Magnetized Tolman-Bondi Collapse

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We investigate the gravitational implosion of magnetized matter by studying the inhomogeneous collapse of a weakly magnetized Tolman-Bondi spacetime. The role of the field is analyzed by looking at the convergence of neighboring particle worldlines. In particular, we identify the magnetically related stresses in the Raychaudhuri equation and use the Tolman-Bondi metric to evaluate their impact on the collapsing dust. We find that, despite the low energy level of the field, the Lorentz force dominates the advanced stages of the collapse, leading to a strongly anisotropic contraction. In addition, of all the magnetic stresses, those that resist the collapse are found to grow faster.

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I. INTRODUCTION

Magnetic fields are common features of almost all astrophysical environments and stellar magnetism is a long established and very active branch of astrophysics. Nevertheless, analytical studies of magnetic fields in strong gravity environments are less developed. Most of the available work addresses the possible gravitational effects on the Maxwell field (e.g. see \cite{2} and references therein) and relatively few look into the implications of magnetic fields for gravitational collapse itself. Perhaps the most intriguing result so far has been obtained by Thorne in his analysis of Melvin’s cylindrical magnetic universe \cite{3}. There, by developing the concept of ‘cylindrical energy’, the author reached the conclusion that “a strong magnetic field along the axis of symmetry may halt the collapse of a finite cylinder before the singularity is reached” \cite{4}. The possible support of the field against the gravitational collapse of massive bounded systems was also studied in \cite{4}. That analysis led to solutions of Einstein-Maxwell equations with no singularities or event horizons, where the gravitational attraction is balanced solely by magnetic stresses. Other work, however, indicated that locally naked singularities may develop in some cases \cite{6}. Studies of contracting charge dust has suggested that the fluid may “rebounce”, thus preventing black-hole formation \cite{7}. It has been argued, on the other hand, that a collapsing spherically symmetric charged dust will inevitably produce naked singularities due to shell-crossing \cite{8}. The latter indicates the intersection of matter flow-lines along certain spacelike hypersurfaces. Although these singularities are considered weak \cite{9}, since the curvature invariants and the tidal forces remain finite, their appearance could also signal that a nonzero Lorentz force and charged spherical collapse are physically incompatible. This tentative ‘conjecture’ is supported by the results of the present paper, which focuses on the role of the magnetic Lorentz force during the gravitational collapse of charged matter.

We consider the gravitational contraction of an inhomogeneous spatially flat Tolman-Bondi model, filled with a pressureless highly conductive fluid, and allow for a perturbing weak magnetic field. This describes to a collapse of a weakly magnetized charged plasma. The weakness of the field is measured by its contribution to the total energy of the system. Our aim is to investigate the fate of the contracting fluid in the presence of a nonzero and without any symmetry constraints Lorentz force. To the best of our knowledge this question has not been addressed analytically. Nonzero Lorentz force means that, despite the absence of fluid pressure, the particle worldlines are no longer timelike geodesics. As a result, we expect the initial spherical symmetry of the collapse to break. We
quantify the consequences of the magnetic presence by looking at the convergence of neighboring particle worldlines. This is done by analyzing the magnetic contribution to the Raychaudhuri equation. The latter describes the volume evolution of a given fluid element and has played a fundamental role in numerous studies of gravitational collapse and also in the formulation of the major singularity theorems (e.g. see [10]). A relatively weak magnetic field contributes to the Raychaudhuri equation primarily via its Lorentz force. The benefit of our approach is that it identifies the effects of Lorentz force on the collapsing matter directly. In particular, the magnetic input splits up into a pair of stresses one of which always supports against the gravitational pull of the matter. Based on the weakness of the field, we ignore its backreaction on the Tolman-Bondi metric and use the latter to evaluate the aforementioned magnetic stresses. We find that the magnetic presence triggers as range of effects with an overall impact that depends on the specifics of the field in a rather involved way. These effects can severely distort the spherical symmetry and completely dominate the advanced stages of the Tolman-Bondi collapse despite the low levels of the magnetic energy input. Interestingly, of all the magnetic stresses, we found that those supporting against the collapse grow faster and we have also identified physically plausible magnetic configurations where this happens. Although one should be very cautious before extrapolating a linear result into the nonlinear regime, our analysis seems to agree with earlier work on magnetized collapse [2,3]. Put together, these studies suggest that a nonzero Lorentz force may be physically incompatible with spherically symmetric collapse and that there might exist situations where the support of the field can outbalance the total gravitational attraction, at least along certain directions.

