Fidelities for transformations of unknown quantum states

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We present a general theoretical formalism to compute the fidelity of transformations of unknown quantum states, and we apply our theory to Gaussian transformations of continuous variable quantum systems. For the case of a Gaussian distribution of displaced coherent states, the theory is readily tractable by a covariance matrix formalism, and a wider class of states, exemplified by Fock states, can be treated efficiently by the Wigner function formalism. Given the distribution of input states, the optimum feed back gain is identified, and analytical results for the fidelities are presented for recently implemented teleportation and memory storage protocols for continuous variables.

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I. INTRODUCTION

In a generic scenario for quantum state transformations, a protocol is applied to an unknown input quantum state taken from a family of states with a certain probability distribution. The quality of the protocol is quantified by a fidelity measure, which in a natural sense extracts the average overlap between the state obtained through use of the protocol and the state expected under ideal circumstances.

In laboratory experiments, one may delegate the handling of the initial random state preparation and the final examination of the output to an independent person or device, Victor, (and appeal that Victor is not leaking information to the other experimenters). Theoretical physics works differently in the sense that what is specified at one point in the theory is specified throughout, and quantum physics adds the further aspect that the quantum state is a state describing our knowledge about the system, influenced by any knowledge that comes to our mind be it in the form of measurement outcomes or information about the preparation procedure for the system - we cannot know something about a system and at the same time describe it by a state vector or density operator that is independent of this knowledge.

It is the purpose of this paper to present a practically useful theory to determine fidelities without leaving doubts about the correct handling of what is known and what is not known about the input states. In Sec. II, we introduce a theoretical formalism which represents Victor both at the stage of preparation of an unknown input state and at the examination of the output. In Sec. III, we consider the case of continuous variable quantum systems, and we show that a hybrid quantum and classical Wigner function can be applied in an explicit calculation of the fidelity. In Sec. IV, we consider quantum state teleportation by the Braunstein and Kimble protocol\textsuperscript{1}, and we identify the optimum operation given the input state distribution, and we obtain analytical results for the teleportation fidelity. In Sec. V, we consider a recently implemented quantum memory protocol\textsuperscript{2}, and we also here identify the optimum operation and fidelities. In Sec. VI, we consider teleportation of non-Gaussian states as exemplified by a distribution of Fock states. Section VII concludes the paper.

II. GENERAL TRANSFORMATION OF AN UNKNOWN QUANTUM STATE

Consider a family of states $\{\ket{\Psi^{in}(\lambda)}\}$, parameterized by a stochastic variable $\lambda$ and used as input to a certain quantum protocol according to the probability distribution $P(\lambda)$. For convenience, we assume the different states be obtained from one reference state $\ket{\Psi_0}$ by means of a family of unitary operators, $\ket{\Psi^{in}(\lambda)} = U_\lambda \ket{\Psi_0}$. The randomness of $\lambda$ is now handled by introducing an auxiliary (fictitious) physical system with no free evolution and with an initial mixed state, $\sum_\lambda P(\lambda) \ket{\lambda}\bra{\lambda}$, where we assume that the quantum states $\ket{\lambda}$ are orthonormal. In our theoretical modelling, we let the auxiliary system interact in a Quantum-Non-Demolition (QND) manner with our physical system prepared in $\ket{\Psi_0}$, $H = \sum_\lambda \ket{\lambda}\bra{\lambda} \otimes H_\lambda$, which we assume will lead, after a suitable interaction time, to the following correlated state,

$$\rho = \sum_\lambda P(\lambda) \ket{\lambda}\bra{\lambda} \otimes \ket{\Psi^{in}(\lambda)}\bra{\Psi^{in}(\lambda)}.$$  

(1)

The partial trace over the auxiliary $\lambda$-degrees of freedom produces a density operator $\sum_\lambda P(\lambda) \ket{\Psi^{in}(\lambda)}\bra{\Psi^{in}(\lambda)}$ describing the mixture of input states to the protocol with the appropriate probabilities, but we note that the fidelity of a protocol is not a measure of how well a mixed state is transformed into its image by the ideal protocol, but a measure of how well each member of the mixture transforms. It should also be noted that Eq.\textsuperscript{11} identifies specific individual pure state components in the input, whereas the reduced density matrix $\sum_\lambda P(\lambda) \ket{\Psi^{in}(\lambda)}\bra{\Psi^{in}(\lambda)}$ does not have a unique unravelling in terms of pure states. In\textsuperscript{11} we have retained the variable $\lambda$ in the auxiliary system,
which enables us to study the fidelity at the pure state level, without selecting a specific pure state input to the protocol.

