GRAVITATIONAL RADIATION,
VORTICITY AND THE ELECTRIC AND
MAGNETIC PART OF WEYL TENSOR

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Abstract

The electric and the magnetic part of the Weyl tensor, as well as the invariants obtained from them, are calculated for the Bondi vacuum metric. One of the invariants vanishes identically and the other only exhibits contributions from terms of the Weyl tensor containing the static part of the field. It is shown that the necessary and sufficient condition for the spacetime to be purely electric is that such spacetime be static. It is also shown that the vanishing of the electric part implies Minkowski spacetime. Unlike the electric part, the magnetic part does not contain contributions from the static field. Finally a speculation about the link between the vorticity of world lines of observers at rest in a Bondi frame, and gravitational radiation, is presented.

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1 Introduction

The study of the electric $E_{\alpha\beta}$ and magnetic $H_{\alpha\beta}$ parts of Weyl tensor has attracted the attention of researchers for many years (see [1]–[19] and references therein).

Particularly intriguing is the eventual relationship of the magnetic part of the Weyl tensor, with rotation [2, 9] and with gravitational radiation [1, 5, 8, 10, 11, 12].

On the other hand, the link has been established between gravitational radiation and vorticity of world–lines of observers at rest in a Bondi space–time [20, 21]. Specifically, it has been shown that the leading term in the vorticity (in an expansion of powers of $1/r$) is expressed through the news function in such a way that it will vanish if and only if there is no news (no radiation). This suggests the possibility of detecting gravitational waves by means of gyroscopes [20, 22].

In order to delve deeper into these issues, we shall calculate in this work the electric and the magnetic part of the Weyl tensor, as well as the two invariants obtained from them, in the field of gravitational radiation.

From the obtained expressions, it follows that the vanishing of the magnetic part, implies the vanishing of the news function and also the vanishing of non–radiative but time dependent field, except for a very peculiar class of solutions, called by Bondi “non–natural, non–radiative moving systems”.
This result, together with the known fact [6] that static Weyl metrics are purely electric, implies, if we exclude by physical reasons the class of solutions mentioned before, that the necessary and sufficient condition for a Bondi spacetime to be purely electric is that such space–time be static.

The vanishing of the electric part of the Weyl tensor is shown to imply that the spacetime is Minkowski. Thus there is no purely magnetic vacuum Bondi space–times, in agreement with the conjecture that purely magnetic vacuum space–times do no exist [6, 7, 9, 18, 19].

It is also obtained that one of the invariants \( Q \equiv E^{\alpha\beta}H_{\alpha\beta} \) vanishes identically whereas the other \( L \equiv E^{\alpha\beta}E_{\alpha\beta} - H^{\alpha\beta}H_{\alpha\beta} \) has a leading term with contributions only from the coefficients in the expansion of the Weyl tensor which contain the static part of the field. Coefficients containing purely radiative and/or non–radiative but time dependent part of the field, do not enter in \( L \).

Finally, we shall speculate that the fact that gravitational radiation produces vorticity of a time–like congruence, might be explained by a mechanism similar to the one suggested to explain the vorticity of a time–like congruence in the field of a charged static magnetic dipole [23].

We shall carry out our calculations using the Bondi’s formalism [24] which has, among other things, the virtue of providing a clear and precise criterion for the existence of gravitational radiation (see also [25]). Namely, if the news function is zero over a time interval, then there is no radiation during
that interval.

The formalism has as its main drawback [26] the fact that it is based on a series expansion which could not give closed solutions and which raises unanswered questions about convergence and appropriateness of the expansion.

However we shall restrain ourselves to a region sufficiently far from the source, so that we shall need in our calculations only the leading terms in the expansion of metric functions. Furthermore, since the source is assumed to radiate during a finite interval, then no problem of convergence appears [27].

A brief resume of Bondi’s formalism is given in the next section, together with the expression of the vorticity of a time-like congruence of observers at rest in a Bondi frame. In section 3 we present the result of the calculations of the electric and a magnetic part as well as the invariants $Q$ and $L$, and in the last section results are discussed.