II. WORLDLINES OF MAGNETIZED MATTER

Assume a general spacetime filled with a highly conductive perfect fluid and allow for a magnetic field. High conductivity means that there is no electric field and that the magnetic field is ‘frozen in’ with the matter. This is the well known MHD approximation (e.g. see [1]). In what follows we will investigate the implications of the pure magnetic component of the Lorentz force on the collapse of such a model. We will do so by testing the convergence of the particle worldlines using the covariant approach to general relativity [11].

Covariantly, the dynamics of gravitational collapse is monitored through the Raychaudhuri equation, which describes the volume evolution of a self-gravitating fluid element. Consider a congruence of timelike worldlines tangent to the 4-velocity field \( u_a \) (with \( u_a u^a = -1 \)) that follows the motion of the fluid. Raychaudhuri’s formula determines the evolution of \( \Theta = \nabla_a u^a \), the scalar measuring the average contraction (or expansion) between a pair of neighboring particle worldlines [11]. In a magnetized environment we have [12]

\[
\dot{\Theta} + \frac{1}{3} \Theta^2 = -\frac{1}{2} \left( \rho + 3p + B^2 \right) - 2 \left( \sigma^2 - \omega^2 \right) + D^a \dot{u}_a + \dot{\sigma}_a u^a,
\]

where \( \rho \) and \( p \) are respectively the energy density and pressure of the fluid, \( B^2 = B_a B^a \) measures the energy density and the isotropic pressure of the magnetic field \( (B_a) \), \( \sigma^2 \) and \( \omega^2 \) are the respective magnitudes of the shear and the vorticity associated with \( u_a \) and \( \dot{u}_a = u^b \nabla_b u_a \) is the 4-acceleration. When the right-hand side of the above is negative definite, an initially converging family of worldlines will focus (i.e. \( \Theta \to -\infty \)) within a finite amount of time [10]. Thus, positive definite terms in the right-hand side of [11] will resist against further gravitational contraction.

The magnetic contribution to the Raychaudhuri equation comes form the gravitational pull of the field which adds to the gravitational pull of the matter and also from the magnetic input to the 4-acceleration. The latter satisfies the momentum-density conservation law, which for a magnetized, highly conductive perfect fluid takes the form [12]

\[
(\rho + p + \frac{2}{3} B^2) \dot{u}_a = -D_a p - \epsilon_{abc} B^b \nabla^c B^c - \Pi_{ab} \dot{u}^b.
\]

In the above \( D_a = h_a \nabla_a \) is the covariant derivative operator orthogonal to \( u_a \), with \( h_{ab} \) representing the projection tensor (i.e. \( h_{ab} u^b \)), while \( \Pi_{ab} = -B_a B_b \) describes the anisotropic pressure of the field [15]. The second term in the right-hand side of Eq. [2] is the Lorentz force, which is always normal to the field vector. Note that we consider non-geodesic worldlines, since the motion of the particles is dictated by the combined Einstein-Maxwell field and not by gravity alone. Also, the fluid flow is generally not hypersurface orthogonal, which explains the presence of the vorticity term in [11].

