Consistent dimensional reduction of five-dimensional off-shell supergravity

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Abstract

There are some points to notice in the dimensional reduction of the off-shell supergravity. We discuss a consistent way of the dimensional reduction of five-dimensional off-shell supergravity compactified on $S^1/Z_2$. There are two approaches to the four-dimensional effective action, which are complementary to each other. Their essential difference is the treatment of the compensator and the radion superfields. We explain these approaches in detail and examine their consistency. Comments on the related works are also provided.

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1 Introduction

Higher dimensional supergravity (SUGRA) appears in many researches of particle physics and cosmology, for example, as an effective theory of the superstring theory or as a setup of the braneworld scenario. For higher dimensional SUGRA on a compact space, some sort of dimensional reduction to lower dimensions such as the Kaluza-Klein reduction is necessary in order to discuss the low-energy physics. Consistent dimensional reduction of SUGRA in diverse dimensions is extensively discussed, e.g., in Refs. [1, 2].

Among higher dimensional SUGRA, five dimensional supergravity (5D SUGRA) compactified on an orbifold $S^1/Z_2$ has been thoroughly investigated since it is shown to appear as an effective theory of the strongly-coupled heterotic string theory [3] compactified on a Calabi-Yau 3-fold. Especially, the Randall-Sundrum model [4] is attractive as an alternative solution to the hierarchy problem, and its supersymmetric (SUSY) version is also discussed in many papers [5, 6]. The low-energy effective theories of such brane-world models are four dimensional (4D). In this paper, we will discuss the dimensional reduction of 5D SUGRA compactified on $S^1/Z_2$ to derive the 4D effective action, mainly focus on the matter couplings including the radion multiplet. In the following, we will concentrate ourselves on the case that the background saturates Bogomol’ny-Prasad-Sommerfield (BPS) bound [7], i.e., preserves $\mathcal{N} = 1$ SUSY.\footnote{$\mathcal{N} = 1$ SUSY denotes supersymmetry with four supercharges in this paper.}

In the global SUSY case, the superfield formalism is useful because it makes SUSY manifest and we can systematically construct the action. Its counterpart in SUGRA is the superconformal gravity formulation, or off-shell SUGRA. (See, for example, Refs. [8, 9, 10, 11].) Since the 4D effective theory has only $\mathcal{N} = 1$ SUSY, it is convenient to decompose each 5D superconformal multiplet into $\mathcal{N} = 1$ multiplets [9]. In fact, we can rewrite the 5D off-shell SUGRA action in terms of $\mathcal{N} = 1$ multiplets in the language of 4D off-shell SUGRA [12, 13]. Thus it seems possible to treat 5D off-shell SUGRA as if 4D off-shell SUGRA. In fact, the corresponding treatment is possible in the global SUSY case. A 5D supersymmetric action can be rewritten in terms of $\mathcal{N} = 1$ superfields on the $\mathcal{N} = 1$ superspace [14, 15], and each 5D $\mathcal{N} = 1$ superfield can be expanded into infinite 4D superfields. Thus we can easily obtain a 4D superspace action by integrating the fifth dimension. Namely, we can perform the Kaluza-Klein (K.K.) dimensional reduction keeping the $\mathcal{N} = 1$ superfield structure. Unfortunately, the corresponding procedure cannot be performed in the SUGRA case. The main obstacle is the fact that 5D off-shell SUGRA has some multiplets that cannot be expanded into K.K. 4D multiplets. For example, it has the 5D compensator multiplet, which is not an independent degree of freedom. Its component fields are rewritten in terms of the physical matter fields after the superconformal gauge fixing. Therefore, we cannot perform the K.K. expansion for this multiplet keeping the $\mathcal{N} = 1$ structure manifest. We have to move to the on-shell action before the dimensional reduction in this approach. We will refer to this approach as the “5D off-shell approach” in the following.

From the 4D point of view, the radius of the orbifold is also a dynamical degree of freedom and behaves as a scalar field, which is called the radion. This belongs to a chiral multiplet for the preserved $\mathcal{N} = 1$ SUSY. Using this radion multiplet, we can construct the 4D off-shell action that is consistent with 5D SUGRA. This is the other way of deriving
the 4D effective action. In contrast to the 5D off-shell approach, the resultant 4D action is expressed in the off-shell formalism. Thus we will refer to this as the “4D off-shell approach” in the following.

In this paper, we will explain these two approaches to the effective theory in detail and examine their consistency. Each approach has both advantage and drawback, and the two approaches are complementary to each other. We will also give a brief comment on the well-known works related to ours [15, 16].

The paper is organized as follows. In the next section, we will explain the 5D off-shell approach after a brief review of the off-shell formalism of 5D SUGRA. Using a simple model, we will demonstrate the derivation of the effective action and show some problems of this approach. In Sect. 3, we will explain the 4D off-shell approach and derive the effective action expressed by superfields. The consistency with the 5D off-shell approach is also examined. Sect. 4 is devoted to the summary. The definitions of the $N = 1$ superfields we use in this paper are collected in Appendix A, and the explicit forms of the mode functions on the slice of AdS$_5$ geometry are listed in Appendix B.

2 5D off-shell approach

In this section, we will briefly review the off-shell formalism of 5D SUGRA and discuss the derivation of the 4D effective theory based on it. Throughout this paper, we will use $\mu, \nu, \cdots = 0, 1, 2, 3, 4$ for the 5D world vector indices, and $m, n, \cdots = 0, 1, 2, 3$ for the 4D indices. The coordinate of the fifth dimension compactified on the orbifold $S^1/Z_2$ is denoted as $y$ ($0 \leq y \leq \pi R$). The corresponding local Lorentz indices are denoted by underbarred indices. Here we will focus on the case where the background preserves 4D Poincaré symmetry. Then the background metric can be written as

$$ds^2 = e^{2\sigma} \eta_{mn} dx^m dx^n - dy^2,$$

where $\sigma(y)$ is a function of only $y$, which is determined by solving the Einstein equation.

2.1 5D Off-shell action

The off-shell formulation of 5D SUGRA was provided by Refs. [9, 10, 11]. This is very systematic and consistent way of constructing the 5D SUGRA action. For the purpose of deriving the 4D effective theory, it is convenient to decompose each 5D superconformal multiplet into $N = 1$ multiplets [9]. In fact we can rewrite the 5D SUGRA action in terms of $N = 1$ multiplets in the language of 4D off-shell SUGRA [12], i.e., the $D$-term and $F$-term formulae of Ref. [8]. Especially, in the case where we do not discuss the fluctuation modes of the gravitational multiplet, the action can be expressed on the $N = 1$ superspace by using the following superfields [13].

From the vector multiplet $V^{I}$ ($I = 0, 1, \cdots, n_V$), we can construct $N = 1$ vector and chiral superfields $V^I$ and $\Phi^I_s$. From the hypermultiplets $\mathcal{H}^{\hat{\alpha}}$ ($\hat{\alpha} = 0, 1, \cdots, n_H$), we

\[\text{The metric convention is chosen as } \eta_{mn} = \text{diag}(1, -1, -1, -1).\]

\[\text{For simplicity, we will consider only abelian gauge groups in this paper.}\]
can construct a pair of chiral superfields ($\Phi^{2\tilde{a}+1}, \Phi^{2\tilde{a}+2}$). The orbifold $Z_2$ parity for each superfield is listed in Table 1. The vector multiplet $\mathcal{V}^{I=0}$ corresponds to the graviphoton multiplet, and the hypermultiplet $\mathcal{H}^{\tilde{a}=0}$ plays a role of the compensator multiplet.\footnote{In this paper, we will consider the case of one compensator multiplet. An extension to the multi- compensator case is straightforward.} In order to focus on the matter couplings, we will fix the gravitational fields to the following background values:}

$$\langle e^4_m \rangle = e^\sigma \delta_m^n, \quad \langle e^4_y \rangle = \text{constant},$$

$$\langle \psi^i_{\mu} \rangle = 0,$$  

(2.2)

where $\psi^i_{\mu}$ is the gravitino ($i = 1, 2$ is an $SU(2)_L$-index). We can always rescale the coordinate $y$ so that $\langle e^4_y \rangle = 1$, but we will keep the $\langle e^4_y \rangle$-dependence in the following expressions for the readers to recall its origin.

We can also introduce the 4D vector and chiral superfields, $U^A$ ($A = 1, 2, \cdots$) and $S^a$ ($a = 1, 2, \cdots$), on the boundaries of the orbifold.

The explicit forms of these $\mathcal{N} = 1$ superfields in the notations of Refs. \cite{9, 10} are collected in Appendix A.