2 The Bondi’s formalism

The general form of an axially and reflection symmetric asymptotically flat metric given by Bondi is [24]

$$ds^2 = \left(\frac{V}{r} e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 + 2 e^{2\beta} dudr + 2 U r^2 e^{2\gamma} dud\theta - r^2 \left(e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\phi^2\right)$$  \hspace{1cm} (1)
where $V, \beta, U$ and $\gamma$ are functions of $u, r$ and $\theta$.

We number the coordinates $x^{0,1,2,3} = u, r, \theta, \phi$ respectively. $u$ is a timelike coordinate such that $u = constant$ defines a null surface. In flat spacetime this surface coincides with the null light cone open to the future. $r$ is a null coordinate ($g_{rr} = 0$) and $\theta$ and $\phi$ are two angle coordinates (see [24] for details).

Regularity conditions in the neighborhood of the polar axis ($\sin \theta = 0$), imply that as $\sin \theta - > 0$

$$V, \beta, U/ \sin \theta, \gamma/ \sin^2 \theta$$

(2)
each equals a function of $\cos \theta$ regular on the polar axis.

The four metric functions are assumed to be expanded in series of $1/r$, then using the field equations Bondi gets

$$\gamma = cr^{-1} + \left(C - \frac{1}{6}c^3\right)r^{-3} + ...$$

(3)

$$U = -(c_\theta + 2c \cot \theta) r^{-2} + \left[2N + 3cc_\theta + 4c^2 \cot \theta\right] r^{-3} ...$$

(4)

$$V = r - 2M$$

$$- \left(N_\theta + N \cot \theta - c_\theta^2 - 4cc_\theta \cot \theta - \frac{1}{2}c^2(1 + 8 \cot^2 \theta)\right) r^{-1} + ...$$

(5)

$$\beta = -\frac{1}{4}c^2r^{-2} + ...$$

(6)

where $c, C, N$ and $M$ are functions of $u$ and $\theta$, letters as subscripts denote derivatives, and
\[ 4C_u = 2c^2c_u + 2cM + N \cot \theta - N_\theta \] (7)

The three functions \( c, M \) and \( N \) are further related by the supplementary conditions

\begin{align*}
M_u &= -c_u^2 + \frac{1}{2} (c_\theta \theta + 3c_\theta \cot \theta - 2c)_u \quad (8) \\
-3N_u &= M_\theta + 3cc_u + 4cc_u \cot \theta + c_u c_\theta \quad (9)
\end{align*}

In the static case \( M \) equals the mass of the system whereas \( N \) and \( C \) are closely related to the dipole and quadrupole moment respectively.

Next, Bondi defines the mass \( m(u) \) of the system as

\[ m(u) = \frac{1}{2} \int_0^\pi M \sin \theta d\theta \] (10)

which by virtue of (8) and (2) yields

\[ m_u = -\frac{1}{2} \int_0^\pi c_u^2 \sin \theta d\theta \] (11)

Let us now recall the main conclusions emerging from Bondi’s approach.

1. If \( \gamma, M \) and \( N \) are known for some \( u = a(\text{constant}) \) and \( c_u \) (the news function) is known for all \( u \) in the interval \( a \leq u \leq b \), then the system is fully determined in that interval. In other words, whatever happens at the source, leading to changes in the field, it can only do so by affecting \( c_u \) and viceversa. In the light of this comment the relationship between news function and the occurrence of radiation becomes clear.
2. As it follows from (11), the mass of a system is constant if and only if there are no news.

Now, for an observer at rest in the frame of (1), the four-velocity vector has components

\[ u^\alpha = \left( \frac{1}{A}, 0, 0, 0 \right) \tag{12} \]

with

\[ A \equiv \left( \frac{V}{r} e^{2\beta} - U^2 e^{2\gamma} \right)^{1/2}. \tag{13} \]