III. TOLMAN-BONDI COLLAPSE

The general inhomogeneous collapse of pressure-free spherically symmetric matter is monitored by means of the Tolman-Bondi metric, which in the synchronous gauge reads

\[
ds^2 = -dt^2 + R^2(1 + kr^2)^{-1} dr^2 + R^2 dr^2,
\]

(3)
where \( R = R(r, t) \), \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) and a prime indicates differentiation with respect to \( r \) (e.g. see 13). The spatial curvature index \( k = 0, \pm 1 \) corresponds to flat, closed and open geometry respectively. Here we will only consider the \( k = 0 \) case and throughout this paper we will use geometrized units with \( \kappa = 1 = c \). In the adopted coordinate system, the energy momentum tensor of a dust cloud is \( T_{ab} = \rho(r, t)u_a u_b \), with \( u^a = (1, 0, 0, 0) \) representing the 4-velocity of the fluid. Solved on the above metric Einstein’s equations give

\[
R = R(r, t) = \left( \frac{9}{4} \right)^{1/3} M^{1/3} (t - \tau)^{2/3} \quad \text{and} \quad \rho = \rho(r, t) = \frac{M'}{8\pi R^2 R'},
\]

with the functionals \( M = M(r) \) and \( \tau = \tau(r) \) describing the spatial distribution of the matter at fixed time. To proceed further we use the residual gauge freedom on the radial coordinate to define \( M = \mu r^3/3 \), which makes \( \mu \) the mass per unit coordinate volume (i.e. \( 4\pi r^3/3 \)). Thus, the only physically free function left to describe the shape of the pressureless fluid at a fixed time is \( \tau(r) \). Note that the point \( t = \tau \) corresponds to a singularity which is reached at a different time for each shell of fixed radius \( r \). Indeed, following (11) and (12), the Ricci scalar \( R^a_a = -8\pi T^a_a = 8\pi \rho \) associated with any given shell of radius \( r = r^* \), will diverge when \( t = \tau(r^*) \). Here, we assume that the arrow of time increases from \( t = 0 \), which marks the beginning of the collapse, to \( t = \tau(r) \). This in turn guarantees that \( \tau - t > 0 \) for each \( r \). Then, the density and the shear magnitude of the Tolman-Bondi solution are

\[
\rho = \frac{1}{2\pi(\tau - t)[3(\tau - t) + 2\tau \tau']}, \quad \text{and} \quad \sigma^2 = \frac{8\tau'^2}{3[3(\tau - t) + 2\tau \tau']^2 (\tau - t)^2},
\]

respectively. As we approach the singularity the behavior of these two variables changes and for \( t \to \tau \) their evolution is monitored by the following approximate expressions

\[
\rho \simeq \frac{1}{4\pi \tau \tau'(\tau - t)}, \quad \text{and} \quad \sigma^2 \simeq \frac{2}{9(\tau - t)^2},
\]

indicating that the shear can dominate the final stages of the collapse. Note that, according to (11), we guarantee a positive definite energy density for the matter by demanding that \( \tau' > 0 \) at all times. The latter also offers a sufficient condition for avoiding cross-shell singularities (see 13 for further discussion). Finally, we stress out that by using the null coordinates, with

\[
dR = R'dr + \dot{R}dt \quad \text{and} \quad dv = dt - (R - 1)^{-1}dR,
\]

the line element (3) reads

\[
ds^2 = -(1 - \dot{R}^2)dv^2 + 2vdvR + R^2d\Omega^2.
\]

Then, the surface where the radial velocities of null congruences vanish (i.e. for \( dR/dv = 0 \)) is a spherically symmetric horizon. Therefore, by putting \( dv/dv = dR/dv = d\Omega/dv = 0 \), the horizon is the surface \( \dot{R}^2 = 1 \). As we are looking for collapsing solutions only (i.e. with \( \dot{R} < 0 \)), the horizon will be on the restricted surface \( \dot{R}_H = -1 \) or, more explicitly, on \( t(t_H) = \tau(r_H) - 2\mu r_H^3/9 \).