A physical transformation of a quantum state must be completely positive and preserve the normalization of the density operator, and it can be written most generally in the Kraus form, \( \rho \rightarrow \sum_s E_s \rho E_s^\dagger \), where the \( E_s \) operators can be any set of operators that fulfills \( \sum_s E_s^\dagger E_s = 1 \). Important examples include (i) unitary evolution, where there is only one unitary operator, \( E_s = U \), (ii) open system dynamics following a Lindblad form master equation with jump and no-jump operators \( E_s \), (iii) von Neumann measurements of a hermitian system observable with orthogonal projections \( E_s \), and (iv) more general measurement positive operator valued measure (POVM) scenarios.

Without loss of generality we assume that the desired protocol takes our quantum state to a final state on a similar Hilbert space (same dimensionality), so that each input state \( |\Psi_{in}(\lambda)\rangle \) is ideally transformed by a unitary operation \( V \) into \( |\Psi_{out}(\lambda)\rangle = V U_\lambda |\Psi_0\rangle \). We assume that physical interaction and measurements take place only on our quantum system of interest and on possibly added quantum systems. The variable \( \lambda \) is not made subject to interaction or direct observation, and we hence write the state after application of the protocol, tracing over possible further unobserved quantum degrees of freedom as

\[
\rho = \sum_{\lambda} P(\lambda) |\lambda\rangle \langle \lambda| \otimes \left( \sum_s E_s |\Psi_{in}(\lambda)\rangle \langle \Psi_{in}(\lambda)| E_s^\dagger \right). \tag{2}
\]

The sum over \( E_s \) terms may represent the result of measurements on the system and it may also incorporate a unitary feedback \( U_{feedback} \) on the system, conditioned on the outcome \( s \) of the measurement. \((E_s \rightarrow U_{feedback} E_s \) also fulfil the required property \( \sum_s (U_{feedback} E_s)^\dagger U_{feedback} E_s = 1 \) of the Kraus form).

Equation (2) is very illustrative. The sum over the different Kraus operators corresponds to averaging over the outcomes of measurements and potential feedback on the system. It shows how each input component transforms into a mixed state \( \rho_{out}(\lambda) = \sum_s E_s |\Psi_{in}(\lambda)\rangle \langle \Psi_{in}(\lambda)| E_s^\dagger \). If we accept the output for all such measurement results, we must carry out this average, and we use the state in Eq. (2) to compute the fidelity of the protocol. To check if \( \rho_{out}(\lambda) \) equals the desired state \( |\Psi_{out}(\lambda)\rangle = V U_\lambda |\Psi_{in}(\lambda)\rangle \) we apply the unitary \( \sum_s |\lambda\rangle \langle \lambda| \otimes (V U_\lambda)^{-1} \) on (2), and check if the quantum system is now in the initial reference state \( |\Psi_0\rangle \). Rather than verifying that the output equals the ideally transformed input state, we check if the inverse of the transform on the output agrees with the fixed reference input state. Ideally, this agreement should be obtained for all \( \lambda \)-components of the system and we thus perform the partial trace over the \( \lambda \)-degrees of freedom and compare the final state density matrix with the pure state \( |\Psi_0\rangle \):

\[
F = \sum_{\lambda} P(\lambda) \langle \Psi_0| U_\lambda^\dagger V^\dagger \rho_{out}(\lambda) V U_\lambda |\Psi_0\rangle. \tag{3}
\]

In this scheme we compute the average value of the overlap without having to specify which initial state is applied.

We observe that this result can also be written

\[
F = \sum_{\lambda} P(\lambda) \langle \Psi_{out}(\lambda)| \rho_{out}(\lambda) |\Psi_{out}(\lambda)\rangle, \tag{4}
\]

where \( |\Psi_{out}(\lambda)\rangle = V |\Psi_{in}(\lambda)\rangle \) is the desired output state, and despite our concerns in the Introduction about a consistent treatment of unknown input states, the fidelity is simply the fidelities obtained for each input state averaged over the input state distribution. A measurement part of the protocol may yield some information about the \( \lambda \)-variable and hence change the probability distribution \( P(\lambda) \), but when we average over the outcomes \( s \), we return to the original distribution. The fact that a feedback may be applied to the system conditioned on the measurement affects only the fidelity through the form of the \( E_s \) operators. We note that Eq. (4) could give the erroneous impression that the \( P(\lambda) \) distribution only enters via the explicit weighted sum. This is not the case; as we shall see below, to yield the highest possible fidelity the optimum feedback, i.e., the operators \( U_{feedback}^\dagger \) should be chosen in a manner that depends on the distribution \( P(\lambda) \).