The 5D off-shell SUGRA action can be written as

$$S = \int d^5x \left( \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{hyper}} + \sum_{\varphi^* = 0, \pi} \mathcal{L}^{(\varphi^*)}_{\text{brane}} \delta(y - \varphi^* R) \right),$$

$$\mathcal{L}_{\text{vector}} = \left[ \int d^2 \theta \frac{3C_{IJK}}{2} \left\{ i \Phi^I_S \mathcal{W}^I \mathcal{W}^K + \frac{1}{12} \bar{D}^2 (V^I D^\alpha \partial_y V^J - D^\alpha V^I \partial_y V^J) \mathcal{W}^K_{\alpha} \right\} + \text{h.c.} \right]$$

$$- e^{2\sigma} \int d^4 \theta V^{-2}_E C_{IJK} V^I_S V^J_S V^K_S,$$

$$\mathcal{L}_{\text{hyper}} = -2e^{2\sigma} \int d^4 \theta V_E d^\alpha_\beta \Phi^\beta \left( e^{-2igV^I t_I} \right)^{\alpha}_{\gamma} \Phi^\gamma$$

$$- e^{3\sigma} \left[ \int d^2 \theta \Phi^\alpha d^\beta_\gamma (\partial_y - 2g\Phi^I t_I)^\gamma_\delta \Phi^\delta + \text{h.c.} \right],$$

$$\mathcal{L}^{(\varphi^*)}_{\text{brane}} = \left\{ \int d^2 \theta f^{(\varphi^*)}_{AB}(S) \mathcal{W}^A \mathcal{W}^B + \text{h.c.} \right\} - e^{2\sigma} \int d^4 \theta \left( |\Phi^\alpha = 2 |^2 \right)^{2/3} \exp \left\{ -K^{(\varphi^*)}(\bar{S}, S, U) \right\}$$

$$+ e^{3\sigma} \left\{ \int d^2 \theta (\Phi^\alpha = 2)^2 P^{(\varphi^*)}(S) + \text{h.c.} \right\},$$

(2.3)

where $C_{IJK}$ is a real constant tensor which is completely symmetric for the indices, $\rho_{\alpha\beta}$ is an antisymmetric tensor defined as $\rho_{\alpha\beta} \equiv i\sigma_2 \otimes 1_{n_{m+1}}$, and $d^\alpha_\beta$ is a metric of the hyperscalar

\footnote{We can always restore the dynamical gravitational modes by replacing $d^4 \theta$- and $d^2 \theta$-integrals in Eq. (2.3) by the $D$- and $F$-term formulae of 4D superconformal gravity \cite{5}.}
Note that the sign of the metric for the compensator superfields ($\Phi^{\alpha=1}, \Phi^{\alpha=2}$) is opposite to that for the matter superfields $\Phi^\alpha$ ($\alpha \geq 3$). The summations for the indices are implicit.

The generators $t_I$ ($I = 0, 1, \cdots, n_V$) are defined as anti-hermitian. The superfield $V_E$ corresponds to the $\mathcal{N} = 1$ real general multiplet $W_y$ in Eq.(4.12) of Ref. [9], and is a spurion-like superfield since the gravitational fields are frozen here.

$$V_E = \langle e^4_y \rangle - \theta^2 \mathcal{F}_V - \bar{\theta}^2 \mathcal{F}_V,$$

where $\mathcal{F}_V$ is an auxiliary field. The superfields $V_I^S$ and $W_I^\alpha$ are gauge-invariant quantities defined as

$$V_I^S \equiv -\partial_y V^I - i\Phi_I^S + i\bar{\Phi}_I^S,$$
$$W_I^\alpha \equiv -\frac{1}{4} \bar{D}^2 D_a V^I.$$

The first line of $\mathcal{L}_{\text{vector}}$ corresponds to the gauge kinetic terms and the supersymmetric Chern-Simons terms. In fact, the variation of these terms under the gauge transformation does not vanish, but becomes total derivative. Thus these terms cannot be expressed by only the above gauge-invariant quantities.

In the boundary action, $f^{(\tilde{\vartheta})}_{AB}(\tilde{S}, S, U)$ and $P^{(\tilde{\vartheta})}(S)$ are the gauge kinetic function, the Kähler potential, and the superpotential, respectively. $S^a$ and $U^A$ can be either superfields localized on the boundaries or induced from the bulk superfields. $W^A$ is a superfield strength of $U^A$. Note that only $\Phi^{\alpha=2}$ can appear in the boundary actions as a chiral compensator superfield because $\Phi^{\alpha=1}$ is odd under the $\mathbb{Z}_2$-parity and vanishes on the boundaries. The powers of $\Phi^{\alpha=2}$ are determined by the Weyl weight counting. Note also that the Weyl weights of the matter multiplets in the 4D off-shell action must be zero. Thus, if $S$ and $U$ are the induced superfields from the $\mathbb{Z}_2$-even 5D superfields $\Phi^{2\hat{\alpha}+2}$ ($\hat{\alpha} \geq 1$) and $V^I$ ($I \neq 0$), they are identified as

$$S^{a=\hat{\alpha}} = \frac{\Phi^{2\hat{\alpha}+2}}{\Phi^{\beta=2}}, \quad (\hat{\alpha} \geq 1)$$
$$U^{A=I} = V^I. \quad (I \neq 0)$$

Eq.(2.3) reproduces the off-shell SUGRA action in Ref. [10]. (See Appendix. B of Ref. [13].) Note that Eq.(2.3) is just a shorthand expression for the full SUGRA action. We can always restore the gravitational fields to the dynamical ones by promoting $d^4\theta$- and $d^2\theta$-integrals to the $D$- and $F$-term formulæ of 4D superconformal gravity [8] and regarding each superfield as the corresponding $\mathcal{N} = 1$ multiplet. (See also Ref. [12].) Thus, we will use both words “$\mathcal{N} = 1$ superfield” and “$\mathcal{N} = 1$ multiplet” without clear distinction in the following.

$^6$The explicit forms of $D$- and $F$-term formulæ are compactly listed in Appendix C of Ref. [9].
2.2 Superconformal gauge fixing and practical form of 5D action

In order to obtain the Poincaré supergravity, we have to fix the extraneous superconformal symmetries. The required gauge-fixing conditions are listed in Appendix A.3. Here we will rewrite those conditions in our notation.

We will denote each component of the superfields as

\[
V^I = \theta_\sigma \bar{\theta} W^I_m + i \bar{\theta}^2 \bar{\theta} \lambda^I - i \bar{\theta}^2 \theta \lambda^I + \frac{1}{2} \theta^2 \bar{\theta}^2 D^I, \\
\Phi^I = \varphi^I_S - \theta \chi^I_S - \theta^2 \mathcal{F}^I, \\
\Phi^\alpha = \varphi^\alpha - \theta \chi^\alpha - \theta^2 \mathcal{F}^\alpha,
\]

where \( I = 0, 1, \cdots, n_V \) and \( \alpha = 1, 2, \cdots, 2(n_H + 1) \).

In terms of these component fields, the gauge-fixing conditions can be rewritten as follows.

\[
\varphi^{\alpha = 1} = 0, \\
\varphi^{\alpha = 2} = \left( M_5^3 + \sum_{\beta = 3}^{2n_H + 2} |\varphi^\beta|^2 \right)^{1/2} = M_5^{3/2} + \frac{\kappa^{3/2} 2n_H + 2}{2} \sum_{\beta = 3} |\varphi^\beta|^2 + \mathcal{O}(\kappa^{9/2}), \\
\chi^{\alpha = 1} = - (\varphi^{\alpha = 2})^{-1} \sum_{\beta, \gamma = 3}^{2n_H + 2} \rho_{\beta \gamma} \varphi^\beta \chi^\gamma, \\
\chi^{\alpha = 2} = (\varphi^{\alpha = 2})^{-1} \sum_{\beta = 3}^{2n_H + 2} \bar{\varphi}^\beta \chi^\beta, \\
\text{Im} \varphi^{I = 0} = \frac{\langle e_y^4 \rangle M_5}{2} + \frac{\kappa}{3 \langle e_y^4 \rangle} \sum_{J = 1}^{n_V} (\text{Im} \varphi^J_5)^2 + \mathcal{O}(\kappa^5), \\
\chi^{I = 0} = \frac{2\kappa}{3 \langle e_y^4 \rangle} \sum_{J = 1}^{n_V} (\text{Im} \varphi^J_5) \chi^J + \mathcal{O}(\kappa^{9/2}), \\
\chi^{J = 0} = \frac{2\kappa}{3 \langle e_y^4 \rangle} \sum_{J = 1}^{n_V} (\text{Im} \varphi^J_5) \chi^J + \mathcal{O}(\kappa^{9/2}),
\]

where \( M_5 \) is the 5D Planck mass and \( \kappa \equiv 1/M_5 \). For simplicity, we have assumed that the vector sector is maximally symmetric.\(^8\) Namely,

\[
C_{IJK} M^I M^J M^K = (M^{I = 0})^3 - \frac{1}{2} M^{I = 0} \sum_{J = 1}^{n_V} (M^J)^2,
\]

where \( M^I \) is the gauge scalar in the 5D vector multiplet \( \mathcal{V}^I \). (See Appendix A.2.) If we want to include the Fayet-Iliopoulos term, the above cubic function of \( M^I \) is corrected by adding an extra term \( \mathcal{O}(\kappa) \).

\(^7\)We have defined \( \theta_\sigma \) and \( \theta^2 \) components of the chiral superfields with minus signs in order to match the notation of Refs. [9, 10].

\(^8\)Extension to more generic case, for example the cases discussed in Refs. [1, 2], will be straightforwardly possible.
Therefore, the compensator multiplet $\Phi^{\alpha=1,2}$ and the graviphoton multiplet $(V^0, \Phi^0_S)$ are not independent fields except for their auxiliary fields and the graviphoton field $W^0$. In order to discriminate these multiplets from the other physical multiplets, we will rewrite the superfields as follows.\footnote{Here we consider the case of $n_H = n_V = 1$ case. The extension to the case of arbitrary $n_H$ and $n_V$ is straightforward.} For the hypermultiplets,

\begin{align}
\Xi & \equiv \kappa^{3/2} \Phi^{\alpha=2} = \xi - \theta \chi_{\xi} - \theta^2 F_{\xi}, \\
\Xi^c & \equiv \kappa^{3/2} \Phi^{\alpha=1} = \xi^c - \theta \chi_{\xi}^c - \theta^2 F_{\xi}, \\
H & \equiv \sqrt{2} \Phi^{\alpha=4} = h - \theta \chi_h - \theta^2 F_h, \\
H^c & \equiv \sqrt{2} \Phi^{\alpha=3} = h^c - \theta \chi_{h}^c - \theta^2 F^c_{h}. \quad (2.11)
\end{align}