**IV. MAGNETIZED TOLMAN-BONDI COLLAPSE**

Observations have long established the widespread presence of astrophysical magnetic fields, while compact stellar objects are capable of supporting considerably strong fields. Neutron stars, for example, can carry magnetic fields that reach up to \( 10^{15} \) G and \( 10^{16} \) G. Despite their strength, however, the energy density of these B-fields is much smaller than that of the supporting matter (i.e. \( B^2/\rho \ll 1 \)). One can therefore use its relative weakness to treat the magnetic field as a perturbation on a matter dominated background. Here, we will employ the Bondi-Tolman metric to study the collapse of an inhomogeneous, weakly magnetized dust cloud. We will do so by adopting the familiar MHD approximation, which is supported by the expected very high conductivity of stellar interiors.

According to (11), the volume evolution of a weakly magnetized fluid element within the irrotational Tolman-Bondi spacetime is monitored by the following version of the Raychaudhuri equation

\[
\dot{\Theta} + \frac{1}{4} \Theta^2 = -\frac{1}{8} \dot{\rho} - 2\sigma^2 + D^a u_a + u_a u_a^a,
\]

given that \( p = 0 \) and \( B^2 \ll \rho \). The last pair of terms in the right-hand side of the above is entirely due to the magnetic presence, since \( u_a = \epsilon_{abc} B^b \epsilon \mu r B^c / \rho \) for \( p = 0 \) (see Eq. (9) and also 12). In what follows, we will employ
the Tolman-Bondi metric to evaluate these two terms, while ignoring the magnetic backreaction on the shear and the vorticity of the background model.

At the MHD limit the electric field vanishes and Maxwell’s equations reduce to a set of one propagation and one constraint equation. In covariant form, these are given by the respective expressions

$$h^a_b F^b_a + \Theta B^a = 0$$

and

$$h^a_b \nabla_a B^b = 0,$$

(10)

where $F^a_b = u^b \nabla_a B^a - B^b \nabla_b u^a$ is the Lie derivative of $B^a$ along the 4-velocity of the fluid. Adopting spherical polar coordinates and recalling that $B_a u^a = 0$, we set $B^a = (0, B^r, B^\theta, B^\phi)$ with $B^a = B^a(t, r, \theta, \phi)$ and $\alpha = r, \theta, \phi$. Then, solving (10b) on our Tolman-Bondi background we obtain

$$B^a = \frac{F_a}{(\tau - t) [3(\tau - t) + 2rr']},$$

(11)

with $F_a$ representing time-independent functionals (i.e. $\partial_t F_a = 0$). Similarly, written relative to a spherical polar coordinate system, constraint (10b) translates into

$$\sin \theta (r F_r + 2F_r) + r (\sin \theta \partial_\theta F_\theta + \cos \theta F_\theta) = 0.$$

(12)

As with the matter density and the shear magnitude before, the behavior of the magnetic field changes as one gets closer to the singularity. In particular, for $t \to \tau$, we find that the magnitude of the above given $B$-field evolves as

$$B^2 \approx \frac{1}{4} \frac{\dot{A} F_r^2}{(\tau - t)^{8/3}},$$

(13)

where $B^2 = B_a B^a$ and $\dot{A} = 3 \sqrt{4\mu^2/81}$. This in turn combines with result (14) to provide a measure of the energy-density ratio $B^2/\rho$ near the Tolman-Bondi singularity

$$\frac{B^2}{\rho} \approx \frac{\dot{A} F_r^2 r^r}{(\tau - t)^5/3},$$

(14)

with $A = \pi \dot{A}$. Since the magnetic density grows faster than that of the collapsing matter (compare (13) to Eq. (14)), the above also allows for a rough upper bound on the ratio $B^2/\rho$. It is therefore clear that the weak-field approximation (i.e. $B^2/\rho \ll 1$) will hold as long as

$$\tau - t \gg (\dot{A} F_r^2 r^r)^{3/5},$$

(15)

The $r$-dependence in the right-hand side of this condition implies that the weak-field approximation can be satisfied at any time during the contraction. For instance, for any arbitrary finite value of $\tau - t$, one can always set $0 < r \ll (\tau - t)^{5/3}/\dot{A} F_r^2 r^r$ to ensure that (15) holds (recall that $r^r > 0$). In other words, we are always able to identify a shell, labeled by its radius $r$, where condition (15) is satisfied.