Then, it can be shown that for such an observer the vorticity vector may be written as (see [20] for details)

\[ \omega^\alpha = \left( 0, 0, 0, \omega^\phi \right) \tag{14} \]

with

\[
\omega^\phi = -\frac{e^{-2\beta}}{2r \sin \theta} \left[ 2\beta \theta e^{2\beta} - \frac{2e^{2\beta}A_\theta}{A} - \left( U e^{2\gamma} r^2 \right)_r \\
+ \frac{2U r^2 e^{2\gamma}}{A} A_r + \frac{e^{2\beta} \left( U r^2 e^{2\gamma} \right)_u - U r^2 e^{2\gamma}}{A^2} - 2\beta e^{2\beta} \right] \tag{15}
\]

and for the absolute value of \( \omega^\alpha \) we get

\[
\Omega \equiv (-\omega_\alpha \omega^\alpha)^{1/2} = \frac{e^{-2\beta - \gamma}}{2r} \left[ 2\beta \theta e^{2\beta} - \frac{2e^{2\beta}A_\theta}{A} - \left( U e^{2\gamma} r^2 \right)_r \\
+ 2U r^2 e^{2\gamma} A_r + \frac{e^{2\beta} \left( U r^2 e^{2\gamma} \right)_u - 2\beta \theta A^2 U e^{2\beta}}{A^2} \right] \tag{16}
\]

Feeding back (3–6) into (16) and keeping only the two leading terms, we obtain

\[
\Omega = -\frac{1}{2r} \left( c_{u\theta} + 2c_u \cot \theta \right)
\]
\[
\frac{1}{r^2} [M_\theta - M(c_{u\theta} + 2c_u \cot \theta) - cc_{u\theta} + 6cc_u \cot \theta + 2c_u c_\theta] \quad (17)
\]

Therefore, up to order \(1/r\), a gyroscope at rest in (1) will precess as long as the system radiates \((c_u \neq 0)\). Observe that if

\[
c_{u\theta} + 2c_u \cot \theta = 0 \quad (18)
\]

then

\[
c_u = \frac{F(u)}{\sin^2 \theta} \quad (19)
\]

which implies

\[
F(u) = 0 \implies c_u = 0 \quad (20)
\]

in order to insure regularity conditions, mentioned above, in the neighbourhood of the polar axis \((\sin \theta = 0)\). Thus the leading term in (17) will vanish if and only if \(c_u = 0\).

The order \(1/r^2\) contains, beside the terms involving \(c_u\), a term not involving news, namely \(M_\theta\). This last term represents the class of non-radiative motions discussed by Bondi [24] and may be thought of as corresponding to the tail of the wave, appearing after the radiation process [26].

Let us now assume that initially (before some \(u = u_0\) =constant) the system is static, in which case

\[
N_u = c_u = 0 \quad (21)
\]

which implies, because of (9),

\[
M_\theta = 0 \quad (22)
\]
and $\Omega = 0$ (actually, in this case $\Omega = 0$ at any order) as expected for a static field (for the electrovacuum case however, this may change [23]). Then let us suppose that at $u = u_0$ the system starts to radiate ($c_u \neq 0$) until $u = u_f$, when the news vanish again. For $u > u_f$ the system is not radiating although (in general) $M_\theta \neq 0$ implying (see for example (9)) time dependence of metric functions (non-radiative motions [24]).

For $u > u_f$ there is a vorticity term of order $1/r^2$ describing the effect of the tail of the wave. This in turn provides “observational” evidence for the violation of the Huygens’s principle, a problem largely discussed in the literature (see for example [24, 26], and references therein).

### 3 The electric and magnetic parts of Weyl tensor

The electric and magnetic parts of Weyl tensor, $E_{\alpha\beta}$ and $H_{\alpha\beta}$, respectively, are formed from the Weyl tensor $C_{\alpha\beta\gamma\delta}$ and its dual $\tilde{C}_{\alpha\beta\gamma\delta}$ by contraction with the four velocity vector given by (12) [28]:

$$
E_{\alpha\beta} = C_{\alpha\gamma\beta\delta} u^\gamma u^\delta \tag{23}
$$

$$
H_{\alpha\beta} = \tilde{C}_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2} \epsilon_{\alpha\gamma\epsilon\delta} C_{\beta\rho}^\epsilon u^\gamma u^\rho, \quad \epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{-g} \, \eta_{\alpha\beta\gamma\delta} \tag{24}
$$

where $\eta_{\alpha\beta\gamma\delta} = +1$ for $\alpha, \beta, \gamma, \delta$ in even order, $-1$ for $\alpha, \beta, \gamma, \delta$ in odd order and 0 otherwise. Also note that
\[ \sqrt{-g} = r^2 \sin \theta e^{2\beta} \approx r^2 \sin \theta \exp \left(-\frac{c^2}{2r^2}\right) \approx r^2 \sin \theta + O(1) \]