For the vector multiplets,

\begin{align}
V^0 & \equiv V^{I=0} = \theta \sigma^m \bar{\partial} W^0_m + i \bar{\partial}^2 \lambda - i \bar{\partial}^2 \theta^2 \lambda + \frac{1}{2} \theta^2 \partial^2 D^0, \\
E & \equiv -2 i \kappa \Phi^s_0 = \varphi_E - \theta \chi_E - \theta^2 F_E, \\
V & \equiv \sqrt{M_5} V^{I=1} = \theta \sigma^m \bar{\partial} W_m + i \bar{\partial}^2 \lambda - i \bar{\partial}^2 \theta^2 \lambda + \frac{1}{2} \theta^2 \partial^2 D, \\
\Phi_S & \equiv \sqrt{M_5} \Phi^{I=1} = \varphi_S - \theta \chi_S - \theta^2 F_S. \quad (2.12)
\end{align}

Then, Eq. (2.9) is rewritten as

\begin{align}
\xi & = \left\{ 1 + \frac{\kappa^3}{2} \left( |h|^2 + |h^c|^2 \right) \right\}^{1/2}, \quad \xi^c = 0, \\
\chi_\xi & = \frac{\kappa^{3/2}}{2 \xi} (h \chi_h + h^c \chi_{h}^c), \quad \chi_\xi^c = - \frac{\kappa^{3/2}}{2 \xi} (h \chi_{h}^c - h^c \chi_h), \\
\lambda^0 & = \frac{4 \kappa^2}{3 \langle e_y^4 \rangle} (\text{Im} \varphi_S) \lambda + \mathcal{O}(\kappa^{9/2}), \\
\varphi_E & = \langle e_y^4 \rangle - i \kappa W^0_y, \quad \chi_E = - \frac{8 i \kappa^3}{3 \langle e_y^4 \rangle} (\text{Im} \varphi_S) \chi_S + \mathcal{O}(\kappa^{9/2}). \quad (2.13)
\end{align}

We have considered the maximally symmetric case in the vector sector, i.e., $C_{IJK}$ is assumed as Eq. (2.10).

The compensator and the physical hypermultiplets can be charged for the graviphoton $W^0_\mu$ and the ordinary gauge field $W_\mu$. Here we will choose the directions of these gaugings to the $\sigma_3$-direction because the gauging along the other direction mixes $\Phi^{2\alpha+1}$ and $\Phi^{2\alpha+2}$, which have opposite $Z_2$-parities. Thus, the anti-hermitian generator $t_0$ and $t_1$ are

\begin{align}
g t_0 = -i \left( \begin{array}{cc} g_0^c & 0 \\ 0 & g_0^h \end{array} \right) \otimes \sigma_3, \quad g t_1 = -i \left( \begin{array}{cc} 0 & g_1^c \\ g_1^h & 0 \end{array} \right) \otimes \sigma_3. \quad (2.14)
\end{align}

The Pauli matrix $\sigma_3$ acts on each hypermultiplet ($\Phi^{2\alpha+1}$, $\Phi^{2\alpha+2}$). We have assumed that the compensator multiplet is neutral for $W_\mu$, for simplicity. Note that the couplings $g_0^c$ and $g_0^h$.
$g^0_h$ are $Z_2$-odd under the orbifold parity since the graviphoton $W^0_m$ is odd. Such $Z_2$-odd couplings can be consistently introduced in the off-shell SUGRA action by the method proposed in Refs. [11, 18].

Then, the Lagrangians in Eq. (2.3) are rewritten as

$$\mathcal{L}_{\text{vector}} = \left[ \int d^2 \theta \left\{ -\frac{3M_5}{4} E \left( \mathcal{W}^0 \right)^2 + \frac{1}{4} FW^2 - \frac{i}{2} \Phi_5 \mathcal{W}^0 \mathcal{W} + C(V^0, V) \right\} + \text{h.c.} \right]$$

$$-e^{2\sigma} \int d^4 \theta V_E^{-2} \left\{ \left( -\partial_y V^0 + \frac{M_5}{2}(E + \bar{E}) \right)^3 \right. - \kappa \left( -\partial_y V^0 + \frac{M_5}{2}(E + \bar{E}) \right) \left( -\partial_y V - i\Phi_S + i\bar{\Phi}_S \right)^2 \right\}$$

$$\mathcal{L}_{\text{hyper}} = e^{2\sigma} \int d^4 \theta V_E \left\{ -2M_5^2 \left( \bar{\Xi} e^{3\kappa \kappa V^0} \Xi + \bar{\Xi} e^{-3\kappa \kappa V^0} \Xi \right) \right. + \bar{H} e^{2\kappa \kappa V^0 + 2gV} H + \bar{H} e^{-2\kappa \kappa V^0 - 2gV} H^c \right\}

+ e^{3\sigma} \left[ \int d^2 \theta \left\{ -2M_5^2 \bar{\Xi} \left( \partial_y + \frac{3}{2} \bar{\sigma} + \frac{3}{2} \kappa \kappa E \right) \Xi \right. \right. \left. \right. \left. \right. \right. \right.$$

$$\left. \left. \left. \left. \left. + \bar{H} c \left( \partial_y + \frac{3}{2} \bar{\sigma} + m \kappa \kappa E - 2ig \Phi_S \right) H \right\}\right\} + \text{h.c.} \right\}$$

$$\mathcal{L}_{\text{brane}}^{(\varphi^* \varphi^*)} = \left\{ \int d^2 \theta f_{AB}^{(\varphi^* \varphi^*)} (S) \mathcal{W}^A \mathcal{W}^B + \text{h.c.} \right\} - e^{2\sigma} M_5^2 \int d^4 \theta \left( \left| \Xi \right|^2 \right)^{2/3} \exp \left\{ -K(\varphi^* \varphi^*) (S, S, U) \right\}$$

$$+ e^{3\sigma} \left\{ \int d^2 \theta \Xi^2 P(\varphi^*) (S) + \text{h.c.} \right\}, \quad (2.15)$$

where $\bar{\sigma} \equiv d\sigma/dy$, $k \kappa \equiv 2g^0_m M_5/3$, $m \kappa \equiv \bar{g}_h M_5$ and $\kappa \equiv \sqrt{2} \kappa g^0_h$. Here, $\epsilon$ is a periodic step function defined as

$$\epsilon(y) = \begin{cases} +1 & \text{for} \ (2n\pi R < y < (2n + 1)\pi R), \\ -1 & \text{for} \ ((2n - 1)\pi R < y < 2n\pi R), \quad (n: \text{integer}) \end{cases} \quad (2.16)$$

Note that the coupling constants $g^0_h$ and $g^0_m$ are $Z_2$-odd. $C(V^0, V)$ corresponds to the second terms in the first line of $\mathcal{L}_{\text{vector}}$ in Eq. (2.3), which is irrelevant to the following discussion. We have rescaled $P(\varphi^*)$ in Eq. (2.15) as $P(\varphi^*) \to \kappa^3 P(\varphi^*)$ from that in Eq. (2.3).

### 2.3 Simple model

To see the role of the superfields $(\Xi, \Xi^c, V^0, E, V_E)$, we will consider a simple model proposed in Ref. [19]. This corresponds to a supersymmetric extension of the Goldberger-Wise model [20] that stabilizes the size of the fifth dimension by the bulk scalar field.

The model consists of only one physical hypermultiplet $(H, H^c)$ with the bulk mass $m$ besides the compensator and the graviphoton multiplets. Since $H$ is $Z_2$-even, it can appear in the boundary actions. In this model, the following tadpole superpotentials are introduced on the boundaries.

$$P^{(0)} \left( \frac{H}{\Xi} \right) = J_0 \frac{H}{\Xi}, \quad P^{(\pi)} \left( \frac{H}{\Xi} \right) = -J_\pi \frac{H}{\Xi}. \quad (2.17)$$
where \( J_0, J_\pi \) are real positive constants. Recall that \( H \) must appear in the boundary action in the form of \( H/\Xi \). (See Eq. (2.7).)

Due to these boundary superpotentials, the \( Z_2 \)-odd field \( h^c \) must satisfy the following boundary conditions.

\[
[h^c]_0 = J_0, \quad [h^c]_\pi = J_\pi, \tag{2.18}
\]

where the symbol \([\cdots]\) is defined as

\[
[\varphi]_{\partial y} \equiv \varphi(y = \vartheta^* R + 0) - \varphi(y = \vartheta^* R - 0). \tag{2.19}
\]

In the following, we will neglect the 4D component of the graviphoton \( W_0^m \). Then, \( V^0 \) has only the auxiliary field \( D_0 \) because \( \lambda^0 = 0 \) in our case, i.e., \((n_V, n_H) = (0, 1)\).

