Diffusion of Cosmic Rays in Expanding Universe. (I)

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ABSTRACT

We present an analytic solution to diffusion equation for high energy cosmic rays in the expanding universe. The particles are assumed to be ultra-relativistic and they can have energy losses arbitrarily dependent on energy and time. The obtained solution generalizes the Syrovatsky solution, valid for the case when energy losses and diffusion coefficient are time-independent.

*Subject headings:* extragalactic cosmic rays, diffusive propagation of cosmic rays.

1. Introduction

The cosmic rays is important component of extragalactic medium, which is responsible for heating of extragalactic gas and for production of various types of radiation in the space. The charged cosmic ray particles (electrons, protons and nuclei) propagate diffusively in extragalactic space due to scattering in the magnetic fields. The regime of diffusive propagation is reached if scattering length, $\ell_{sc}$, is much less than the size of the considered region.

Typically, diffusion takes place in the magnetized plasma, where the particle deflections occur due to scattering off the turbulent pulsation and hydromagnetic waves. This process proceeds in the resonant regime (see Lifshitz & Pitaevskii 2001, Sections 55 and 61), when a particle giro-radius is equal to a wave length. The calculation of the diffusion coefficient for the resonant scattering of particles off the hydromagnetic waves is presented in the book by Berezinsky et al. (1990a).

The value of diffusion coefficient is quite different for three distinct regions in extragalactic space: voids, filaments and clusters of galaxies. The diffusion in these regions occurs only
at energies when the diffusion length, $\ell_{\text{diff}}$, is much smaller than the size of the region $L$. In the opposite extreme case particles propagate (quasi)rectilinearly.

For propagation of high energy particles from a single source at point $\vec{r}_g$ the diffusion equation reads

$$\frac{\partial}{\partial t} n_p(E, \vec{r}, t) - \text{div} \left[ D(E, \vec{r}, t) \nabla n_p \right] - \frac{\partial}{\partial E} \left[ b(E, \vec{r}, t) n_p \right] = Q(E, \vec{r}, t) \delta^3(\vec{r} - \vec{r}_g),$$

where $n_p(E, \vec{r}, t)$ is the space density of particles $p$ with energy $E$ at time $t$ and at the point $\vec{r}$, $D(E, \vec{r}, t)$ is the diffusion coefficient, $b(E, \vec{r}, t) = -dE/dt$ describes the continuous energy losses, and $Q(E, \vec{r}, t)$ is the source generation function.

In the case when $D$, $b$ and $Q$ depend only on energy, the method of exact analytic solution to the diffusion equation has been suggested by Syrovatsky (Syrovatskii 1959). For a single-source diffusion equation (1) the solution for spherically-symmetric case can be presented (Berezinsky et al. 1990b) as

$$n_p(E, r) = \frac{1}{b(E)} \int_{E}^{\infty} dE_g Q(E_g) \exp \left[ -\frac{\lambda(E, E_g)}{4\pi \lambda(E, E_g)} \right],$$

where

$$\lambda(E, E_g) = \int_{E}^{E_g} d\varepsilon \frac{D(\varepsilon)}{b(\varepsilon)},$$

is the Syrovatsky variable which has the meaning of the squared distance traversed by a particle in the observer direction, while its energy diminishes from $E_g$ to $E$.

The other Syrovatsky variable,

$$\tau(E, E_g) = \int_{E}^{E_g} d\varepsilon \frac{D(\varepsilon)}{b(\varepsilon)},$$

has a meaning of time, during which the particle energy diminishes from $E_g$ to $E$.

In this paper we shall present an analytic solution to the diffusion equation in the expanding universe for the case when $D$, $b$ and $Q$ are arbitrary functions of energy and time. We use the computation method which differs from that of (Syrovatskii 1959). While Syrovatsky used method of the Green functions, we solve the equation for a single source. Its position is described by the delta function $\delta^3(\vec{r} - \vec{r}_g)$, and thus the equation for the Fourier component does not contain $\delta$-functions. This is the first order linear equation with partial derivatives, which can be solved by a standard method, introducing auxiliary characteristic equation.
As far as physical applications are concerned, the solution for a single source is most general, because all other cases, e.g. with homogeneously distributed sources or with single non-stationary source (see (Berezinsky et al. 1990b)) can be straightforwardly obtained from it.