Since the obtained expressions are fairly long, in order to check our results we have calculated the magnetic and electric parts in two different ways and then compared results, excluding thereby any possible error. On the one hand we have calculated with Maple the components of Weyl tensor and from them, “by hand” its electric and magnetic part from (23) and (24). We do not include here the detailed expressions of Weyl components, but they are available upon request. The only non–vanishing components of Weyl tensor are

\[ C_{0101}, C_{0102}, C_{0112}, C_{0212}, C_{0303}, C_{0313}, C_{0323}, C_{1212}, C_{1313}, C_{1323}, C_{2323}. \]

However they are not independent, since the following relations between them exist:

\[ \frac{r^4 \sin^2 \theta}{e^{2\beta}} C_{1010} = e^{2\gamma} (V - r^4 e^{2\gamma - 2\beta}) C_{1313} - 2r^2 e^{2\gamma} C_{0313}, \quad (25) \]

\[ \frac{r^2 \sin^2 \theta}{e^{2\gamma}} C_{0112} = e^{2\beta} C_{1323} - r^2 e^{2\gamma} C_{1313}, \quad (26) \]

\[ \frac{2r^2 \sin^2 \theta}{e^{2\gamma}} C_{0212} = e^{2\beta} C_{2323} - r e^{2\gamma} C_{1313}, \quad (27) \]

\[ \sin^2 \theta C_{1212} = -e^{4\gamma} C_{1313}. \quad (28) \]

On the other hand, we have calculated the magnetic and electric part using GRTensor.
Thus, one has for the components of the magnetic Weyl tensor, up to the order $1/r^3$:

\[ H_0^0 = H_1^0 = H_2^0 = H_3^0 = H_0^1 = H_1^1 = H_2^1 = H_3^1 = H_0^2 = H_1^2 = H_2^2 = H_3^2 = H_0^3 = H_1^3 = H_2^3 = 0 \quad (29) \]

\[
H_3^0 = -\frac{1}{r} \left( 2c_u \cos \theta + c_{\theta u} \sin \theta \right) \\
+ \frac{1}{r^2} \left\{ 4c_u(c - M) \cos \theta + \left[ \frac{3}{2}(N_u + M_\theta + c_u c_\theta) + \frac{7}{2} c_{\theta u} - 2M c_{\theta u} \right] \sin \theta \right\} \\
+ \frac{1}{r^3} \left\{ -\frac{N}{\sin \theta}(1 + 2c_u) + \left[ 8M c_u(c - M) + N_\theta(1 - 2c_u) + \frac{5}{2} c^2 c_u - N c_{\theta u} - P_u - 4M c \right] \cos \theta \\
+ \left[ 2(N - Mc_\theta) - c_{\theta u}(7M c - 4M^2 - N_\theta - \frac{7}{4}c^2) + 3M(N_u + M_\theta) - \frac{1}{2}P_{\theta u} \\
- 3cM_\theta + N_{\theta \theta} + c_u(8N + 3Mc_\theta + \frac{5}{2}c_{\theta u}) \right] \sin \theta \right\}, \quad (30)
\]

\[
H_3^1 = \frac{1}{r} \left[ (c_\theta c_{uu} + c_{\theta u}) \sin \theta + 2(c_{uu} + c_u) \cos \theta \right] \\
+ \frac{1}{r^2} \left\{ \frac{4c_{uu} \cos \theta}{\sin^2 \theta} + \frac{2c_{u}c_{\theta} - cc_{\theta u}}{\sin \theta} \right\} \\
+ \left[ -\frac{1}{2}c_\theta c_{\theta u} + (2M - 3c)c_\theta c_{uu} - \frac{5}{2} (c_{\theta u} + c_u c_\theta) - 2N c_{uu} - \frac{3}{2}(N_u + M_\theta) \right] \sin \theta \\
+ \left[ -6c_{uu} + 4c(M - c)c_{uu} - \frac{1}{2}c_\theta c_{\theta u} - cc_{\theta u} \right] \cos \theta \\
+ \frac{1}{r^3} \left\{ \frac{8c_{uu}(M - c) \cos \theta}{\sin^2 \theta} \right\} + \frac{1}{\sin \theta} \left[ c_u c_\theta(4M - 3c) + N - 2cN_u + 2N c_{uu} + \right.
\]
\[ + c_\theta u(7c - 2M) - 4c_u N - cM_\theta \right] + \left[ \frac{1}{2}P_{\theta u} - N_{\theta \theta} - 2N + 4cM_\theta - 4c_u N + 2cN_u \right.
\]
\[ + c_u c_\theta(-\frac{9}{2}c - M + cc_u + \frac{1}{2}c_{\theta u}) + c_\theta (2M - cM_u + \right] + \frac{1}{2}M_{\theta u}) \right\}
\]