From the equation of motions for the auxiliary fields, we obtain

\[
D^0 = -e^{2\sigma} M_5 \left\{ \dot{\sigma} + k\epsilon \left| \xi \right|^2 - \frac{\kappa^3 m \epsilon}{3} \left( \left| h \right|^2 - \left| h^c \right|^2 \right) \right\},
\]

\[
{\mathcal{F}_E} = 2{\mathcal{F}_V} - \frac{2}{3} e^{\sigma} \kappa^3 m \epsilon \bar{h}^c \bar{h},
\]

\[
{\mathcal{F}_E} = 2{\mathcal{F}_V} + 
\]

\[
{\mathcal{F}_E} = 2{\mathcal{F}_V} + \frac{2}{3} \bar{F}_\xi - \frac{\kappa^3}{3} \left( {\mathcal{F}_h} \bar{h} + {\mathcal{F}_h^c} \bar{h}^c \right),
\]

\[
{\mathcal{F}_E} = \bar{F}_\xi - \frac{\kappa^3}{2} \sum_{\vartheta^* = 0, \pi} e^{\sigma e^{i\vartheta^*} J_{\vartheta^*} \delta(y - \vartheta^* R)},
\]

\[
{\mathcal{F}_E} = \bar{F}_h - e^{\sigma} \left( \partial_y + \frac{3}{2} \dot{\sigma} - m \epsilon \right) \bar{h} + \sum_{\vartheta^*} e^{\sigma} \bar{\xi} e^{i\vartheta^*} J_{\vartheta^*} \delta(y - \vartheta^* R),
\]

\[
{\mathcal{F}_E} = \bar{F}_h - e^{\sigma} \left( \partial_y + \frac{3}{2} \dot{\sigma} + m \epsilon \right) \bar{h}. \tag{2.20}
\]

We will search the background that preserves \( \mathcal{N} = 1 \) SUSY. The BPS condition can be expressed as

\[
\langle D^0 \rangle = \langle {\mathcal{F}_E} \rangle = \langle {\mathcal{F}_V} \rangle = \langle {\mathcal{F}_\xi} \rangle = \langle {\mathcal{F}_h^c} \rangle = \langle {\mathcal{F}_h} \rangle = 0. \tag{2.21}
\]

Solving this condition, we obtain the following BPS background.

\[
\sigma(y) = -k \left| y \right| - \frac{l^2}{24} e^{(3k+2m)\left| y - \pi^* R \right|} + \mathcal{O}(l^4),
\]

\[
h_{cl} = 0,
\]

\[
h_{cl}^c = \frac{J_0}{2} \epsilon(y) e^{(\frac{3k+m}{2})|y|} \left\{ 1 + \mathcal{O}(l^2) \right\}, \tag{2.22}
\]

\[\text{Since } W_0^m \text{ is } Z_2 \text{-odd, it does not have a zero-mode.}
\]

\[\text{They are obtained by taking the variation of } D^0, {\mathcal{F}_E}, {\mathcal{F}_V}, {\mathcal{F}_\xi}, {\mathcal{F}_h^c}, {\mathcal{F}_h} \text{ and } {\mathcal{F}_h^c}, \text{ respectively.}
\]

\[\text{We can show that this is equivalent to the Killing spinor equations, which is the conditions that the SUSY variations of all fermionic fields are zero.}\]
where \( l \equiv \kappa^{3/2} J_\pi \) is a dimensionless constant, which parametrizes the backreaction on the metric from the nontrivial configuration of the bulk scalar \( h^c \). We will assume that \( J_0, J_\pi \ll M_5^{3/2} \), i.e., \( l \ll 1 \) in the following. We have used the first condition of Eq. (2.18) to determine the normalization of \( h^c_\Omega \). From the second condition of Eq. (2.18), we obtain

\[
J_0 = J_\pi e^{-\left(\frac{2k+m}{3}\right)\pi R}.
\]

(2.23)

Therefore, the radius \( R \) is stabilized with a finite value.

The physical fields \((h, \chi_h)\), \((h^c, \chi_h^c)\) are expanded into K.K. modes around the background \( h, \chi \). First, let us consider the limit of \( l \to 0 \). In this case, \( h^c_\Omega \) vanishes, and the background geometry becomes a slice of AdS\(_5\). The radius \( R \) is not stabilized and the radion is massless. The radion mode resides only in the fluctuation of the fünfbein \( e_\mu^f \), which we have dropped here. Since \( H \) and \( H^c \) are even and odd under the \( Z_2 \)-parity, only \((h, \chi_h)\) have zero modes.

Now we will turn on the parameter \( l \). Then \( h^c \) has a nontrivial background configuration and stabilizes the radius. In this case, the radion obtains a nonzero mass of order \( l \), and has its component also in the fluctuation of \( h^c \) although the most part is still in the fünfbein. The zero modes of \((h, \chi_h)\) also obtain nonzero masses of \( O(l) \). We will calculate these masses in the following. The hyperscalars are mode-expanded as

\[
h(x, y) = C_h e^{\left(\frac{2k-m}{3}\right)\pi y} \left\{ 1 + O(l^2) \right\} h_{(0)}(x) + \text{(massive modes)},
\]

\[
h^c(x, y) = h^c_\Omega(y) + \text{(massive modes)},
\]

(2.24)

where the normalization factor \( C_h \) is defined as Eq. (B.5).

Substituting this background into Eq. (2.20), the on-shell values of the auxiliary fields are

\[
F_E = \frac{2}{3} \kappa^3 m J_0 \epsilon^2(y) C_h e^{2k|y|} \bar{h}_{(0)} - \kappa^3 J_0 C_h \sum_{\vartheta^* = 0, \pi} e^{i\vartheta^*} e^{2k|y|} \bar{h}_{(0)} \delta(y - \vartheta^* R) + \cdots,
\]

\[
F_V = \frac{\kappa^3}{2} m J_0 \epsilon^2(y) C_h e^{2k|y|} \bar{h}_{(0)} - \frac{\kappa^3}{2} J_0 C_h \sum_{\vartheta^* = 0, \pi} e^{i\vartheta^*} e^{2k|y|} \bar{h}_{(0)} \delta(y - \vartheta^* R) + \cdots,
\]

\[
F_\xi = -\frac{\kappa^3}{2} m J_0 \epsilon^2(y) C_h e^{2k|y|} \bar{h}_{(0)} + \cdots,
\]

(2.25)

where the ellipses denote terms involving massive modes or higher order terms for \( l \)-expansion. The on-shell values of the other auxiliary fields start from the quadratic terms for \( h_{(0)} \). Note that \( \epsilon^2(y) \) equals one in the bulk, but its boundary values depend on the regularization of the orbifold singularity.

Eliminating the auxiliary fields by Eq. (2.25) and performing the \( y \)-integral, we can calculate the mass of \( h_{(0)} \). Since it comes from the linear terms for \( h_{(0)} \) in Eq. (2.25), the result involves the integrals of \( \epsilon^2(y) \delta(y - \vartheta^* R) \) or \( \delta^2(y - \vartheta^* R) \). These quantities must be

---

\(^{13}\)The \( Z_2 \)-odd fields \((h^c, \chi_h^c)\) also have light modes whose masses are \( O(l) \) in this case. In fact, they correspond to the radion mode and its superpartner. As explained in Sect. 2.2 of Ref. [21], their normalized mode-functions are suppressed by \( l \), and thus we will neglect these modes in the following.
evaluated under some regularization of the orbifold singularity. If we pick up only the regularization-independent part of the result, we obtain
\[ m_h^2 = \frac{l^2}{6} m^2 \left( 1 - \frac{2m}{k} \right) e^{-2k\pi R} \frac{1 - e^{-2k\pi R}}{1 - e^{-(k-2m)\pi R}} + \mathcal{O}(l^4). \] (2.26)

Next, let us calculate the mass of \( \chi_{h(0)} \), the (pseudo-) zero mode of \( \chi_h \). For this purpose, we have to start from the full SUGRA action \([10]\) because the hyperino \( \chi_h \) has mixing terms with the gravitino that we have dropped in the superspace action (2.15). In this calculation, we do not encounter the auxiliary fields in Eq.(2.25) and the result is independent of the orbifold regularization. The resulting mass term in the 4D effective Lagrangian is
\[ \mathcal{L}_{\text{fermi}}^{(4)} = i \bar{\chi} \gamma^m \partial_m \chi - m_\chi \bar{\chi} \chi + \cdots, \] (2.27)
where \( \chi \) is the following Dirac spinor
\[ \chi \equiv \begin{pmatrix} \chi_T \\ \chi_{h(0)} \end{pmatrix}. \] (2.28)
Here, \( \chi_T \) is the lightest mode of \( \psi_y \equiv i(\psi_{yR}^{(2)} + \psi_{yL}^{(1)}) \), which corresponds to the superpartner of the radion. The mass \( m_\chi \) is
\[ m_\chi = \frac{l}{\sqrt{6}} \left( 2k + m \right) \left( 1 - \frac{2m}{k} \right)^{1/2} e^{-k\pi R} \left( \frac{1 - e^{-2k\pi R}}{1 - e^{-(k-2m)\pi R}} \right)^{1/2} + \mathcal{O}(l^3). \] (2.29)

Since \( \chi_{h(0)} \) has a Dirac mass with \( \chi_T \), these two fermionic modes are degenerate. Note that \( m_\chi \) in Eq.(2.29) equals the radion mass \( m_{\text{rad}} \) calculated in Ref. [21]. This is a trivial result because \( \chi_T \) is a superpartner of the radion. Therefore, we can conclude that the correct value of the mass of \( h_{(0)} \) is \( m_{\text{rad}} \) due to the \( \mathcal{N} = 1 \) supersymmetry preserved by the background (2.22). Comparing Eq.(2.26) to Eq.(2.29), we can see that terms proportional to \( km \) and \( k^2 \) are missing in Eq.(2.26). These missing terms are expected to be provided from terms involving \( \epsilon^2(y)\delta(y - \vartheta^* R) \) and \( \delta^2(y - \vartheta^* R) \), respectively. Therefore, a regularization scheme that respects the preserved supersymmetry is necessary to calculate \( m_h^2 \) correctly. This is a generic problem in the 5D off-shell approach if we work in a model with boundary terms. Especially, in the case that the background is non-BPS, we cannot estimate quantities involving the regularization-dependent terms without the information about the more fundamental theory, such as the string theory.

### 2.4 Comment on Marti-Pomarol action

Before ending this section, we will briefly comment on the relation to the superspace action in Ref. [13]. In the 5D off-shell approach, there are four dependent superfields \( (\Xi, \Xi^c, V^0, E) \) \(^{14}\) and one \( \mathcal{N} = 1 \) vector superfield \( V_E \) which originates from the fifth-dimensional component of the 5D gravitational multiplet. All the other superfields are the ordinary matter superfields. In fact, the five superfields
\[ \Xi, \, \Xi^c, \, V^0, \, E, \, V_E \] (2.30)

\(^{14}\)Their scalar and spinor components are rewritten by the other physical fields. See Eq.(2.4).
characterize the theory as the supergravity. Besides their auxiliary fields and the graviphoton field $W^{\mu}_{\rho}$, all the components of the five multiplets become constant at the leading order of $\kappa$. Beyond the leading order, they induce all the SUGRA interactions which are generically suppressed by $\kappa$. In this sense, they play a similar role to the chiral compensator superfield $\Sigma$ in 4D off-shell SUGRA \cite{8}. So we will refer to the superfields (2.30) as “$\Sigma$-like superfields” here.

