted total mass. The difference in the phase fluctuation between \(f_1\) and \(f_2\) in the chirp signal is important. The phase fluctuation \(S\) is reduced to the time delay \(\omega t_d\) in the geometrical optics limit from Eq.(8). But, we note that the time delay is physically unimportant, since it means an arrival time shift and hence it does not change the waveform. Hence we use \(S - \omega t_d\) instead of \(S\) as the phase fluctuation. This quantity \(S - \omega t_d\) has larger value for smaller frequency. We define \(\Delta_S\) as the rms phase difference between the two frequencies, \(\omega_1\) and \(\omega_2\):

\[
\Delta_S^2(\omega_1, \omega_2) \equiv \langle |(S(\omega_1) - \omega_1 t_d) - (S(\omega_2) - \omega_2 t_d)|^2 \rangle.
\]

(17)

This is the same as \(\Delta_K^2\) in Eq.(11) but the filter function is replaced with

\[
F_S = \left[ \frac{\cos(r_{F_1} k^2/2)}{r_{F_1} k^2/2} - 1 - \frac{\cos(r_{F_2} k^2/2)}{r_{F_2} k^2/2} - 1 \right]^2,
\]

(18)

where \(r_{F_1}\) and \(r_{F_2}\) are the Fresnel scales in Eq.(12) for \(\omega_1\) and \(\omega_2\), respectively.

We show the behavior of first and second terms in Eq.(15), \(\left| \frac{\cos(r_{F_2} k^2/2)}{r_{F_1} k^2/2} - 1 \right| / \langle \cos(r_{F_2} k^2/2) \rangle\), as a function of \(r_Fk\) in Fig.5. From this figure, the function peaks at \(r_Fk \sim 1\). Hence the phase fluctuation \(S - \omega t_d\) is affected by the density perturbation of the Fresnel scale. In the chirp signal, as the frequency increases, the GW feels the perturbation of the smaller scale.

4.2. Results

In table I, we show the rms phase differences \(\Delta_S\) in Eq.(17) for the LISA frequency band (\(10^{-5}\) to \(10^{-1}\) Hz) with \(z_s = 1, 3\) and 10. We consider the frequency to be swept from \(f_1\) to \(f_2\) in the chirp signal. The values are in units of radian. This table shows that the typical values of \(\Delta_S\) are \(\approx 10^{-3}\) radian. The results weakly depend on \(f_2\) if \(f_1 \ll f_2\). This is because \(S - \omega t_d\) in Eq.(17) is larger (smaller) value for smaller (higher) frequency.

In Fig.6, \(\Delta_S\) is shown as a function of \(f_1\) in the limit of \(f_2 \rightarrow \infty\). The source redshifts are \(z_s = 1\) (dotted line), 3 (solid) and 10 (dashed). In order to study the behavior of \(\Delta_S\), we assume a single power law for the power spectrum, \(P(r, k) \propto k^n\). The index is \(n \approx -2.7\) at \(k^{-1} \sim 100\) pc. Inserting this \(P(r, k)\) into Eqs.(11) and (18), we have \(\Delta_S \propto \omega^{(n+2)/4} \propto \omega^{0.18}\). With this argument and the results in table I \(\Delta_S\) in \(f_1 \ll f_2\) is roughly fitted by

\[
\Delta_S \approx 3 \times 10^{-3} \text{ rad} \left( \frac{f_1}{10^{-4}\text{Hz}} \right)^{-0.18}
\]

(19)

for \(z_s = 3\). The above value, \(3 \times 10^{-3}\) rad, is replaced by \(1(5) \times 10^{-3}\) rad for \(z_s = 1(10)\).

4.3. Implications for GW Observations

In the matched filtering analysis, the phase of waveform can be measured to an accuracy approximately equal to the inverse of the signal-to-noise ratio, \(S/N\)^{-1}. Here, \(S/N\) is typically \(\approx 10^3\) for the SMBH mergers detected by LISA. But including the effect of the phase fluctuation, the phase cannot be determined with an accuracy of less than \(\Delta_S\). Hence if \(S/N\) is larger than \(\Delta_S^{-1} = 10^3(\Delta_S/10^{-3}\text{rad})\), the phase fluctuation becomes important and the phase of the waveform can be determined with an accuracy of \(\sim \Delta_S^{-1}\).

4.4. Diffraction effect in the phase fluctuation

Similar to Sec. 3.3, we rewrite the phase fluctuation as

\[
S(\omega) = -2\omega \int_0^{r_s} dr \cos \left( \frac{1}{2} r_F^2 \nabla_\perp^2 \right) U(r, \frac{r}{r_s}) \nabla_\perp.
\]

(20)
In $\omega \to \infty$, the above result is $\omega t_d$ in Eq. [11]. To study the behavior of the correction terms, we consider a simple model. We assume the density fluctuation is confined in a thin lens plane at distance $r_l$ and is fitted by a single power law near the Fresnel scale: $U(r, k) \propto k^\alpha \delta(r-r_l)$ where $\alpha$ is an index. Then, from Eqs. [6] and [7], we have

$$S(\omega) = \omega t_d + C\omega^{\alpha/2+2}.$$  

(21)

The coefficient $C$ is constant and the second term is of the order of $10^{-3}$ rad at the LISA frequency band. Hence, by measuring the phase fluctuation directly, one would obtain the constant $C$ and the index $\alpha$.