12
\[ c_\theta u \left( -\frac{19}{4} c^2 + 2c^2 + 2Mc \right) + c_\theta \theta u \left( N + 3cc_\theta - Mc_\theta \right) + c_\theta c_{uu} \left( -6Mc + \frac{1}{2} c^2 + N_\theta + 4M^2 \right) - 2Nc_{uu} \left( c + 2M \right) \sin \theta \]
\[ + \left[ 2c(2M + N_\theta + P_{uu} - cM_u + \frac{1}{2} M_\theta) + c_u(-2Mc - \frac{3}{2} c^2 + 2c^2 c_u + \frac{3}{2} c_\theta^2 + cc_\theta) \right. \]
\[ + c_\theta u (N - Mc_\theta + 8cc_\theta) + c_{uu}(c^3 + 2cN_\theta + Nc_\theta - 8Mc^2 + 8M^2 c) \]
\[ + cc_\theta \theta u (5c - 2M) - c_\theta \left( \frac{1}{2} M_\theta + N_u + P_u - N_\theta \left) \cos \theta \right\}, \tag{31} \]

\[ H_3^2 = \frac{1}{r} \left( \sin \theta c_{uu} \right) + \frac{1}{r^2} \left\{ \frac{2c_u}{\sin \theta} - \frac{1}{2} c_\theta \cos \theta + \left[ 2c_{uu}(M - c) - c_u - \frac{1}{2} c_\theta u \right] \sin \theta \right\} \]
\[ + \frac{1}{r^3} \left\{ 4c_u \frac{M - c}{\sin \theta} + \left[ \frac{3}{2} c_u c_\theta - N_u - \frac{1}{2} M_\theta + Nc_{uu} + \left( \frac{7}{2} c - M \right) c_{uu} \right] \cos \theta + \right. \]
\[ \left. \left[ \left( \frac{5}{2} c - M \right) c_\theta \theta u + \left( \frac{1}{2} c_\theta - M \right) c_u + 2c_\theta c_\theta u + c_{uu}(2c^2 + 4M^2 - 4Mc + N_\theta) + \frac{1}{2} M_\theta - cM_u + N_\theta + P_{uu} + cc^2 \right] \sin \theta \right\}, \tag{32} \]

\[ H_1^3 = \frac{1}{r^3} \frac{2c_u \cos \theta + c_\theta u \sin \theta}{\sin^2 \theta}, \tag{33} \]

\[ H_2^3 = \frac{1}{r \sin \theta} \left[ 2c_{uu}(c + M) - c_u - \frac{1}{2} c_\theta u - \frac{1}{2} \cot \theta c_\theta u + 2 \frac{c_u}{\sin^2 \theta} \right] \]
\[ + \frac{1}{r^3 \sin \theta} \left\{ \frac{4M c_u}{\sin^2 \theta} + \cot \theta \left[ -N_u + Nc_{uu} - \frac{1}{2} c_u c_\theta - \left( \frac{1}{2} c + M \right) c_\theta u - \frac{1}{2} M_\theta \right] \right. \]
\[ + c_{uu}(N_\theta + 4Mc + 4M^2 + 2c^2) + cc^2 + \left( \frac{1}{2} c_\theta - M \right) c_u + \left( \frac{1}{2} c - M \right) c_\theta u \]
\[ + P_{uu} + \frac{1}{2} M_{\theta \theta} + c_{\theta} c_{\theta u} - c M_u + N_{\theta u} \]. \quad (34)