Remaining within the weak-field limit, we will now employ the Tolman-Bondi metric to evaluate the magnetic impact on the contracting spacetime. Focusing on the later stages of the collapse (i.e. allowing $t \to \tau$), the dominant components of last two terms in the right-hand side of the Raychaudhuri equation (see (9)) read

$$D^a u_a \approx -\frac{5}{3} \frac{\dot{A} F_r^2 r^r}{(\tau - t)^{11/3}}$$

and

$$u_a \dot{u}^a \approx \frac{4}{9} \frac{\dot{A} F_r^2 (\partial_\theta F_r)^2 r^2}{(\tau - t)^{14/3}},$$

(16)

respectively. Thus, as long as $F_r \neq 0$ and $\partial_\theta F_r \neq 0$ the former of the above assists the contraction and the latter acts against it. Note that both terms grow as we approach the Tolman-Bondi singularity, compared to the matter density and the shear magnitude given by (14) and (15) respectively. In this case the global spherical symmetry of the collapse will be destroyed by the Lorentz force. Note that, as the supporting magnetic stress (16b) is the fastest growing near the singularity, the overall contribution of the field will tend to resist the collapse and this could cause the converging worldlines to bounce. In addition, when cross-shell singularities are not allowed, halting the gravitational contraction of a shell of proper radius $r = r_0$ means that all shells with $r > r_0$ will also cease collapsing. This is possible while still within the weak-field limit because condition (15) can be satisfied arbitrarily close to the singularity. In other words, although the magnetic energy density associated with a given collapsing shell has negligible contribution to the total energy-momentum tensor, the Lorentz force dictates the symmetries of the collapse. The latter results from the generic inhomogeneity of the Tolman-Bondi spacetime. It is still possible,
however, that nonlinear contributions to the metric, mainly caused by the non-geodesic motion of the matter, could overwhelm the magnetic resistance and push the system into further (anisotropic) contraction.

In what follows we will demonstrate that fields with the aforementioned properties that support against gravitational collapse can be obtained as solutions of Maxwell’s equations on the Tolman-Bondi background. Indeed, imposing the condition $\partial_\theta F_\tau \neq 0$ on the $B$-field and then solving Eqs. (10) we arrive at the functionals

$$F_\tau = -\frac{f}{2} \cos \theta \quad \text{and} \quad F_\theta = \frac{f}{r} \sin \theta,$$

(17)

where $f$ is a constant. Therefore, both components of the associated magnetic field have a clear $\theta$-dependence. Also, despite the fact that $B_\theta$ diverges at $r = 0$, the energy density of the field is perfectly regular there because $B^2 \propto g_{\theta \theta} F_\theta^2$ and $\partial_\theta B_\theta \propto r^2$. Following [17], however, the radial component of the field and its $\theta$-derivative vanish at $\theta = \pi/2$ and at $\theta = 0$, $\pi$ respectively. Hence, along these directions the supporting magnetic effect disappears and the contraction will proceed uninhibited by the presence of the field. This behavior, which is probably typical (see [16]), could be seen as a direct consequence of the generically anisotropic nature of the field. The result is an extremely distorted collapse. In particular, while certain directions will continue collapsing, the gravitational contraction of most of the magnetized particles will face strong resistance by the Lorentz force.

At this point we should also emphasize that the $\theta$-dependence of the radial magnetic component is crucial for the future of the converging worldlines. To be precise, when $\partial_\theta F_\tau = 0$ the last term in Eq. (10) approaches the expression

$$\dot{u}_a \dot{u}^a \propto \frac{1}{4} A F^2_r (F^2_\theta + F^2_\phi \sin^2 \theta) \frac{\sin \theta}{(\tau - t)^{8/3}},$$

(18)

instead of (10b). In this case the later stages of the magnetized collapse are dominated by (10b) and therefore the contraction will proceed unimpeded.