2. Diffusion equation in expanding universe

We shall use the Friedmann-Robertson-Walker metric for the flat space and radial direction, following Weinberg (1972)

\[ ds^2 = c^2 dt^2 - a^2(t) dx^2 = -g_{\mu\nu} dx^{\mu\nu}, \]

where \( \text{diag} ~ g_{\mu\nu} = (-1, a^2, a^2, a^2) \) and \( \text{diag} ~ g^{\mu\nu} = (-1, 1/a^2, 1/a^2, 1/a^2) \), \( \vec{x} \) is the spatial coordinate, corresponding to comoving distance, and \( a(t) \) is the scaling factor of expanding universe, normalized as \( a(t_0) = 1 \) at present age of the universe \( t_0 \). The redshift \( z \) is given by \( 1 + z = 1/a(t) \) and \( dt/dz \) by

\[ -\frac{dt}{dz} = \frac{1}{H_0(1+z)\sqrt{\Omega_m(1+z)^3 + \Lambda}}, \]

where \( H_0 \) is the Hubble parameter at \( z = 0 \) and \( \Omega_m \) and \( \Lambda \) are cosmological mass density and vacuum energy in units of the critical density.

The physical and proper distances are locally determined as \( d\vec{r} = a(t)d\vec{x} \). For the proper distance it is assumed that \( a(t) \) is not changed in the process of distance measurement between \( \vec{x} = 0 \) and \( \vec{x} \), and thus the proper distance between these two coordinates is \( r_{\text{prop}} = a(t)x \), and velocity of the universe expansion is \( \vec{u} = \dot{a}\vec{x} = H(t)r_{\text{prop}} \).

For the physical distance it is assumed that measurement is performed with help of the light signal \( (ds^2 = c^2dt^2 - a^2(t)dx^2 = 0) \) and the distance between the object with redshift \( z \) and observer with \( z = 0 \) is

\[ r_{\text{ph}} = \int_0^x a(t)dx = c \int_0^z \frac{dt}{dz} \frac{dz}{H_0(1+z)\sqrt{\Omega_m(1+z)^3 + \Lambda}}. \]

Following Peebles (1980) we shall use in this paper the proper distance, denoting it by \( r \) and \( r \) without subscript. Being the formal quantity, the proper distance coincides locally with the physical distance, and in most applications we shall in fact use only local properties of the proper distance. The positions of the particles will be described by both proper distances \( r \) and by comoving distances \( \vec{x} \), the latter can be considered as the set of fixed coordinates in the expanding universe.
Expansion of the universe affects diffusion, and we shall take it into account using the proper formalism.

The local observer sees the particle flux density produced by diffusion

\[ j_k = -D \frac{\partial}{\partial x^k} n(\vec{x}, t). \]  

(8)

In case of isotropic diffusion (the diffusion coefficient is rotation invariant), \( j_k \) is the covariant space vector, which together with proper space density of particles \( n \) forms the covariant 4-vector \( j_\mu(\vec{x}, t) = (n, j_k) \) (the Latin indices run through 1 - 3, and the Greek ones through 0 - 3).

The conservation of current \( j^\mu \) can be written as (Weinberg 1972)

\[ \frac{\partial}{\partial x^\mu} (\sqrt{g} j^\mu) = 0, \]  

(9)

where \( g = |\text{det} g_{\mu\nu}| \) and \( \sqrt{g} = a^3(t) \). Differentiating Eq. (9) with using definition of the Hubble parameter \( H(t) = \dot{a}(t)/a(t) \), then transforming the contravariant component \( j^k \) into covariant one as \( j^k = g^{km} j_m \), with \( \text{diag} g^{km} = 1/a^2(t) \), and finally using Eq. (8) for

\[ \frac{\partial}{\partial x^k} j^k = -D g^{ik} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^k} n(\vec{x}, t) = -\frac{D}{a^2} \nabla^2 x n(\vec{x}, t), \]  

(10)

one obtains the diffusion equation

\[ \frac{\partial}{\partial t} n(\vec{x}, t) + 3H(t) n(\vec{x}, t) - \frac{D}{a^2} \nabla^2 x n(\vec{x}, t) = 0, \]  

(11)

where \( 3H(t)n \) term describes expansion of the universe.

The derived equation can be obtained from conservation equation (9.15) of Peebles (1980) excluding there the advection term and adding the diffusion term.

The derivation of the diffusion equation above is rather formal, and now we shall present it in the transparent and physically clear way.

We shall operate here and everywhere below with particle density \( n \) in the expanding volume using two sets of the coordinates \((\vec{r}, t)\) and \((\vec{x}, t)\). This density is given by

\[ n(\vec{r}, t) = n(\vec{x}, t), \]  

(12)

with \( \vec{r}(t) = a(t) \vec{x} \). Differentiating \( n(\vec{r}, t) \),

\[ \frac{d}{dt} n(\vec{r}, t) = \left( \frac{\partial n}{\partial t} \right)_r + \frac{\partial n}{\partial \vec{r} \cdot \frac{d\vec{r}}{dt}} = \left( \frac{\partial n}{\partial t} \right)_r + H(t) \vec{r} \frac{\partial n}{\partial \vec{r}}, \]  

(13)
and using $dn(\vec{r},t)/dt = \partial n(\vec{x},t)/\partial t$ from (12) one obtains the Peebles relation given by Eq. (9.13) from (Peebles 1980):

$$
\left(\frac{\partial n}{\partial t}\right)_x = \left(\frac{\partial n}{\partial t}\right)_r + H(t)\vec{r}\vec{\nabla}_r n,
$$

(14)

where subscripts $x$ and $r$ indicate basis $(\vec{x},t)$ and $(\vec{r},t)$, respectively.