Regarding the electric part, one gets, up to the order \(1/r^3\):

\[ E_0^0 = E_3^0 = E_0^1 = E_3^1 = E_2^2 = E_3^2 = E_0^3 = E_1^3 = E_2^3 = 0 \] \quad (35)

\[ E_1^0 = \frac{2(cc_u + M)}{r^3}, \] \quad (36)

\[ E_2^0 = \frac{2c_u \cos \theta + c_{\theta u} \sin \theta}{r \sin \theta} \]
\[ + \frac{1}{2 \sin \theta r^2} \left\{ 8M c_u \cos \theta + [c_{\theta u}(4M - 3c) - 3(M_{\theta} + c_{\theta} c_{\theta} + N_u)] \sin \theta \right\} \]
\[ + \frac{1}{4r^3} \left\{ (1 + 2c_u) \frac{4N}{\sin^2 \theta} + \cot \theta \left[ 4(N c_{\theta u} + P_u - N_{\theta}) + c_u \left( 32M^2 + 8N_{\theta} - 42c^2 \right) \right] \]
\[ + 4(N - 3c N_u - N_{\theta u}) - 12M (M_{\theta} + N_u) + c_{\theta u} \left( 4N_{\theta} + 16M^2 - 13c^2 - 12M c \right) \]
\[ - c_u \left( 30c c_{\theta} + 12M c_{\theta} + 32N \right) + 2P_{\theta u} \}, \quad (37) \]

\[ E_1^1 = - \frac{2cc_u(1 + \cos^2 \theta) + (c_{\theta} c_{\theta u} + 2M) \sin^2 \theta + 2 \sin \theta \cos \theta(c_u c_{\theta} + c_{\theta u})}{r^3 \sin^2 \theta}, \] \quad (38)

\[ E_2^1 = - \frac{2(cc_{uu} + c_u) \cos \theta + (c_{\theta} c_{uu} + c_{\theta u}) \sin \theta}{r \sin \theta} \]
\[ + \frac{1}{2r^2} \left\{ 3(N_u + M_{\theta}) + c_{uu} (2c_{\theta} - 4M c_{\theta} + 4N) + 5c_{\theta} c_u + c_{\theta} c_{\theta u} + c c_{uu} \right\} \]
\[ + \cot \theta \left[ 2c(c_{\theta u} + 2c_u - 4M c_{uu}) + c_{\theta} c_{uu} \right] + \frac{2}{\sin^2 \theta} \left( c_{\theta u} - 2c_{uu} \right) - \frac{8}{\sin^2 \theta} c_{uu} \cos \theta \}\]
\[ - \frac{1}{r^3} \left\{ 8cc_u (M - c) \frac{\cos \theta}{\sin^3 \theta} + \frac{1}{\sin^2 \theta} \left[ (4M - 7c) c_{\theta} c_{\theta} - 4c_u N + 2N c c_{uu} + (c^2 - 2M c) c_{\theta u} \right] \right\}, \]
\[ + N - c(M_\theta + 2N_\theta) \] + \cot \theta \left[ c_u \left( \frac{1}{2} c_\theta^2 - 2Mc + cc_\theta + 2c^2c_u - \frac{15}{2} c^2 \right) + 
\]
\[ c_\theta \left( c_{uu}N + (3c - M)c_{\theta u} - N_u - \frac{1}{2} M_\theta \right) + c_{uu}(8M^2 - 3c^2 + 2N_\theta) + c_{\theta \theta u}(3c - 2M) \]
\[ + c_{\theta u}N + cM_{\theta \theta} + P_u - N_\theta + 2c(P_{uu} + N_{\theta u} - M - cM_u) \]
\[ + c_{uu} \left[ c_\theta \left( \frac{-7}{2} c^2 + N_\theta - 2Mc + 4M^2 \right) - 6Nc - 4MN \right] + c_u \left[ c_\theta \left( \frac{1}{2} c_{\theta \theta} - \frac{9}{2} c - M + cc_u \right) - 4N \right] \]
\[ + c_{\theta u} \left( c_\theta^2 + 2Mc - \frac{15}{4} c^2 \right) + c_{\theta \theta u} \left( N - M c_\theta + 2cc_\theta \right) + c_\theta \left( \frac{1}{2} M_{\theta \theta} - M + N_{\theta u} + P_{uu} - M_uc \right) \]
\[ - N_{\theta \theta} + \frac{1}{2} P_{\theta u} - cN_u + N + cM_\theta \right \}, \quad (39) \]