If the transformation can be applied with a non-unit success probability, i.e., if the output state is only accepted conditioned on a specific outcome \( \{ s' \} \) of the measurement on the system, we must go back to the joint state \( |\Psi_{in}(\lambda)\rangle \otimes (V U_\lambda)^{-1} \), and multiply the same probabilities together. The trace of the un-normalized state is precisely the probability of acceptance. The resulting state is now a weighted sum of density operator terms with non-unit trace \( q_\lambda = Tr(\sum_s E_s^\dagger E_s |\Psi_{in}(\lambda)\rangle \langle \Psi_{in}(\lambda)| E_s^\dagger) \). We can renormalize the density operators with \( 1/q_\lambda \) and multiply the same \( q_\lambda \) factors on \( P(\lambda) \) which represents then the updated probability distribution of the input states conditioned on the measurement result. Our fidelity calculation proceeds as above with the inverse transformations and the final comparison with the initial reference state, and in this case, the fidelity is again given by the state-to-state transformation fidelities but now weighted by both their initial state probability and their individual probabilities \( q_\lambda \) for the acceptable measurement outcome.

### III. APPLICATION TO CONTINUOUS VARIABLE SYSTEMS

Quantum information protocols with continuous variable systems have been the focus of intense research since
it was suggested and demonstrated that existing squeezed light sources, beam splitters and photodetectors suffice to enable quantum state teleportation of light. The collective atomic population of different internal states in a macroscopic gas sample also provides effectively continuous degrees of freedom, and efficient atomic entanglement protocols that make use of classical light sources and photodetection only were proposed and demonstrated. The work on entangled atomic gases was followed by theoretical and experimental work on quantum state transfer between light and matter (a quantum memory for light and ideas for atomic state teleportation are currently being pursued).

These continuous variable systems can be described in terms of canonically conjugate harmonic oscillator variables $x$ and $p$, and states can be described in terms of Wigner phase space distribution functions in place of the general density matrix notation of the previous section. We consider the case where the ensemble of input states is obtained by displacements of the reference state (the vacuum state in Secs. IV and V) by arguments $x_{cl}$ and $p_{cl}$ according to a probability distribution $P(\lambda = (x_{cl}, p_{cl}))$. Such a displacement of a Wigner function simply amounts to a translation of its argument $W(x,p) \rightarrow W(x-x_{cl},p-p_{cl})$, but as in Sec. II we shall introduce an auxiliary set of QND variables in the modeling of the input state ensemble. We thus treat the real arguments $x_{cl}$ and $p_{cl}$ as two independent variables, e.g., momenta for free particles, or simply as classical variables in a quantum-classical hybrid Wigner function for the total system, which is consistent with Heisenberg's uncertainty relation for the quantum degrees of freedom, but has no such constraints on the classical degrees of freedom. If we take the zero amplitude coherent state with a Wigner function $W_{0}(x,p)$ and displace it by the classical arguments $x_{cl}$ and $p_{cl}$ according to a classical probability distribution $P(x_{cl},p_{cl})$, the joint Wigner function of the quantum and classical variables become in analogy with (1),

$$W_{in}(x,p,x_{cl},p_{cl}) = W_{0}(x-x_{cl},p-p_{cl})P(x_{cl},p_{cl}). \tag{5}$$

Some quantum information protocols make use of additional quantum systems and we shall hence work with a multi-variable Wigner function for all the quantum systems and classical variables involved in the protocol. In teleportation, for example, the communication channel is described by a joint Wigner function of the entangled state of two quantum systems $W_{in}(x_{1},p_{1},x_{2},p_{2})$. The total Wigner function is thus a function of 8 variables $W(x,p,x_{cl},p_{cl},x_{1},p_{1},x_{2},p_{2})$. The linear mode mixing transformations of the teleportation protocol amount to the application of linear transformations on the variables within the original distribution function; measurements of a given quantum variable amounts to evaluating the function with the corresponding argument attaining the measured value (and integrating over the canonical conjugate variable which is accordingly completely undetermined), and finally a joint distribution of the output quantum state and the classical variables is obtained. The verification of the protocol consists in comparing the output state with the desired one (which for teleportation is the same as the input state) and this is done by displacing the quantum system with the negative of the classical variables (inverse of Eq. (5)), integrating over the unknown classical variables and comparing the ensuing quantum state with the reference state $W_{0}(x,p)$.

In the most general case one has to deal with a multi-variable function, and one has to carry out integrals of this function with respect to a number of the variables. We shall now turn to examples where the initial quantum states and the classical distribution function are all Gaussian. This situation is of practical relevance in the above mentioned experiments and it offers a significant simplification of the problem. Gaussian states are fully characterized by the mean values and the covariance matrix for the variables, and quantum state overlap integrals are given explicitly by these quantities. The present approach to the fidelity problem, involving joint probabilities for classical and quantum variables, is related to recent applications of the general theory of parameter estimation and Kalman filtering with quantum systems and, in particular, to our recent application of this theory to atomic magnetometry. The formal treatment of the fidelity issue is actually simpler than the magnetometry analysis, and we now present the details of such a calculation.

### IV. TELEPORTATION OF AN UNKNOWN COHERENT STATE

We treat the case of teleportation of a physical system 3 by use of an entangled pair of systems, 1 and 2. It has been argued, that a general positive map can be viewed as "teleporting a state through a gate" and, hence this operation has both specific and more general interest.