Finally, let us consider the homogeneous limit $\tau' \to 0$, which corresponds to FRW geometry. In this special case the shear contribution to the Raychaudhuri vanishes, while $\rho \propto (\tau - t)^{-2}$ (see Eqs. (5)). Also, for $\tau' = 0$ the right-hand side of both expressions in (10) vanish, which means that one needs to evaluate the magnetic stresses at higher order. Then, the total magnetic effect near the initial singularity is given by the sum

$$\dot{u}^a \dot{u}_a + D^a \dot{u}_a \propto K (\tau - t)^{-8/3},$$

(19)

where $K = K(r, \theta, \phi)$ is an involved function of the $F_\alpha s$ and their derivatives and vanishes for an homogeneous magnetic field. The interesting point is that, as $t \to \tau$, the above stress grows faster than the matter density and therefore it is expected to dominate the final stages of the collapse. In addition the functional $K$ does not have a definite sign and therefore its effect on the collapsing dust depends on the particular magnetic configuration.

V. DISCUSSION

Past studies of magnetized gravitational contraction have indicated that the presence of the field could affect the outcome of the collapse in nontrivial ways. In [4], for example, magnetism was found capable of halting the contraction of a finite cylinder, while in [3] the authors provided a static solution of the Einstein-Maxwell equations with a pressure-free matter component. It has also been shown that the spherically symmetric collapse of a charged star could bounce [7], although the bounce seems to cause naked shell-crossing singularities [8]. More recently, it was argued that magneto-curvature tension stresses could also affect the collapse of a conventional magnetized fluid [14]. All these claims suggest that the Maxwell field could play a key role during the gravitational collapse of a bounded system. Motivated by that we have considered the inhomogeneous contraction of a weakly magnetized Tolman-Bondi spacetime filled with a highly conductive pressureless fluid. The advantage of the Tolman-Bondi metric is that it offers the most general mathematical framework for studying dust collapse. Here, we did so by analyzing the magnetic contribution to the Raychaudhuri equation, which monitors the convergence of the particle worldlines in a covariant manner. We show that the magnetic input splits into two stresses, one of which always supports against contraction. Assuming that the energy density of the field is only a small fraction of the matter density, we have used the Tolman-Bondi metric to evaluate the input of the aforementioned stresses to Raychaudhuri’s formula. Such an approximation was unavoidable given the fully analytic nature of our study. Within this limit, we found that the magnetic presence can severely distort the spherical symmetry of the collapse. Interestingly, we found that the magnetic stress which resists the collapse is the fastest growing one and we identified physically plausible magnetic configurations which allow this to happen. It should be noted, of course, that by ignoring the backreaction effects, mainly those due to the non-geodesic motion of the magnetized fluid, we have limited the range of our results.
Nevertheless, the tendency of the Lorentz force to dominate the collapse, even when the energy level of the field is relatively low, should not depend on the approximation level. This argument is supported by earlier studies, showing that spherically symmetric charged collapse produces naked singularities \cite{8}. Thus, given that all known stars support magnetic fields of various strengths, we believe that these fields can play a protagonist’s role in the evolution of such gravitationally bound systems.

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\item Assuming a solution of (11) with $\partial_\theta F_\theta = 0$, we find that $F_\theta = [\alpha(r, \phi) + \beta(r, \phi) \cos \theta] / \sin \theta$. However the latter is singular at $\theta = 0$, $\pi$, which means that $\partial_\theta F_\theta$ cannot vanish identically. Nevertheless, $F_\varphi$ will vanish for some $\theta \neq 0$. In fact, there are always some $\theta$s such that $\sin \theta \partial_\theta F_\theta + \cos \theta F_\theta = 0$, independently on $r$. For these $\theta$s the only regular solution (i.e. non proportional to $1/r^2$ and therefore non-divergent) of (12) is $F_\varphi = 0$. Clearly, this proof does not apply when $\partial_\theta F_\theta = 0$. In that case, however, the solution has $F_\varphi \propto 1/\tan \theta$, which diverges at $\theta = \pi/2$, and it is therefore discarded.
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