If we neglect $\kappa$-suppressed quartic or higher couplings, the fermionic components of the $\Sigma$-like superfields can be dropped. Roughly speaking, the equation of motion for $D^0$ is used to determine the background geometry, i.e., $\sigma(y)$. After the geometry is determined, $V^0$ and $\Xi^c$ contribute to only $\kappa$-suppressed terms we are neglecting here. Furthermore, if the bulk scalar fields do not have any nontrivial background configurations, $V_E$ is related to $E$ as \cite{13}

$$V_E \simeq \frac{E + \bar{E}}{2},$$

(2.31)

up to the $\kappa$-suppressed terms. This can be seen from the second equation in Eq. (2.20). The second term in the right-hand side of that equation leads to only $\kappa$-suppressed quartic terms if $h_{cl} = h_{cl}^c = 0$. Thus, $\mathcal{F}_E \simeq 2\mathcal{F}_V$. This suggests Eq. (2.31).

Therefore, there remain only two $\Sigma$-like superfields $\Xi$ and $E$ in this case. They correspond to the superfields called as the compensator and the radion superfields in Ref. \cite{15}. Using the relation (2.31), our superspace action becomes a similar form to that of Ref. \cite{15}. As we have pointed out in Ref. \cite{13}, however, there is a difference between them that cannot be removed by the superfield redefinition. Namely, the action in Ref. \cite{15} is not consistent with the 5D SUGRA action of Refs. \cite{9} \cite{10}. Furthermore, we should treat $\mathcal{F}_V$ and $\mathcal{F}_E$ as independent auxiliary fields in the case that the bulk scalars have nonzero background configurations, as in the case of Sect. 2.3. In fact, the relation $\mathcal{F}_E \simeq 2\mathcal{F}_V$ does not hold in Eq. (2.25).

Of course, all five $\Sigma$-like superfields are necessary for realizing the full SUGRA action including the $\kappa$-suppressed quartic or higher terms.

3 4D off-shell approach

In this section, we will explain another approach to derive the 4D effective theory in the case that the background preserves $\mathcal{N} = 1$ SUSY. In this case, the effective theory is expected to be 4D SUGRA. Thus it can be expressed within the framework of 4D off-shell SUGRA \cite{8}. In this framework, there is only one superfield other than the ordinary matter superfields, i.e., a chiral compensator superfield $\Sigma$. This $\Sigma$ is not obtained from the 5D compensator superfields by the usual K.K. decomposition. Since the 5D compensator superfields are not independent of the matter fields, the usual K.K. decomposition cannot be performed for them. Therefore, we need a different approach to obtain the 4D off-shell action. The key of this approach is the introduction of the radion superfield. The resultant action becomes a 4D superspace action. Thus we will refer to this approach as the 4D off-shell approach in this paper.

In order to obtain the complete effective action, we have to take all the gravitational interactions into account. In this section, however, we will focus on the matter-radion
couplings among them since the other gravitational interactions are generically suppressed by the large 4D Planck mass and are irrelevant to the phenomenological discussion in many realistic brane-world models. The matter-radion couplings, on the other hand, cannot be neglected in such models because they play an important role in the mediation of SUSY breaking effects when we introduce some SUSY breaking mechanism.\textsuperscript{15} Namely the $F$-term of the radion superfield mediates the SUSY breaking effects to the visible sector through the matter-radion couplings.

3.1 Radion superfield in 5D action

We should note that the physical matter multiplets can be expanded into K.K. towers of 4D multiplets in the ordinary manner at least the leading order for the $\kappa$-expansion. For a 5D matter superfield $\Phi$, the mode expansion can be performed keeping the $\mathcal{N} = 1$ superfield structure as

$$\Phi(x, y, \theta) = \sum_{n=0}^{\infty} f^{(n)}(y) \varphi^{(n)}(x, \theta), \quad (3.1)$$

where $f^{(n)}(y)$ are appropriate mode functions that are solutions of the mode equations. Since the only obstacle for the naive K.K. expansion is the existence of the compensator and graviphoton multiplets, we can derive the 4D superspace action just like the global SUSY case if we drop them. Of course, the action derived in such a way lacks the radion superfield not only the 4D gravitational interaction terms. Although the radion multiplet behaves as a chiral matter multiplet in the 4D effective theory, its dependence of the effective action is not determined by the ordinary K.K. dimensional reduction in contrast to the other matter multiplets. This stems from the fact that the radion originally belongs to the 5D gravitational field.

In our previous paper \cite{21}, we have clarified the appearance of the radion superfield $T$ in the 5D superspace action as follows.\textsuperscript{16}

\begin{align*}
\mathcal{L} &= \mathcal{L}_{\text{bulk}} + \sum_{\vartheta^s = 0, \pi} \mathcal{L}_{\text{brane}}^{(\vartheta^s)} \delta(y - \vartheta^s R), \\
\mathcal{L}_{\text{bulk}} &= \left\{ \int d^2 \theta \, G(T) W^2 + \text{h.c.} \right\} + e^{2\sigma} \int d^4 \theta \, G_R^{-2}(T) \left( \partial_y V + i \Phi_S - i \bar{\Phi}_S \right)^2 \\
&\quad + e^{2\sigma} \int d^4 \theta \left\{ G_R^2(T) \left( \dot{H} e^{2gV} + \ddot{H} e^{-2gV} H^c \right) - 3 M^3_5 \ln G_R \right\} \\
&\quad + e^{3\sigma} \left\{ \int d^2 \theta \, H^c \left( \partial_y + \frac{3}{2} \dot{\sigma} + m G(T) - 2 i g \Phi_S \right) H + \text{h.c.} \right\}, \\
\mathcal{L}_{\text{brane}}^{(\vartheta^s)} &= \left\{ \int d^2 \theta \, f^{(\vartheta^s)}_{AB}(S) W^A W^B + \text{h.c.} \right\} - e^{2\sigma} M^2_5 \int d^4 \theta \, G_R^{-1}(T) \exp \left\{ - K^{(\vartheta^s)}(S, \bar{S}, U) \right\} \\
&\quad + e^{3\sigma} \left\{ \int d^2 \theta \, G^{-\frac{3}{2}}(T) P^{(\vartheta^s)}(S) + \text{h.c.} \right\}, \quad (3.2)
\end{align*}

\textsuperscript{15}The 4D off-shell approach remains applicable even in such a case if the SUSY breaking scale is much lower than the Kaluza-Klein mass scale, which is a cutoff scale of the 4D effective theory.

\textsuperscript{16}In the following, we will neglect the backreaction on the metric, \textit{i.e.}, $\sigma(y) = -k |y|$.
where
\[ G(T) \equiv \left(1 + e^{2k|y|}e^{-k\pi R \sinh k\pi (R - T)} \right)^{-1}, \]
\[ G_R(T) \equiv \text{Re} G(T). \] (3.3)

The 4D vector and chiral superfields \( U \) and \( S \) can be localized or induced superfields on the boundaries. In the case that \( S \) is an induced superfield from a bulk hypermultiplet \((H, H^c)\), it is understood as \(^{17}\)
\[ S = G^4(T)H. \] (3.4)

The power of \( G(T) \) in Eq. (3.2) is determined by the Weyl-weight matching and the condition for the radion mode to be promoted to a chiral superfield. (See Sect.3.2 in Ref. [21].) The last term in the second line of \( \mathcal{L}_{\text{bulk}} \) corresponds to the kinetic term for the radion superfield. Roughly speaking, the scalar component of \( G(T) \) corresponds to \( e^4y \).\(^{18}\) The radion dependence of the metric for our result agrees with that of Ref. [23], where the \( T \)-dependent effective action for 5D pure SUGRA is derived.

Note that Eq. (3.2) should not be understood as a 5D action. This is because \( T \) is introduced as a 4D superfield from the beginning in this approach. We cannot eliminate the auxiliary field of \( T \) at this stage because it has no \( y \)-dependence. Thus, Eq. (3.2) is not a final form yet. Recall that the K.K. expansion can be performed in the ordinary way for the other matter superfields. In order to obtain the 4D effective action, we have to expand them into K.K. modes first, drop the heavier modes than the cut-off scale of the effective theory, and perform the \( y \)-integral. Then we can obtain the desired effective action.