We note that for coherent states with an amplitude of a given absolute value but with a random choice of complex phases, Ide et al. studied how to optimize the teleportation fidelity by a proper choice of the strength of the feedback on the output fields. Furasek has applied the covariance matrix formalism, similar in spirit to our work, but rather than optimizing he assumed a fixed value for the feedback strength, and then he turned to a study of the effect of further local Gaussian operations.

#### A. Covariance matrix method

We shall be working with Gaussian states, and hence the state is fully characterized by the mean values $m_{i} = \langle y_{i} \rangle$ and the covariance matrix $\gamma_{ij} = 2 \text{Re}(\langle (y_{i} - \langle y_{i} \rangle)(y_{j} - \langle y_{j} \rangle) \rangle)$ of all quadrature variables $y_{i}$. For a more detailed
description of the covariance matrix formalism and its practical implementation of linear transformations and measurement processes, see, e.g., [14].

Following [1], we use the entanglement in the 12-system to teleport an unknown coherent state of system 3 drawn from an ensemble of states with a Gaussian distribution of the mean amplitude onto system 1 by performing displacements of system 1 conditioned on the output of joint measurements on systems 2 and 3. We introduce the auxiliary classical variables \((x_{cl}, p_{cl})\) with zero mean and variance given by \(v_c = 2\text{Var}(x_{cl}) = 2\text{Var}(p_{cl})\). The classical variables represent an agent Victor who’s role is to transform the system to the following eight variables via a displacement. Experimentally, one is interested in the case with sizable \(v_c\) (truly unknown input states), but we shall obtain general expressions for arbitrary \(v_c\). Note that since \(v_c\) describes classical variables, it is not limited by the Heisenberg uncertainty relation, and \(v_c = 0\) corresponds to the case, where the input is the vacuum state with certainty. The covariance matrix of system 3 and Victors classical variables prior to the displacement is \(\gamma_{3, V} = \text{diag}(1, 1, v_c, v_c)\), and the displacement leads to the transformation \(\gamma_{3, V} \rightarrow \gamma_{3, V}' = S_{d}^{\gamma_{3, V}} S_{d}^T\), where the matrix \(S_d\) describes the linear mapping \(x_3 \rightarrow x_{3} + x_{cl}, p_3 \rightarrow p_3 + p_{cl}, x_{cl} \rightarrow x_{cl}, p_{cl} \rightarrow p_{cl}\). The mean values are also formally transformed by this mapping, but since the classical distribution and the quantum state are assumed to have vanishing mean values initially, this is also the case after the action of \(S_d\).

After this preparation of a quantum input state correlated with classical stochastic variables as described by \(\gamma_{3, V}'\), we obtain the complete \(8 \times 8\) covariance matrix \(\gamma = \text{blockdiag}(\gamma_{12}, \gamma_{3, V}')\), with \(\gamma_{12}\) the covariance matrix for the initially entangled 12 system. The commuting pair of non-local variables \(x_{23}^{(23)} = (x_2 - x_3)/\sqrt{2}\), \(p_{23}^{(23)} = \frac{p_2 + p_3}{\sqrt{2}}\) is measured. It is useful to transform the system to the following eight variables: \((x_1, p_1, x_{23}^{(23)}, p_{23}^{(23)}, x_{2}, p_{2}, x_{cl}, p_{cl})\) with the covariance matrix \(\gamma \rightarrow \gamma' = T\gamma T^T\), where the block diagonal matrix \(T = \text{blockdiag}(I_2, T^{(23)}, I_2)\) is the \(2 \times 2\) identity matrix (system 1 and the classical displacements are not affected by this transformation), and \(T^{(23)}\) the matrix effecting the change of basis from system 2 and 3 variables to the joint variables \(x_{23}^{(23)}, p_{23}^{(23)}\). A measurement of a single variable from a joint Gaussian distribution results in an updated Gaussian distribution for the remaining unknown variables. This update is readily accounted for in terms of the mean values and the covariance matrix of the variables. First, we reorganize the variables in the order \((x_1, p_1, x_{cl}, p_{cl}, x_{2}, p_{2}, x_{23}, p_{23})\), so that the covariance matrix \(\gamma'\) is decomposed into \(4 \times 4\) dimensional matrix blocks:

\[
\gamma' = \begin{pmatrix}
A & C \\
C^T & B
\end{pmatrix},
\]

where \(A\) is the covariance matrix for the unmeasured quantum and classical components, \(B\) is the covariance matrix for the measured variables, and \(C\) and \(C^T\) describe the correlations between the measured and unmeasured variables. The effect of the measurement on a subsystem on the covariance matrix of the remaining, unmeasured variables is given by the update formula [14, 12]

\[
A \rightarrow A' = A - C(\pi B \pi)^{-1}C^T,
\]

where \(\pi = \text{diag}(0, 1, 1, 0)\) with unity at the entrances of \(p_{23}^{(23)}, x_{23}^{(23)}\) and \((\ldots)^-\) denotes the Moore-Penrose pseudoinverse.