\[ E_1^2 = - \frac{2c_u \cos \theta + c_{\theta u} \sin \theta}{r^3 \sin \theta}, \quad (40) \]

\[ E_2^2 = - \frac{c_{uu}}{r} + \frac{1}{2r^2} \left( c_{\theta \theta u} - 4Mc_{uu} + 2c_u + \cot \theta c_{\theta u} - \frac{4c_u}{\sin^2 \theta} \right) \]
\[ + \frac{1}{r^3} \left\{ M_u c - \frac{1}{2} M_{\theta \theta} + M - N_{\theta u} - P_{uu} \right\} \]
\[ + \cot \theta \left[ M c_{\theta u} + \frac{1}{2} M_\theta + N_u - \frac{1}{2} cc_\theta + \frac{1}{2} c_\theta c_u - N c_{uu} \right] \]
\[ + c_u \left[ \frac{4(c - M)}{\sin^2 \theta} + M - c - cc_u - \frac{1}{2} c_{\theta \theta} \right] - c_{uu}(4M^2 + N_\theta) - c_\theta c_{\theta u} + c_{\theta \theta u}(M - \frac{3}{2} c) \right \}, \quad (41) \]

\[ E_3^3 = \frac{c_{uu}}{r} - \frac{1}{2r^2} \left( c_{\theta \theta u} - 4Mc_{uu} + 2c_u + \cot \theta c_{\theta u} - \frac{4c_u}{\sin^2 \theta} \right) \]
\[ + \frac{1}{r^3} \left\{ M + N_{\theta u} + P_{uu} + \frac{1}{2} M_{\theta \theta} - cM_u \right\} \]
\[ + \cot \theta \left[ \frac{5}{2} cc_\theta u - \frac{1}{2} M_\theta - N_u + N c_{uu} - M c_{\theta u} + \frac{3}{2} c_u c_\theta \right] \]
\[ + c_u \left[ \frac{4(c - M)}{\sin^2 \theta} + M - c - cc_u - \frac{1}{2} c_{\theta \theta} \right] - c_{uu}(4M^2 + N_\theta) - c_\theta c_{\theta u} + c_{\theta \theta u}(M - \frac{3}{2} c) \right \], \quad (41) \]
\[\begin{align*}
&+ c_u \left[ \frac{4M}{\sin^2 \theta} + cc - M + \frac{1}{2} c \theta - c \right] + c_{uu} (4M^2 + N_\theta) + 2c_\theta c_\theta u + \left( \frac{3}{2} c - M \right) c_{\theta \theta u} \right],
\end{align*}\]

(42)

where

\[P \equiv C - \frac{c^3}{6}.\]

(43)

We shall now provide expressions for the two algebraic invariants associated to the electric and magnetic parts of the Weyl tensor, namely

\[Q = H^\alpha_\beta E^\beta_\alpha, \quad L = E^\alpha_\beta E^\beta_\alpha - H^\alpha_\beta H^\beta_\alpha.\]

(44)

As it turns out, the invariant \(Q\) vanishes identically; i.e.:

\[Q = 0\]

(45)

whereas the first non-vanishing order in \(L\) is \(1/r^6\) and one then has:

\[L = \frac{2}{r^6} \left[ 3(cc + M)^2 + (c^3 + 6P)c_{uu} + 6N(c_{\theta u} + 2c_u \cot \theta) \right] + O(1/r^7)\]

(46)

4 Discussion

We are now ready to try to answer to the main questions which motivated this work, in the context of the Bondi metric, namely:

- What consequences do emerge from the vanishing of the magnetic part of the Weyl tensor?
• What consequences do emerge from the vanishing of the electric part of the Weyl tensor?

• How do different types of fields (radiative, non radiative but time dependent, and static) enter into the electric and magnetic part of the Weyl tensor, and into the corresponding invariants?

• Why does gravitational radiation produce vorticity?