5. VALIDITY OF THE BORN APPROXIMATION

Throughout this paper we assume the weak density fluctuation and employ the Born approximation to discuss the lensing effects on the waveform. Since the variances of the amplitude and the phase fluctuations are much smaller than one, this approximation is valid in almost all cases. But in a few cases, the GW may pass through the strong density fluctuation or pass near massive compact objects (i.e. strong lensing). Hence we have some comments about the validity of the Born approximation.

For the amplitude fluctuation $K$, the maximum of $K$ is the convergence $\kappa$ from Fig. 8. For the thin lens plane at $r_l$, $\kappa$ is the surface density of the lens $\Sigma$ divided by the critical density $\Sigma_{cr} = (1/4\pi)r_s/\ell r_{ls} \approx 2 \times 10^3 M_\odot pc^{-2} (r_s/\ell r_{ls})/(Gpc^{-1})$ (here $r_{ls} = r_s - r_l$).

Hence if the GWs pass through high density region $\Sigma > \Sigma_{cr}$ such as a core of galaxy or cluster, the Born approximation breaks down and one should use the Kirchhoff diffraction integral to obtain the exact waveform (Schneider, Ehlers & Falco 1992, Sec.4.7 and 7).

If the gravitational potential is a Gaussian random field, one can obtain exact solutions of the correlation functions of the lensed waveform (Macquart 2004; see also Ishimaru 1978, Ch.20). If there are many compact lens objects and the GWs are scattered several times, the exact lensed waveform can be obtained by using the multiple lens-plane theory in the wave optics (Yamamoto 2003).

6. CONCLUSION & DISCUSSION

We have discussed the lensing effects on the amplitude and phase of a waveform. The rms amplitude fluctuation is $1 - 10\%$ for $f > 10^{-10}$ Hz, which is the same as the result in weak lensing. But for a very low frequency of $f < 10^{-12}$ Hz, it decreases to less than 5%. In the chirp signal, the phase fluctuation is typically $10^{-3}$ radian at the LISA frequency band. The phase cannot be measured with an accuracy less than this value.

The power spectrum $P$ has been measured down to a small scale $k^{-1} \sim \text{several} \times 0.1$ Mpc from the Ly$\alpha$ forest (e.g. Zaroubi et al. 2005). In this paper, we assume the formula of $P$ is valid down to the Fresnel scale $k^{-1} \sim 100$ pc. If the amplitude or the phase fluctuation is measured in future, the constraints for $P$ at $\sim 100$ pc could be obtained.

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APPENDIX

CONFORMAL TRANSFORMATION

The relation between the Einstein tensor $G_{\mu\nu}$ for the metric $g_{\mu\nu}$ and $\tilde{G}_{\mu\nu}$ for $\tilde{g}_{\mu\nu} (= a^2 g_{\mu\nu})$ is given by (Wald 1984),

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - 2\nabla_\mu \nabla_\nu \ln a + 2(\nabla_\mu \ln a)(\nabla_\nu \ln a)$$

$$+ g_{\mu\nu} (\nabla^\sigma \ln a)(\nabla_\sigma \ln a) + 2 g_{\mu\sigma} \nabla^\sigma \nabla_\nu \ln a.$$  

(A1)

Let us consider the linear perturbation $\tilde{h}_{\mu\nu} (= a^2 h_{\mu\nu})$ in the background metric $\tilde{g}_{\mu\nu}^B (= a^2 g_{\mu\nu}^B)$ given in Eq. [11]:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}^B + h_{\mu\nu} = a^2 (g_{\mu\nu}^B + h_{\mu\nu}).$$  

(A2)

Inserting Eq. [A2] into [A1], we obtain the linearizing Einstein equation

$$\delta \tilde{G}_{\mu\nu} = \delta G_{\mu\nu} + 2 a' a \delta \Gamma^\sigma_{\mu\nu} + \left[ 2 a'' - \frac{a'}{a^2} \right] \delta (g_{\mu\sigma} g^{\alpha\nu} - \frac{1}{2} g_{\mu\nu} g^{\sigma\rho} \Gamma^\rho_{\alpha\sigma}) - 2 a' a \delta (g_{\mu\sigma} g^{\sigma\rho} \Gamma^\rho_{\alpha\nu}),$$  

(A3)

where $a' = da/d\eta$, $\Gamma$ is the Christoffel symbol, and $\delta$ means the perturbed component. The first term $\delta G_{\mu\nu}$ is of the order of $|h_{\mu\nu}|/\lambda^2$ ( $\lambda$ is the wavelength of the gravitational wave ). The other terms on the right-hand side are roughly

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{$\Delta s$ as a function of $f_1$ in the limit of $f_2 \rightarrow \infty$. The source redshifts are $z_s = 1$ (dotted line), 3 (solid) and 10 (dashed).}
\end{figure}
\[ |h_{\mu\nu}|/\lambda H \] or \[ |h_{\mu\nu}|/\lambda_H^2 \] where \( \lambda_H \) is the horizon scale, since \( a' = a^2 H \sim a^2 / \lambda_H \). Hence, if \( \lambda \ll \lambda_H \) the propagation equation for the gravitational wave is conformally invariant, i.e. \( \delta \tilde{G}_{\mu\nu} = \delta G_{\mu\nu} = 0 \).

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