The measurement outcome affects the mean values: conditioned on a positive readout \(\xi_{23}\) in the measurement of \(x_{23}\) our knowledge about \(x_3\) and hence of \(x_{cl}\) is biased towards negative values and \(x_2\) is biased towards positive values (and hence \(x_1\) is biased towards positive values). Precisely how much, is determined by the variables initially ascribed to these variables, and we have the following formula [14, 12] for the vector of mean values, \(m = (x_1, p_1, x_{cl}, p_{cl})^T\):

\[
m \rightarrow C(\pi B \pi)^{-1}(\cdot, \xi_{23}, \xi_{23}, \cdot)^T,
\]

where the dots replace unmeasured quantities, which do not need to be specified due to the zeros in the projector \(\pi\), and \(\eta_{23}\) denote the random outcome of the measurements of \(p_{23}^{(23)}\) and \(x_{23}^{(23)}\). In Sec IV B, we shall give the expressions for the change of mean values conditioned on the random measurement outcome. It is optimal to apply a feedback that brings the mean value of the quantum variables in exact agreement with the mean value of the classical displacement terms. This is so, because the covariance matrix does not depend on the measurement outcome, hence at the end of the calculation we shall compare the vacuum Gaussian state with either a single Gaussian state with vanishing mean or with a distribution of Gaussian states with the same covariance matrix but with different displacements with respect to the desired state.

The feedback, just described is part of the teleportation protocol. As part of our verification or quality assessment of the protocol, we displace the final quantum state by the negative of the classical parameters and compare the outcome, known to have vanishing mean amplitudes, with the vacuum state. Correspondingly, we apply the inverse of the classical displacement \(S_{d}^{-1}\) on the quantum and classical variables \(x_1, p_1, x_{cl}, p_{cl}\), and obtain their resulting covariance matrix:

\[
V = S_{d}^{-1} A'(S_{d}^{-1})^T.
\]

The \(2 \times 2\) block \(\gamma_{out} = V (1 : 2, 1 : 2)\) describing the covariances of the quantum variables is the covariance matrix for the quantum system, when the unknown classical displacements are integrated out, and it should ideally be the identity matrix describing the vacuum state.

The fidelity of the protocol is the overlap of the Wigner functions. For a single mode state the Wigner function is given by \(W = \frac{1}{(\pi \sqrt{\det \gamma})} \exp(-\chi^T \gamma^{-1} \chi)\) with \(\chi^T = (x, p)\). In terms of Wigner functions, the average fidelity is defined as \(F = 2\pi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp W_{in}(x, p) W_{out}(x, p)\).
The vacuum state is Gaussian with a covariance matrix equal to the identity $I_2$, and the integrand is thus a Gaussian function $\propto \exp(-\chi^T \gamma^{-1}_{\text{res}} \chi)$ with $\gamma_{\text{res}} = (\gamma^{-1}_{\text{out}} + I_2)^{-1}$ so that the integral follows directly from the standard expressions for Gaussian normalization integrals,

$$F = 2 \sqrt{\frac{\det(\gamma_{\text{res}})}{\det(\gamma_{\text{out}})}}$$  \hspace{1cm} (9)

**B. Results**

For simplicity we consider the symmetric case where the joint covariance matrix of the variables $(x_1, p_1, x_2, p_2)$ for systems 1 and 2 is given by

$$\gamma_{12} = \begin{pmatrix}
  n & 0 & k & 0 \\
  0 & n & 0 & 0 \\
  k & 0 & n & 0 \\
  0 & -k & 0 & n \\
\end{pmatrix},$$  \hspace{1cm} (10)

where $n$ describes twice the variance of the variables of system 1 and 2, and where $k$ describe the correlations between the systems. The collective variables $(x_1, x_2)$ and $(p_1, p_2)$ have the variances $(n \pm k)$, and the Heisenberg uncertainty relation implies that $n^2 - k^2 \geq 1$. Realizations of such a bipartite entangled state include the atom-light setup $[5]$ and the EPR-light source channel $[3]$. The matrix operations just described are straightforward, and we readily obtain analytical results at all steps of the calculation. The measurement process yields random outcomes $\xi_{23}, \eta_{23}$, and inserting the initial covariance matrices described above and carrying out the matrix operations, we obtain the conditioned mean values $\langle x_1 \rangle = \frac{k + v_c}{1 + n + v_c} (\sqrt{2} \xi_{23})$ and $\langle x_{cl} \rangle = \frac{-v_c}{1 + n + v_c} (\sqrt{2} \xi_{23})$, and similar expression for $p_1, p_{cl}$ in terms of the measured quantity $\eta_{23}$. At this point in the teleportation protocol, the aim is to have a state with $\langle x_1 \rangle = \langle x_{cl} \rangle, \langle p_1 \rangle = \langle p_{cl} \rangle$, and this is obtained by applying a feedback on system 1, in form of a displacement for both the $x_1$ and $p_1$ variables:

$$x_1 \rightarrow x_1 - \frac{k + v_c}{1 + n + v_c} (\sqrt{2} \xi_{23}),$$

$$p_1 \rightarrow p_1 + \frac{k + v_c}{1 + n + v_c} (\sqrt{2} \eta_{23}).$$  \hspace{1cm} (11)