Let us demonstrate this procedure in the model of Ref. [19]. The hypermultiplet \((H, H^c)\) is expanded as
\[ H(x, y, \theta) = \sum_{n=0}^{\infty} f(n)(y)H_n(x, \theta), \]
\[ H^c(x, y, \theta) = h^c_{cl}(y) + \sum_{n=1}^{\infty} f^c_{(n)}(y)H^c_{(n)}(x, \theta), \] (3.5)

where \( h^c_{cl}(y) \) is defined in Eq. (2.22). Here we will neglect the subleading contribution for \( k \)-expansion. In this case, \( H^c \) has no light mode, while \( H \) has a (pseudo) zero mode whose mode function is
\[ f_{(0)}(y) = C_he^{\frac{4k}{2}k-m)|y|}, \] (3.6)
where \( C_h \) is defined in Eq. (B.5). The mode functions of the massive modes for \( H \) and \( H^c \) are expressed by the Bessel functions \(^5\). (See Appendix B)

The function \( G(T) \) is expanded as
\[ G(T) = 1 + \alpha e^{2k|y|\tilde{T}} + \mathcal{O}(\tilde{T}^2), \] (3.7)
\(^{17}\)\( H^c \) is \( Z_2 \)-odd and cannot appear in the boundary actions.
\(^{18}\)Here, we have chosen the coordinate \( y \) so that \( \langle e^4y \rangle = 1 \). Thus, the parameter \( R \) is the radius of the orbifold. Namely, \( \langle T \rangle = R \) by definition of \( T \).
where $\tilde{T} \equiv T - R$ is a fluctuation of the radion superfield around the background value, and

$$\alpha \equiv \frac{2k\pi}{e^{2k\pi R} - 1}. \quad (3.8)$$

Plugging these expressions into Eq. (3.2) and performing the $y$-integral, we obtain the effective action,

$$L^{(4)} = \int d^4\theta \left\{ \frac{3}{4} M_5^2 \pi \alpha |\tilde{T}|^2 + |H_{(0)}|^2 + \mathcal{O}(\tilde{T}^3) \right\}
+ \left[ \int d^2\theta \left\{ \frac{J_0 C h \pi}{2} \left( \frac{3}{2} k + m \right) \tilde{T} H_{(0)} + \mathcal{O}(T^2) \right\} + \text{h.c.} \right] + \cdots, \quad (3.9)$$

where the ellipsis denotes terms involving the massive modes.

From Eq. (3.9), we can easily see that $\tilde{T}$ and $H_{(0)}$ has the following degenerate mass.

$$m_{\text{rad}} = m_{H_{(0)}} = \frac{l}{\sqrt{6}} \left( \frac{3}{2} k + m \right) \left( 1 - \frac{2m}{k} \right)^{1/2} e^{-k\pi R} \left( \frac{1 - e^{-2k\pi R}}{1 - e^{-(k-2m)\pi R}} \right)^{1/2}. \quad (3.10)$$

This certainly agrees with the result (2.29) in the previous section. Note that $\chi_T$ and $\chi_{h(0)}$ in Eq. (2.28) are the fermionic components of $\tilde{T}$ and $H_{(0)}$, respectively.

In contrast to the 5D off-shell approach, we have not encountered any regularization-dependent quantities such as $\epsilon^2(y) \delta(y - \vartheta R)$ or $\delta^2(y - \vartheta R)$. This is because we have not eliminated the auxiliary fields until we derive the 4D superspace action (3.9). Therefore, we don’t have to search for a supersymmetric regularization of the orbifold singularity. This is one of the advantage of the 4D off-shell approach.

In general, the radion Kähler potential $K_{\text{rad}}^{(4)}(T, \bar{T})$ becomes a complicated function because of the nontrivial $y$ dependence of $G(T)$ in Eq. (3.3). However, in the flat limit (i.e., $k \to 0$), $G(T)$ becomes independent of $y$ and reduces to a simple form,

$$G(T) = \frac{T}{R} = 1 + \frac{\tilde{T}}{R}. \quad (3.11)$$

In this case, the radion Kähler potential has the following no-scale form up to a constant.

$$K_{\text{rad}}^{(4)}(T, \bar{T}) = -3M_4^2 \ln(T + \bar{T}), \quad (3.12)$$

where $M_4 \equiv (\pi RM_5^3)^{1/2}$ is the 4D Planck mass.

### 3.2 Consistency with 5D off-shell approach

Now we will check the consistency of this 4D off-shell approach with the 5D off-shell approach discussed in the previous section. The agreement of Eqs. (2.29) and (3.10) provides a nontrivial cross-check for the powers of $G(T)$ in the boundary superpotential and in Eq. (3.3). For further checks, we will introduce an extra hypermultiplet $(Q, Q^c)$ with a bulk
mass $m_Q$ and a vector multiplet $(V, \Phi_S)$ to the model of Ref. \[19\]. The mode expansions of these additional superfields are as follows.

$$Q(x, y, \theta) = \sum_{n=0}^{\infty} f_Q(n)(y)Q_n(x, \theta),$$

$$Q^c(x, y, \theta) = \sum_{n=1}^{\infty} f_{Q^c}(n)(y)Q^c_n(x, \theta),$$

$$V(x, y, \theta) = \sum_{n=0}^{\infty} v_n(y)V_n(x, \theta),$$

$$\Phi_S(x, y, \theta) = \sum_{n=1}^{\infty} u_n(y)\Phi_{S_n}(x, \theta).$$

(3.13)

Note that $Q^c$ and $\Phi_S$ do not have zero-modes because they are odd under $Z_2$-parity. The explicit forms of the mode functions are collected in Appendix B.

In 5D off-shell SUGRA, there are five superfields (2.30) that play the similar role to that of the chiral compensator $\Sigma$ in 4D off-shell SUGRA, as mentioned in Sect. 2.4. They induce all the gravitational interactions, which partly become the couplings to the radion superfield. Thus, to check the consistency between the two approaches, we will focus on the contributions from the superfields (2.30) in the 5D off-shell approach and those from $T$ in the 4D off-shell approach because the other parts are identical in both approaches.

In the 5D off-shell approach, we can see from Eq. (2.25) that $F_E$, $F_V$ and $F_\xi$ have linear terms for $h_{(0)}$ in their on-shell values. There are singular terms among them that are proportional to $\delta(y - \vartheta^* R)$ ($\vartheta^* = 0, \pi$). They lead to the regularization-dependent terms. Note that the non-singular terms in Eq. (2.25) are proportional to the bulk mass parameter $m$ while the singular terms do not. This means that the $m$-dependent part of each quantity is completely calculated only from the non-singular terms. Thus we will focus on the $m$-dependent contributions here. The linear terms in Eq. (2.25) lead to the following cubic interactions.

$$L^{(4)}_{\text{cubic}} = \frac{\kappa^3 mJ_0C_p}{6} \left\{ \sum_{n,l} \left( a^\lambda_{nl} \bar{h}_{(0)} \lambda_{(n)} \lambda_{(l)} + 2a^\chi_{nl} \bar{h}_{(0)} \chi_{S(n)} \chi_{S(l)} \right) 
+ \sum_{n,l} 3b_{nl}h_{(0)}q^c_{(n)}q_{(l)} + \text{h.c.} \right\} + \cdots$$

(3.14)

where $\lambda_{(n)}$, $\chi_{S(n)}$ and $q_{(l)}$, $q^c_{(l)}$ are the spinor components of $V_{(n)}$, $\Phi_{S(n)}$ and the scalar components of $Q_{(l)}$ and $Q^c_{(l)}$, respectively. The ellipsis denotes the $m$-independent contributions.
coming from the singular terms in Eq. (2.25). The constants $a_{nl}^\lambda$, $a_{nl}^\chi$ and $b_{nl}$ are defined as

$$a_{nl}^\lambda \equiv \int_0^{\pi R} dy \ e^{2ky} v(n)(y) v(l)(y),$$

$$a_{nl}^\chi \equiv \int_0^{\pi R} dy \ u(n)(y) u(l)(y),$$

$$b_{nl} \equiv \left\{ m^q_n d_{nl} + m^c_l d^c_{nl} - \frac{4mQ}{3} c_{nl} \right\} ,$$

where $m^q_n$ are degenerate mass eigenvalues of $(Q(n), Q^c(n))$, and

$$c_{nl} \equiv \int_0^{\pi R} dy \ e^{-ky} f_Q(n)(y) f_Q(l)(y),$$

$$d_{nl} \equiv \int_0^{\pi R} dy \ f_Q(n)(y) f_Q(l)(y),$$

$$d^c_{nl} \equiv \int_0^{\pi R} dy \ f_Q^c(n)(y) f_Q^c(l)(y).$$

In the 4D off-shell approach, the corresponding interaction terms come from the contribution of $F_T$, the auxiliary field of $T$. The 4D effective Lagrangian is calculated as follows by repeating the procedure from Eq.(3.2) to Eq.(3.9) including $(Q, Q^c)$ and $(V, \Phi_S)$. The result is

$$L^{(4)} = \frac{1}{4} \left\{ \int d^2 \theta \ \sum_n W^2_n + \alpha \sum_{n,l} a_{nl}^\lambda \bar{T} W_n W_l + \text{h.c.} \right\}$$

$$+ \int d^4 \theta \ \left[ \sum_{n \neq 0} |\Phi_S(n)|^2 + \alpha \sum_{n,l} \left\{ a_{nl}^\chi \bar{T} \Phi_S(n) \Phi_S(l) + \text{h.c.} \right\} \right]$$

$$+ \int d^4 \theta \ \left[ \frac{3}{4} M_5^3 \pi \alpha \ |\bar{T}|^2 + |H(0)|^2 + \sum_n |Q(n)|^2 + \sum_{n \neq 0} |Q^c(n)|^2 \right.$$

$$\left. + \frac{3}{2} \alpha \sum_{n,l} \text{Re} \bar{T} \left\{ d_{nl} \bar{Q}(n) Q(l) + d^c_{nl} \bar{Q}^c(n) Q^c(l) \right\} \right]$$

$$+ \left[ \int d^2 \theta \ \left\{ \frac{\pi J_0 C_h}{2} \left( \frac{3}{2} k + m \right) \bar{T} H(0) + \sum_{n \neq 0} m^q_n Q^c(n) Q(n) \right. \right.$$

$$\left. + m_Q \alpha \sum_{n,l} c_{nl} \bar{T} Q^c(n) Q(l) \right\} + \text{h.c.} \right] + \cdots ,$$

where the ellipsis denotes irrelevant terms to the discussion here.

