How is chiral symmetry restored at finite density?

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Received

Abstract. Taking into account pseudoscalar as well as scalar condensates, we reexamine the chiral restoration path on the chiral manifold. We shall see both condensates coherently produce a density wave at a certain density, which delays chiral restoration as density or temperature is increased.

Keywords: chiral restoration, spin density wave, quark matter
PACS: 11.30.Rd; 75.10.-b; 26.60.+c

1. Introduction

Condensed matter physics of QCD is now one of the active fields in nuclear physics and the main purpose is to figure out the phase diagram in the temperature-density plane. Color superconductivity is realized by forming the condensation of the quark Cooper pairs. There should be also the instability by creating particle-hole (p-h) pairs near the Fermi surface. This is a condensation of p-h pairs, where the standing wave with the wave number $q$ is developed in the ground state. The condensate with $q = 0$ is uniform, and, e.g., ferromagnetic instability corresponds to this case\textsuperscript{1}. A typical example of the condensate with $q = 2k_F$ ($k_F$: Fermi momentum) is the Overhauser state or the spin-density wave state\textsuperscript{2}. In quark matter the scalar or tensor density may form a standing wave with a finite momentum about $2k_F$, called the chiral density wave\textsuperscript{3}. We consider here another type of the density wave due to chiral symmetry in QCD \textsuperscript{4}.

2. Dual Chiral Density Wave

We are concentrated on the flavor-$SU(2)$ case, and introduce a dual chiral-density wave (DCDW) state, where both the scalar and pseudo-scalar densities always reside on the hypersphere with a constant modulus $\Delta$, $(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau_i\psi)^2 = \Delta^2$, while
each density is *spatially non-uniform*; we consider the following configuration in quark matter,

\[
\begin{align*}
\langle \bar{\psi}\psi \rangle &= \Delta \cos \theta(r) \\
\langle \bar{\psi}i\gamma_5\tau_3\psi \rangle &= \Delta \sin \theta(r).
\end{align*}
\]

Taking the simplest but nontrivial form for the chiral angle \( \theta \) such that \( \theta(r) = q \cdot r \), we call this configuration DCDW.

It should be noted that we can construct the DCDW state by acting a space-dependent chiral transformation such as \( \psi \rightarrow \psi_W = \exp(i\gamma_5\tau_3\theta(r)/2)\psi \), on the usual spontaneously symmetry breaking (SSB) state where only the scalar density condenses. When the chiral angle has some spatial dependence, there should appear an extra term in the Lagrangian by the local chiral transformation, \( \bar{\psi}_W(\gamma_5\tau_3\gamma \cdot \nabla \theta/2)\psi_W \), due to the non-commutability of \( \theta(r) \) with the kinetic (differential) operator. Consequently some terms are added in the effective action by this contribution. One is the interaction term between quarks inside the Fermi sea and DCDW. The other is rather nontrivial and comes from the vacuum-polarization effect: it should give an additional term, \( \propto (\nabla \theta)^2 \) in the lowest order. This can be regarded as an appearance of the kinetic energy term for DCDW through the vacuum polarization [5]. Therefore, when the interaction energy is superior to the kinetic energy, quark matter becomes unstable to form DCDW.

We will see that the mechanism is quite similar to that for the spin density wave suggested by Overhauser [2], and entirely reflects the nesting of the Fermi surface. In the higher dimensional systems, however, the nesting is incomplete and the density wave should be formed provided the interaction of a relevant \( p-h \) channel is strong enough.

### 3. Phase diagram within the NJL model

We explicitly demonstrate that quark matter becomes unstable for a formation of DCDW at moderate densities, using the Nambu-Jona-Lasinio (NJL) model.

Assuming here the form (1) for the mean-fields, we define a new quark field \( \psi_W \) by the Weinberg transformation. In terms of the new field the NJL Lagrangian renders

\[
\mathcal{L}_{MF} = \bar{\psi}_W[i\partial - M - 1/2\gamma_5\tau_3\theta]\psi_W - G\Delta^2,
\]

which appears to be the same as the usual one except an "external" axial-vector field, \( q \); the amplitude of DCDW generates the dynamical quark mass \( M = -2G\Delta \) in this case. The Dirac equation for \( \psi_W \) then gives a *spatially uniform* solution \(^1\) with the eigenvalues

\[
E^\pm(k) = \sqrt{E_k^2 + |q|^2/4 \pm \sqrt{(k \cdot q)^2 + M^2|q|^2}}, \quad E_k = (M^2 + |k|^2)^{1/2}
\]

\(^1\)This feature is very different from refs. [3], where the wave function is no more plane wave.
for positive energy (valence) quarks with different polarizations denoted by the sign $\pm$. For negative energy quarks in the Dirac sea, they have an energy spectrum symmetric with respect to null line because of charge conjugation symmetry in the Lagrangian. The energy spectra show a salient feature: they break rotation symmetry, and lead to the axial-symmetrically deformed Fermi seas.