In the limit of infinite $v_c$ the feedbacks (11) are $\sqrt{2}$ times the measured values themselves, but for states chosen from a finite width distribution, we see that a non-trivial gain factor

$$g = \frac{k + v_c}{1 + n + v_c}$$  \hspace{1cm} (12)

should be applied in the feedback.

The resulting explicit expression for the fidelity reads

$$F = \frac{2 (1 + n + v_c)}{(1 + 2n + n^2 - k^2 + 2v_c (1 + n - k))}.$$  \hspace{1cm} (13)

If the input state is the vacuum state with certainty, $v_c = 0$, according to (12), the optimum feedback gain is $g = k/(1 + n)$, and we note that for $n^2 - k^2 = 1$, which characterizes a pure two-mode squeezed state, system 1 is restored in the vacuum state with unit fidelity. In the opposite limit $v_c \rightarrow \infty$, $F$ simplifies to the result

$$F = \frac{1}{1 + \Delta},$$  \hspace{1cm} (14)

where $\Delta = n - k = \text{Var}(x_1 - x_2) = \text{Var}(p_1 + p_2)$ is also known as the EPR variance of systems 1 and 2 $[16]$. The fidelity approaches unity when this variance approaches zero corresponding to a maximally entangled channel. Finally, if the quantum channel is in the vacuum state with $n = 1$ and $k = 0$, the optimum gain is $g = v_c/(1 + n + v_c)$, and our general relation (13) reduces to $F = (2 + v_c)/(2 + 2v_c) = (1 + \lambda)/(2 + \lambda)$ where $\lambda = 2/v_c$, which is exactly the best result that can be obtained with a classical strategy $[17]$.

**V. QUANTUM STATE STORAGE**

Let us now turn to another example: an atomic quantum memory, as demonstrated in a recent experiment $[2]$. In this protocol the aim is to store the quantum state of a light pulse in the collective spin degrees of freedom of a spin polarized atomic sample. The transverse quantum degrees of freedom of the collective spin can be effectively described by canonical conjugate variables (their commutator, the polarized spin component, can be treated as a constant). In the protocol investigated in $[2]$, the optical Faraday rotation provides the light-atom interaction, described by a bilinear interaction Hamiltonian $\propto p_L p_A$. In the Heisenberg interaction picture this causes a change in the conjugate variables

$$x_A \rightarrow x_A + \kappa p_L,$$

$$x_L \rightarrow x_L + \kappa p_A,$$  \hspace{1cm} (15)

where $\kappa$ is the dimensionless integrated interaction strength $[2]$. The interaction thus encodes the field variable $p_L$ onto the atomic $x_A$, and by subsequently detecting the $x_L$ component of the field and displacing $p_A$ according to the measurement result, also this field component is read onto the atoms.

A theoretical analysis of the fidelity of this approach, applied to an unknown coherent state of light taken from a Gaussian distribution of coherent state amplitudes follows the above discussion of teleportation. We introduce classical variables $x_{cl}, p_{cl}$ with variance parameter $v_c$ and zero mean and quantum variables for the atoms and light in zero mean field coherent initial states, so that the Wigner function is a function of six variables, and the covariance matrix is $6 \times 6$. We apply the linear transformation between the field variables and classical variables to initialize the ensemble, and we apply the time evolution due to the atom-light interaction $[15]$. These operations
We wish to encode a feedback on the atomic field component $x_L$ leads to an output value $\xi$, and conditioned on this output, we obtain the mean values, $\langle p_A \rangle = \kappa / (1 + \kappa^2 + v_c) \xi$ and $\langle x_A \rangle = v_c / (1 + \kappa^2 + v_c) \xi$. We wish to encode $-x_L$ in $p_A$, and shall hence apply a feedback on the atomic $p_A$ variable $p_A \rightarrow p_A - g \xi$ with the non-trivial (optimal) gain factor

$$g = (\kappa + v_c) / (1 + \kappa^2 + v_c)$$ \hspace{1cm} (16)

As for teleportation, in the $v_c \rightarrow \infty$ limit the optimum feedback gain is unity, but for finite width distributions it depends explicitly on the variance $v_c$. The state stored is now guaranteed to have the same mean amplitudes as the classical variables, and to check if we managed to store $p_A$ in $x_A$ and $-x_L$ in $p_A$, we follow the procedure from above and displace $x_A$ by $-p_A$ and $p_A$ by $x_A$, and compare the resulting Gaussian state covariance matrix with the vacuum state as in Eq. (10).