From this action, the on-shell value of $F_T$ is calculated as

$$F_T = \frac{2 \kappa^3 J_0 C_h}{3 \alpha} \left( \frac{3}{2} k + m \right) \bar{h}_{(0)} + \cdots ,$$
where the ellipsis denotes terms beyond the linear order for fields or involving the massive modes \( h^{(n)} (n \neq 0) \). The linear term for \( h^{(0)} \) in Eq. (3.18) induces the following cubic couplings.

\[
\mathcal{L}^{(4)}_{\text{cubic}} = \frac{\kappa^3 J_0 C_h}{6} \left( \frac{3}{2} k + m \right) \left\{ \sum_{n,l} \left( a^\lambda_n \bar{h}^{(0)} \lambda(n) \lambda(l) + 2 a^\chi_n h^{(0)} \chi_S(n) \chi_S(l) \right) + \sum_{n,l} 3 b_{nl} \bar{h}^{(0)} q^n \chi q^l + \text{h.c.} \right\}.
\] (3.19)

This result is consistent with that obtained in the 5D off-shell approach (3.14). Note that we have encountered no regularization-dependent terms.

The matching of the masses and the cubic interactions obtained in both approaches provides nontrivial cross-checks for their consistency. Note that these mass and cubic terms come from the linear terms for \( h^{(0)} \) in \( \mathcal{F}_E, \mathcal{F}_V \) and \( \mathcal{F}_\xi \) in the 5D off-shell approach (see Eq. (2.25)), while they come from the linear term in \( \mathcal{F}_T \) in the 4D off-shell approach (see Eq. (3.18)). This means that the role of the superfields (2.30) in the 5D off-shell approach is partially played by the radion superfield \( T \) in the 4D off-shell approach.

Finally, we will comment on the result of Ref. [16], in which the authors constructed the 4D off-shell effective action of 5D SUGRA. However, their method is based on the naive replacement of the radius of the orbifold \( R \) with the dynamical radion field \( r(x) \). We have pointed out in our previous work [21, 22] that this naive replacement does not lead to the correct action. This is because the mode function of the radion is not taken into account by this replacement. In fact, our result obtained in the 4D off-shell approach does not agree with the result of Ref. [16].

4 Summary

We have discussed the dimensional reduction of 5D off-shell SUGRA to derive the 4D effective theory. There are two approaches to derive the 4D effective action. We call them the 5D and the 4D off-shell approaches in this paper. The essential difference between them is the treatment of the compensator and the radion superfields in the action.

The 5D off-shell approach is based on the 5D superconformal gravity formulated in Refs. [9, 10, 11]. Since each 5D superconformal multiplet can be decomposed into \( \mathcal{N} = 1 \) multiplets [9], we can express the 5D SUGRA action in the language of 4D off-shell SUGRA [12, 13]. Therefore, it seems possible to treat 5D off-shell SUGRA as if 4D off-shell SUGRA at first sight. However, there are five \( \mathcal{N} = 1 \) multiplets (2.30) that have no counterpart in 4D off-shell SUGRA. These five \( \mathcal{N} = 1 \) multiplets play a similar role to the chiral compensator multiplet \( \Sigma \) in 4D off-shell SUGRA. If the bulk scalars do not have any nontrivial background configurations, only \( \Xi \) and \( E \) among them remain in the action if we neglect \( \kappa \)-suppressed quartic or higher terms. (See Sect. 2.4) This corresponds to the situation considered in Ref. [15]. There, \( \Xi \) and \( E \) are called as the compensator and the radion superfields, respectively. As we have pointed out in Ref. [13], however, their dependences of the action in Ref. [15] is not consistent with the 5D SUGRA action.
of Refs. [9, 10]. Note also that the relation (2.31) does not hold if the bulk scalars have a nontrivial background configuration. For example, $F_V$ and $F_E$ should be treated as independent auxiliary fields in the model of Ref. [19] discussed in Sect.2.3. Therefore, we should start from the action (2.15), which includes all the five superfields (2.30).

The important point in the 5D off-shell approach is that all the auxiliary fields must be eliminated before the dimensional reduction. This is because we cannot perform the naive K.K. expansion for the five superfields (2.30). Thus the 4D effective action derived in this approach inevitably becomes the on-shell action. The on-shell values of the auxiliary fields generically have singular terms proportional to $\delta(y - \vartheta^* R)$ ($\vartheta^* = 0, \pi$) if the boundary terms exist. On the other hand, some quantities in the bulk are odd under the $Z_2$ parity. Therefore, some terms in the on-shell 5D action depend on the regularization of the orbifold singularity. This means that there are some quantities we cannot calculate without the orbifold regularization. This is one of the drawbacks of the 5D off-shell approach.

In the case that the background preserves $\mathcal{N} = 1$ SUSY, there is another approach to derive the 4D effective action, which we call the 4D off-shell approach. The key of this approach is an introduction of the radion superfield $T$. The starting point is Eq.(3.2), which shows the $T$-dependence of the 5D action. Note that all 5D $\mathcal{N} = 1$ superfields in Eq.(3.2) can be expanded into K.K. modes keeping the structure of $\mathcal{N} = 1$ superfields. Performing the $y$-integral, the 4D effective action written in the superspace is obtained. Although Eq.(3.2) is not the final form of the above derivation, it is useful because we can easily read off integrands of the overlap integrals that appear as coupling constants in the effective action. Since we obtain the 4D effective action as a superspace action, the preserved $\mathcal{N} = 1$ SUSY is manifest. In addition, we do not suffer from the regularization-dependent terms in contrast to the 5D off-shell approach. Recall that all such terms originate from the elimination of the 5D auxiliary fields. In the present case, on the other hand, the elimination of the auxiliary fields does not cause such terms because the $y$-dependence is already integrated out. We have focus on the radion couplings among the gravitational interactions in this approach. In order to incorporate all the interactions into the discussion, we have to find some method to derive Eq.(3.2) directly from the original 5D off-shell SUGRA action. We will discuss this issue in the forthcoming paper.

We have checked the consistency between the effective actions derived by the above two approaches. Comparing Eqs.(2.15) and (3.2), the difference between the two approaches is the treatment of the compensator and the radion superfields. The role of the five superfields (2.30) in the 5D off-shell approach is partially inherited by $T$ in the 4D off-shell approach.

When we use the word “the radion superfield”, we have to specify the context in which we work. It denotes $E$ (and $V_E$) in the 5D off-shell approach, while it denotes $T$ in the 4D off-shell approach. For example, Ref. [15] corresponds to the former case, and Ref. [16] to the latter case although their actions should be modified. Here note that only the 4D off-shell approach can treat it as a dynamical superfield. In the 5D off-shell approach, it becomes a spurion-like superfield since the gravitational fields are fixed to the background values in compensation for expressing the action on the superspace.\footnote{If we use the $F$- and $D$-term formulae instead of the superspace, we can treat “the radion multiplet” $E$ (and $V_E$) as dynamical fields also in the 5D off-shell approach, but the action becomes quite complicated.}
One of the advantages of the 5D off-shell approach is that it can also deal with non-BPS backgrounds. Especially, the Scherk-Schwarz SUSY breaking mechanism \cite{24} is clearly understood in this approach \cite{13, 25}. Its disadvantage is the appearance of the regularization-dependent quantities. The regularization of the orbifold depends on the fundamental theory at higher energies. Namely, such quantities are UV sensitive and cannot be calculated without specifying the UV theory. On the other hand, the 4D off-shell approach can derive the effective action without any regularization-dependent quantities with the aid of the radion superfield. In this approach, the preserved SUSY is manifest since the effective action is expressed by superfields. However, it should be noted that the 4D off-shell approach is available only when the background is BPS. For example, it cannot deal with the Scherk-Schwarz SUSY breaking.

As a future work, we will plan to extend the discussion to the case where the bulk is six or higher dimensions. Unfortunately, it is known that no off-shell description of such higher dimensional SUGRA with matter multiplets exists. Therefore, only the 4D off-shell approach is available. In such a case, more moduli superfields will appear and the structure of the moduli space becomes rich and complicated. Thus, the investigation of such structure is also an intriguing subject.

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A Superfields and gauge fixing

Here, we will collect the explicit forms of the $\mathcal{N} = 1$ superfields in terms of the superconformal notation of Refs. \cite{9,10}.

A.1 4D superfields

4D superconformal multiplets can appear in the boundary action. The construction of $\mathcal{N} = 1$ superfields from them is straightforward. The only thing to note is the existence of the warp factor. This comes from the $y$-dependence of the Killing spinor for the preserved SUSY.

From a vector multiplet $(B^A_m, \lambda^A, D^A) \ (A = 1, 2, \cdots)$, we can obtain the following vector superfield.

\[
U^A \equiv \theta \sigma^m \bar{\theta} B^A_m + i e^{\frac{3}{2}\sigma} \theta^2 \bar{\theta} \lambda^A - i e^{\frac{3}{2}\sigma} \bar{\theta}^2 \theta \lambda^A + \frac{1}{2} e^{2\sigma} \theta^2 \bar{\theta}^2 D^A, \tag{A.1}
\]

where $\sigma(y)$ is estimated at the boundary $y = 0, \pi R$. The corresponding superfield strength is defined as

\[
W^A_\alpha \equiv -\frac{1}{4} \bar{D}^2 D^A \theta^\alpha. \tag{A.2}
\]

From a chiral multiplet $(s^a, \chi^a, \mathcal{F}^a_s) \ (a = 1, 2, \cdots)$, we can obtain the following chiral superfield.

\[
S^a \equiv s^a - e^{\bar{\theta}^2} \theta \chi^a - e^{\sigma} \theta^2 \mathcal{F}^a_s. \tag{A.3}
\]
A.2 5D superfields

From the 5D Weyl multiplet, we can construct the following $\mathcal{N} = 1$ superfield.

$$V_E = \langle e_y^4 \rangle + ie^1 \theta^2 \left( V_y^{(1)} + iV_y^{(2)} \right) - ie^2 \bar{\theta}^2 \left( V_y^{(1)} - iV_y^{(2)} \right),$$  \hspace{1cm} (A.4)

where $V_y^{(r)}$ is an $SU(2)_U$ gauge field, which is an auxiliary field in 5D off-shell SUGRA. This superfield corresponds to the $\mathcal{N} = 1$ multiplet $W_y$ in Ref. [9].