Hereafter, we choose $q/\hat{z}$, $q = (0, 0, q)$ with $q > 0$, without loss of generality. The effective potential is then given by summing up all the energy levels,

$$\Omega_{\text{total}} = \Omega_{\text{val}} + \Omega_{\text{vac}} + M^2/4G,$$

(4)

The first term $\Omega_{\text{val}}$ is the contribution by the valence quarks filled up to the Fermi energy in each Fermi sea, while the second term $\Omega_{\text{vac}}$ is the vacuum contribution that is formally divergent and should be properly regularized. Note that we cannot apply the usual momentum cut-off regularization scheme to $\Omega_{\text{vac}}$, since the energy spectrum has no more rotation symmetry. Instead, we adopt the proper-time regularization scheme, which may be a most suitable one for our purpose. Then $\Omega_{\text{vac}}$ results in a function of $q$ and its lowest order contribution is proportional to $q^2$.

We can expect that the optimal value of $q$ is $2k_F$ in $1+1$ dimensions. First, consider the energy spectra with $k = (0, 0, k_z)$ for massless quarks, $M = 0$. Two energy spectra are essentially reduced to the usual ones $E^\pm(k) = |k|$ in this case. Then we can see a level crossing at $k_z = 0$. Once the mass $M$ is taken into account, this degeneracy is resolved and the energy gap appears there. Hence we have always an energy gain by filling only the lower-energy spectrum $E^{-}(k)$ up to the Fermi energy, if the relation $q = 2k_F$ holds. However, we shall see that $q < 2k_F$ in $3+1$ dimensions by numerical evaluations. This mechanism is very similar to that of spin density wave by Overhauser.

The left panel of Fig. 1 demonstrates the behaviors of the order-parameters $M$ and $q$ as functions of chemical potential $\mu$ at zero temperature for a parameter set $GA^2 = 6, \Lambda = 860$ MeV. It is found that the magnitude of $q$ becomes finite just

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Left panel: the wave number $q$ and the dynamical mass $M$ are plotted as functions of $\mu$. Solid (dotted) line for $M$ with (without) DCDW, and dashed line for $q$. Right panel: the phase diagram in the temperature-density plane.}
\end{figure}

2The similar mechanism also works in the context of the chiral density wave.
before the ordinary chiral-symmetry restoration, and DCDW survives in the finite range of $\mu$, which corresponds to the baryon-number densities $\rho_b/\rho_0 = 3.62 - 5.30$. The wave number $q$ increases with density, while its value is always smaller than the canonical value of $2k_F$ due to the higher dimensional effects.

The right panel shows a phase diagram in the temperature-density plane. We can see that DCDW develops outside the phase boundary for the usual SSB phase. We thus conclude that DCDW is induced by finite-density contributions, and has the effect to extend the SSB phase ($M \neq 0$) to high density region, which suggests another path for chiral-symmetry restoration by way of the DCDW state at finite density.

Finally we would like to indicate an interesting magnetic aspect of the DCDW state. The magnetic moment is evaluated to be spatially oscillating like the spin density wave \[ \text{[4].} \]

4. Summary and concluding remarks

We have seen that quark matter becomes soft for producing DCDW at a certain density just near the usual chiral restoration, which gives another path for the chiral restoration. This is due to the nesting effect of the Fermi sea.

It would be interesting to recall that DCDW is similar to pion condensation \[ \text{[6], where meson condensates take the same form as Eq. \text{[1].} \] So we might be tempted to connect pion condensation with DCDW by a symmetry consideration (hadron-quark continuity).

If DCDW phase is developed, we can expect a Nambu-Goldstone mode as phason. Such phasons may affect the thermodynamic properties of quark matter.

This work is supported by the Japanese Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Culture, Sports, Science and Technology (13640282, 16540246).

References