The resulting fidelity is a function of the variance of the classical variables and the coupling strength $\kappa$:

$$F = 2 \sqrt{\frac{1 + \kappa^2 + v_c}{(1 + v_c \kappa^2 - 2 v_c \kappa + v_c + \kappa^2 + 2 v_c + 1)(1 + v_c \kappa^2 - 2 v_c \kappa + v_c + \kappa^2 + 1)}}.$$ \hspace{1cm} (17)

This general expression for the storage fidelity has several interesting limits. First, we observe that for a completely unknown initial state with $v_c \rightarrow \infty$, the fidelity vanishes unless $\kappa = 1$, in which case one gets the value $F = \sqrt{2/3} \sim 0.8165$, reported in the literature \cite{2}. In the opposite limit of a known vacuum input, the choice $\kappa = 1$ yields $F = 2 \sqrt{2/3} \sim 0.9428$ for the storage fidelity. The optimum strategy for finite $v_c$, however, is to adjust the value of $\kappa$ so as to maximize the fidelity, and this leads to unit storage fidelity for $\kappa = 0, v_c = 0$.

In the storage protocol, the field variables are mapped onto the atomic ones, but part of the initial atomic noise in the $x_A$ variable remains in the atomic system whereas the feedback manages to cancel the $p_A$ component exactly. It has therefore been suggested to use an initially squeezed atomic state. This is readily analyzed in our description. We simply take the values $(1/r, r)$ with $r$ a squeezing parameter larger than unity for the initial diagonal elements of the atomic covariance matrix in the $(x_A, p_A)$ basis and go through all of the above steps again. In this case, we find the optimum feedback gain factor

$$g = (\kappa r + v_c) / (1 + \kappa^2 r + v_c)$$ \hspace{1cm} (18)

The state stored is again guaranteed to have the same mean amplitudes as the classical variables, and the fidelity of the memory storage is a function of the initial atomic squeezing, the variance of the classical variables and the coupling strength $\kappa$:

$$F = 2 \sqrt{\frac{r(1 + \kappa^2 r + v_c)}{(r + rv_c \kappa^2 - 2 rv_c \kappa + rv_c + \kappa^2 r + 2 rv_c + 1)(r + rv_c \kappa^2 - 2 rv_c \kappa + rv_c + \kappa^2 r + 1)}}.$$ \hspace{1cm} (19)

\section{VI. Transformation of Non-Gaussian States}

For quantum computing it is necessary to be able to store logical states $0$ and $1$ and their superposition states, and one may not restrict an analysis to Gaussian states only, but still the above mentioned protocols may be useful. A Gaussian entangled state may be used to teleport also non-Gaussian states \cite{13}, and a quantum memory protocol which transforms Gaussian states into Gaussian states also applies to qubit states, encoded in a two-dimensional subspace of the continuous variable Hilbert space.
space. A wide class of non-Gaussian states can be obtained by application of non-Gaussian operations on Gaussian states, e.g., photon counting on squeezed states. The corresponding non-Gaussian Wigner functions can, in turn, be expressed in terms of simple mathematical operations on Gaussian functions. This implies that results explicitly derived for Gaussian states can be used as generating functions for quantities of relevance also for non-Gaussian states. Closer to the spirit of the present paper, we shall, however, give a few examples, where we explicitly apply the method of Sec. III, i.e., we assume that the Wigner functions are known for the input states to the protocol, and we carry transformations on the joint Wigner function of the entire physical system.

We shall focus on teleportation by use of a two-mode squeezed state, and we shall apply the same protocol as above, but the input states will be number states and randomly displaced number states.

The entanglement channel of modes 1 and 2 is described by a covariance matrix \( \gamma_{12} \), and hence by the two-mode Wigner function,

\[
W_{12}(x_1,p_1,x_2,p_2) = \frac{1}{\pi \sqrt{\det(\gamma_{12})}} \exp(-\chi^T \gamma_{12}^{-1} \chi),
\]

where \( \chi^T = (x_1,p_1,x_2,p_2) \). We assume the covariance matrix in Eq. (10), with the allowed values of the parameters \( n \) and \( k \), listed in Sec. IV.B.

The \( N = 1 \) Fock state input Wigner functions is given by

\[
W_{N=1}(x_3,p_3) = \frac{1}{\pi} (2x_3^2 + 2p_3^2 - 1) \exp(-x_3^2 - p_3^2).
\]

The Wigner functions for the Fock states are all products of a polynomial in the arguments and a Gaussian. The beam splitter operations, the evaluation at the arguments measured, and the integration over conjugate and unmeasured variables, specified for the teleportation protocol in Sec. III, all preserve this mathematical form of the Wigner function, and it is hence possible to obtain analytically the outcome of the protocol and its fidelity. We shall now summarize the results of this analysis.