A vector multiplet $V^I$ ($I = 0, 1, \ldots, n_V$) consists of

$$M^I, \ W^I_\mu, \ \Omega^I_i, \ Y^{I(r)},$$  \hspace{1cm} (A.5)

which are a gauge scalar, a gauge field, a gaugino and an auxiliary field, respectively. The indices $i = 1, 2$ and $r = 1, 2, 3$ are doublet and triplet indices for $SU(2)_U$. From this multiplet, we can construct the following chiral superfields.

$$V^I \equiv \theta \sigma^m \bar{\theta} W^I_m + i \theta^2 \bar{\theta} \lambda^I - i \bar{\theta}^2 \theta \lambda^I + \frac{1}{2} \theta^2 \bar{\theta}^2 D^I,$$

$$\Phi^I_S \equiv \varphi^I_S - \theta \chi^I_S - \theta^2 \mathcal{F}^I_S,$$  \hspace{1cm} (A.6)

where

$$\lambda^I \equiv 2e^2 \bar{\theta}^2 \Omega^I_R,$$

$$D^I \equiv -e^2 \left\{ \langle e_y^4 \rangle^{-1} \partial_y M^I - 2Y^{I(3)} + \langle e_y^4 \rangle^{-1} \bar{\theta} M^I \right\},$$

$$\varphi^I_S \equiv \frac{1}{2} \left( W^I_y + i \langle e_y^4 \rangle M^I \right),$$

$$\chi^I_S \equiv -2e^2 \left( \langle e_y^4 \rangle \Omega^I_R \right)^2,$$

$$\mathcal{F}^I_S \equiv e^2 \left\{ \left( \langle V_y^{(1)} + iV_y^{(2)} \rangle M^I - i \langle e_y^4 \rangle \left( Y^{I(1)} + iY^{I(2)} \right) \right) \right\}.$$  \hspace{1cm} (A.7)

The hypermultiplets consist of complex scalars $A_0^\alpha$, spinors $\zeta^\alpha$ and auxiliary fields $\mathcal{F}_i^\alpha$. They carry a $USp(2, 2n_H)$ index $\alpha = 1, 2, \ldots, 2n_H + 2$ on which the gauge group can act. These are split into $n_H + 1$ hypermultiplets as

$$\mathcal{H}^{\bar{\alpha}} = (A_i^{2\bar{\alpha}+1}, \ A_i^{2\bar{\alpha}+2}, \ \zeta^{2\bar{\alpha}+1}, \ \zeta^{2\bar{\alpha}+2}, \ \mathcal{F}_i^{2\bar{\alpha}+1}, \ \mathcal{F}_i^{2\bar{\alpha}+2}), \quad (\bar{\alpha} = 0, 1, \ldots, n_H)$$  \hspace{1cm} (A.8)

From these multiplets, we can construct the following chiral superfields.

$$\Phi^\alpha \equiv \varphi^\alpha - \theta \chi^\alpha - \theta^2 \mathcal{F}^\alpha,$$  \hspace{1cm} (A.9)

where

$$\varphi^\alpha \equiv A_2^\alpha,$$

$$\chi^\alpha \equiv -2e^2 \bar{\theta}^2 \zeta^\alpha,$$  \hspace{1cm} (A.10)

$$\mathcal{F}^\alpha \equiv e^2 \langle e_y^4 \rangle^{-1} \left\{ \partial_y A_1^\alpha + i \left( V_y^{(1)} + iV_y^{(2)} \right) A_2^\alpha + i \left( \langle e_y^4 \rangle + \frac{iW_y^0}{M_y^0} \right) \tilde{\mathcal{F}}_1^\alpha \right. \right.$$
A.3 Superconformal gauge-fixing conditions

The gauge fixing conditions for the extraneous superconformal symmetries, i.e., the dilatation $D$, $SU(2)_U$, the conformal supersymmetry $S$ are as follows.\(^{20}\)

The $D$-gauge is fixed by
\[
\mathcal{N} \equiv C_{IJK} M^I M^J M^K = M_5^3,
\]
\[
\mathcal{A}_i^\alpha d_\alpha \mathcal{A}_i^\beta = 2 \left\{ - \sum_{\alpha=1}^2 |A_2^\alpha|^2 + \sum_{\alpha=3}^{2n_H+2} |A_2^\alpha|^2 \right\} = -2M_5^3, \tag{A.11}
\]

where $\mathcal{N}$ is called the norm function.

The $SU(2)_U$ is fixed by
\[
A_\alpha^i \propto \delta_\alpha^i. \quad (\alpha = 1, 2) \tag{A.12}
\]

The $S$-gauge is fixed by
\[
\mathcal{N}_I \Omega^{li} = 0, \quad A_i^\alpha d_\alpha \zeta_\beta = 0, \tag{A.13}
\]

where $\mathcal{N}_I \equiv \partial \mathcal{N}/\partial M^I$.

B Explicit forms of mode functions

Here we will collect the explicit forms of the mode functions for the matter fields. On the Randall-Sundrum background \[^{[4]}\], i.e., $\sigma(y) = -k |y|$, they are expressed by the Bessel functions \[^{[5]}\].

Let us define the following functions first.
\[
\mathcal{M}_\alpha(\lambda) \equiv J_\alpha(\lambda) Y_\alpha(\lambda e^{k\pi R}) - Y_\alpha(\lambda) J_\alpha(\lambda e^{k\pi R}),
\]
\[
A_\alpha(\lambda, z) \equiv N_{\alpha-1}(\lambda) \left\{ Y_{\alpha-1}(\lambda) J_\alpha(\lambda z) - J_{\alpha-1}(\lambda) Y_\alpha(\lambda z) \right\},
\]
\[
B_\alpha(\lambda, z) \equiv N_\alpha(\lambda) \left\{ Y_\alpha(\lambda) J_\alpha(\lambda z) - J_\alpha(\lambda) Y_\alpha(\lambda z) \right\}, \tag{B.1}
\]

where
\[
N_\alpha(\lambda) \equiv \frac{\sqrt{k\pi} \lambda}{\sqrt{2}} \left\{ \frac{Y_\alpha^2(\lambda)}{Y_\alpha^2(\lambda e^{k\pi R})} - 1 \right\}^{-1/2}. \tag{B.2}
\]

For the hypermultiplet $(H, H^c)$, the mode expansions are as follows.
\[
H(x, y, \theta) = \sum_{n=0}^{\infty} f_{(n)}(y) H_{(n)}(x, \theta),
\]
\[
H^c(x, y, \theta) = \sum_{n=1}^{\infty} f_{(n)}(y) H^c_{(n)}(x, \theta). \tag{B.3}
\]

\[^{20}\text{The special conformal transformation } K \text{ is already fixed in our superspace formalism } \[^{[13]}\].\]
For the zero mode $H(0)$,
\[ f(0)(y) = C_h e^{(\frac{3}{2}k-m)\vert y \vert}, \] (B.4)
where the normalization constant $C_h$ is
\[ C_h = \left( \frac{k - 2m}{e^{(k-2m)\pi R} - 1} \right)^{1/2}. \] (B.5)

For the massive modes $H(n)$, $H^c(n)$ $(n \neq 0)$,
\[ f(n)(y) = e^{2k\vert y \vert} A_{c+\frac{1}{2}}(\lambda_n, e^{k\vert y \vert}), \]
\[ f^c(n)(y) = e^{2k\vert y \vert} B_{c-\frac{1}{2}}(\lambda_n, e^{k\vert y \vert}) \epsilon(y), \] (B.6)
where $c \equiv m/k$, $\epsilon(y)$ is the periodic step function defined in Eq.(2.16), and $\lambda_n$ is a solution of
\[ M_{c-\frac{1}{2}}(\lambda_n) = 0. \] (B.7)
The corresponding mass eigenvalue is given by $m(n) = k\lambda_n$. Note that $H(n)$ and $H^c(n)$ are degenerate for each nonzero $n$.

For the vector multiplet $(V, \Phi_S)$, the mode expansions are as follows.
\[ V(x, y, \theta) = \sum_{n=0}^{\infty} v(n)(y)V(n)(x, \theta), \]
\[ \Phi_S(x, y, \theta) = \sum_{n=1}^{\infty} u(n)(y)\Phi_S(n)(x, \theta). \] (B.8)

For the zero mode $V(0)$, the mode function is
\[ v(0)(y) = \frac{1}{\sqrt{\pi R}}. \] (B.9)

For the other massive modes $V(n)$, $\Phi_S(n)$ $(n \neq 0)$,
\[ v(n)(y) = e^{k\vert y \vert} A_1(\mu_n, e^{k\vert y \vert}), \]
\[ u(n)(y) = \frac{e^{2k\vert y \vert}}{\sqrt{2}} B_0(\mu_n, e^{k\vert y \vert}) \epsilon(y), \] (B.10)
where $\mu_n$ is a solution of
\[ M_0(\mu_n) = 0. \] (B.11)
The corresponding mass eigenvalue is given by $m(n) = k\mu_n$. Note that $V(n)$ and $\Phi_S(n)$ are degenerate for nonzero $n$.

The above mode functions satisfy the following orthonormal relations.
\[ \int_0^{\pi R} dy \, e^{-2ky} f(n)(y)f(l)(y) = \delta_{nl}, \]
\[ \int_0^{\pi R} dy \, e^{-2ky} f^c(n)(y)f^c(l)(y) = \delta_{nl}, \]
\[ \int_0^{\pi R} dy \, v(n)(y)v(l)(y) = \delta_{nl}, \]
\[ \int_0^{\pi R} dy \, u(n)(y)u(l)(y) = \frac{\delta_{nl}}{2}. \] (B.12)
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