Let us start with the first question. If we put $H^\alpha_\beta = 0$ then it follows from the coefficient of $1/r$ in (30) that

$$c_\theta u \sin \theta + 2c_u \cos \theta = 0$$

which according to (19) and (20) implies

$$c_u = 0.$$

Thus the field is non-radiative. Next, the vanishing of the coefficient $1/r^2$ in (30) implies in turn that

$$M_\theta = N_u = 0,$$

where we have used (9).

Finally from the vanishing of the coefficient of the $1/r^3$ in (30) it follows that

$$N_\theta \sin^2 \theta + N_\theta \sin \theta \cos \theta - N \cos 2\theta - (2cM \sin^2 \theta) \theta = 0$$
whose general solution is

\[ N = \left( \int \frac{2cM}{\sin \theta} d\theta + \sigma \right) \sin \theta \tag{51} \]

where \( \sigma \) is a constant.

Feeding back (51) into (7) and using (48), it follows that

\[ C_u = 0. \tag{52} \]

It can be easily checked that no further information can be obtained from (31–34). Therefore up to order \( 1/r^3 \) in \( \gamma \), the metric is static, and the mass, the “dipole” \( (N) \) and the “quadrupole” \( (C) \) moments correspond to a static situation. However, the time dependence might enter through coefficients of higher order in \( \gamma \), giving rise to what Bondi calls “non–natural non–radiative moving system” (nnnrms). In this later case, the system keeps the three first moments independent of time, but allows for time dependence of higher moments. As unlikely as this situation may be from the physical point of view, we were not able not rule it out mathematically. On the other hand it is known that static space–times are purely electric. Accordingly, we conclude that, excluding nnnrms, the necessary and sufficient condition for a Bondi metric to be purely electric is to be static.

The second question has a simple answer. Indeed assuming \( E_\beta^\gamma = 0 \) and using regularity conditions, we find from the coefficient of order \( 1/r \) in (37)

\[ c_u = 0, \tag{53} \]
then, it follows at once from (36) that \( M = 0 \). If we exclude the possibility of negative masses, then the spacetime must be Minkowski, giving further support to the conjecture that there are no purely magnetic vacuum spacetimes [9].

The third question has also a simple answer. The electric part contains all kinds of contributions, radiative, non–radiative but time dependent (nrtd), and static. At order \( 1/r \) only radiative contributions appear, whereas nrtd terms appear at order \( 1/r^2 \), and higher and contributions from the static field enter in the \( 1/r^3 \) order, and higher. However the magnetic part does not contain contribution from the static field. Only radiative (at order \( 1/r \) and higher) and nrtd terms (at order \( 1/r^2 \) and higher) appear.

On the other hand, the only non–vanishing invariant \((L)\) has a leading term of order \( 1/r^6 \) implying that purely radiative and nrtd terms in the Weyl tensor do not contribute to \( L \). This is in agreement with the fact that for purely radiative spacetimes, both invariants \( Q \) and \( L \) vanish [8]

Finally, let us consider the last question. With this purpose in mind, it is worth recalling a result obtained by Bonnor [23] concerning the dragging of inertial frames by a charged magnetic dipole. To explain the appearance of vorticity in such space–times, Bonnor notices that the corresponding electromagnetic Poynting vector has a non–vanishing component, describing a flow of electromagnetic energy round in circles where frame–dragging occurs [29]. He then suggests that such a flow of energy affects inertial frames by
producing vorticity of congruences of particles, relative to the compass of inertia.

One could speculate about a similar mechanism in our case, i.e. a flow of gravitational radiation in the $\phi$ direction. However for testing such a conjecture we should have available a unique expression for a “gravitational” Poynting vector, which is still an open question in general relativity.

Thus for example, the super–Poynting vector based on the Bel–Robinson tensor, as defined in [15], is

$$P_\alpha = \epsilon_{\alpha \beta \gamma \delta} E^\beta_\rho H^{\gamma \rho} u_\delta$$

(54)
giving in our case $P^\phi = 0$. However, besides the ambiguity problem in the definition of energy, this negative result may be caused by the reflection symmetry of the Bondi metric, which intuitively seems to be incompatible with the presence of a circular flow of energy in the $\phi$ direction. In order to clarify this situation, $P^\phi$ should be calculated for the general radiative metric without reflection symmetry [30], but this of course is out of the scope of the present work.

References