We have evaluated the output state for different degrees of entanglement of the teleportation channel. For \( N = 0 \) we reproduce the results of Sec. IV.B with \( v_e = 0 \), and for higher \( N \) we compare our results with Ref. [18]. Our parameters \( n \) and \( k \) are equivalent to \( c \) and \( s \) in Ref. [18], and in the expression for the fidelity, our parameter \( \Delta = n - k \) is equivalent to \( t/2 \) in the notation of [18]. (The expression \( t = 2/(c + s) \) in [18] only applies for the pure state case, and should in the general case be replaced by \( 2(c - s) \) for the ensuing results to be correct). With these modifications, we reproduce the expression for the fidelities in [18], and in particular the result

\[
F_{N=1} = \frac{1 + \Delta^2}{(1 + \Delta)^3}.
\]

The analysis in [18] assumes a unit feedback. Applying instead a variable feedback gain \( g \) as in the previous sections, we are able to optimize the teleportation protocols also for non-Gaussian states: For the \( N = 1 \) Fock state, we find that for strong entanglement (small \( \Delta \)) unit gain is favored, but for weaker entanglement, it is advantageous to reduce the gain factor continuously to the value \( g = 1/\sqrt{2} \) for \( n = 1 \) and \( k = 0 \). This is summarized by the numbers

\[
\{(n, F_{N=1,g=1}, F_{N=1,g=opt})\} = \{(8, 0.8364, 0.8524), (4, 0.7098, 0.7346),
\]

\[
(2, 0.5258, 0.5602), (1, 1/4 = 0.25, 8/27 = 0.2963)\}
\]

for pure state channels with \( n^2 = k^2 + 1 \).

Teleportation of a known quantum state can in principle be replaced by a local production of the given state with much higher fidelity. Let us therefore proceed with teleportation of unknown states, and let us begin with the teleportation of a Fock state, taken from an exponential distribution of Fock states, \( p_N \propto \exp(-N/N) \), with mean value \( \langle N \rangle = N \). This distribution can also be written \( p_N = (1 - \lambda)\lambda^N \) with \( \lambda = \exp(-1/N) \). The average fidelity, according to our Eq. (4) is the mean value of the fidelities weighted with the probability distribution for the input states. In [18], the Fock state teleportation fidelities are derived from a generating function, which is, apart from a factor, precisely this mean value. We therefore readily obtain the mean fidelity as function of the channel EPR variance and the parameter \( \lambda \)

\[
F(\lambda, \Delta) = \frac{1 - \lambda}{\sqrt{(1 + \Delta)^2 - 2\lambda(1 + \Delta^2) + \lambda^2(1 - \Delta)^2}}.
\]

The exponential distribution of \( N \)-values is equivalent to a Gaussian distribution of amplitudes, hence the ensemble of Fock states has the same density matrix as the ensemble of displaced vacuum states treated in Sec. IV, but since we are dealing with the state-to-state teleportation fidelity, the results are very different. In particular, we found in (14), that when the variance of the distribution of coherent input states diverges, the fidelity approaches \( F = 1/(1 + \Delta) \), whereas [18] vanishes for fixed \( \Delta \) and \( N = (v_e - 1)/2 \rightarrow \infty \). It requires a very strongly entangled channel, \( \Delta < 1/N \), to reliably handle the difference between highly excited Fock states.

In contrast, we have also implemented the scenario in which the input to the teleportation channel is the \( N = 1 \) Fock state displaced by an unknown amount, similar to the displacements of the vacuum \( (N = 0) \) state in Sec. IV. With unit gain, in this case, we find that the result [22] holds irrespectively of the variance \( v_e \) of the distribution of displacements.

VII. CONCLUSION

In conclusion, we have presented a theory to determine the fidelity of a general quantum state transformation on an unknown quantum system. The result of this analysis
is that, as long as the protocol has been definitely determined in terms of the actions on the system conditioned on the measurement outcomes, one can compute the fidelity as a simple weighted average of the state-to-state fidelities over the incoming set of states.

We have introduced a formalism which incorporates the preparation of the input states, and showed that our use of a fictitious system in a mixed state which is correlated with the input state to the protocol may indeed be convenient for practical calculations. We demonstrated this last point in the case of Gaussian transformations of Gaussian states, where we showed that the covariance matrix formalism readily identifies the optimum performance and provides simple analytical results for the fidelity of teleportation and quantum memories. The optimal use of non-trivial feed-back gain in these protocols was a particularly interesting result brought out clearly by the analysis. Finally we showed that more general states can also be handled by their appropriate Wigner functions.

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