We shall obtain the diffusion equation from conservation of number of particles. Consider a sphere of radius $x$, which expands in the basis $(\vec{r},t)$ as $r(t) = a(t)x$. The number of particles inside this sphere is changing only due to diffusive flux, which is defined as $\vec{j} = -D(\vec{r},t)\vec{\nabla}_r n(\vec{r},t)$. The corresponding equation reads:

$$
\frac{d}{dt}\int_{V(t)} dV n(\vec{r},t) = -\int_{S(t)} \vec{j} d\vec{s} = \int_{V(t)} dV \text{div}[D(\vec{r},t)\vec{\nabla}_r n(\vec{r},t)],
$$

(15)

where $S(t)$ is the expanding sphere and $V(t)$ is the volume inside it. In Eq. (15) the Gauss theorem was used.

Performing differentiation in the lhs of Eq. (15) with respect to time, and taking into account expanding of the elemental volume $dV$ with time,

$$
\frac{d}{dt}\delta V = \frac{d}{dt} [a^3(t)(\delta V)_{\text{comov}}] = 3H(t)\delta V,
$$

and using for $dn/dt$ Eq. (13), one obtains

$$
\left(\frac{\partial n}{\partial t}\right)_r + H(t)\vec{r}\vec{\nabla}_r n + 3H(t)n - \text{div} [D(r,t)\vec{\nabla}_r n(\vec{r},t)] = 0.
$$

(16)

This is the diffusion equation in the physical basis $(\vec{r},t)$. One may add to the rhs the source term $Q_0\delta^3(\vec{r} - \vec{r}_g)$.

The sum of the second and third terms in Eq. (16) merges into $\vec{\nabla}_r(n\vec{u})$, where in our case $\vec{u} = H(t)\vec{r}$ is expansion velocity of the universe, and in absence of the diffusion term we obtain Eq. (9.11) from (Peebles 1980).

In the basis $(\vec{x},t)$ the first two terms in Eq. (16) give $\partial n(\vec{x},t)/\partial t$ (see Eq. 14), and we arrive at the diffusion equation in the form

$$
\left(\frac{\partial n}{\partial t}\right)_x + 3H(t)n(\vec{x},t) - \frac{D(t)}{a^2(t)}\nabla^2_{\vec{x}} n(\vec{x},t) = \frac{Q_0}{a^3(t)}\delta^3(\vec{x} - \vec{x}_g),
$$

(17)

in accordance with Eq. (11). Deriving Eq. (17) we assumed that $D$ is a function of time only.
Eqs. (11) and (17) are the simplified diffusion equations without energy loss and full source terms, which should be included additionally. Introducing these terms as $-\frac{\partial}{\partial E}(nb)$ and $Q(E, t)\delta^3(\vec{r} - \vec{r}_g)$ we arrive at the diffusion equation for expanding universe in the basis $(\vec{x}, t)$ convenient for its solution:

$$\frac{\partial n}{\partial t} - b(E, t)\frac{\partial n}{\partial E} + 3H(t)n - n\frac{\partial b(E, t)}{\partial E} - \frac{D(E, t)}{a^2(t)}\nabla_x^2 n = \frac{Q(E, t)}{a^3(t)}\delta^3(\vec{x} - \vec{x}_g),$$  \hspace{1cm} (18)

where the total energy losses $dE/dt = -b(E, t)$ may be presented as a sum of collisional energy losses $b_{\text{int}}(E, t)$ and adiabatic energy losses $H(t)E$, and $Q(E, t)$ is the number of particles with energy $E$ produced at time $t$ per unit time. In Eq. (18) $b(E, t)$ and $D(E, t)$ are the arbitrary functions of time and energy, and $n = n(t, \vec{x}, p)$ must depend in fact on $\vec{x} - \vec{x}_g$.

Note, that $n(t, \vec{x}, p)$ in Eq. (18) is not the distribution function $f(t, \vec{x}, p)$, which is defined in statistical physics (see e.g. (Lifshitz & Pitaevskii 2001)) as the density in phase space. The relation between them is given by $n(t, \vec{x}, p) = 4\pi p^2 f(t, \vec{x}, p)$. We remind that here is considered the ultra-relativistic case $p \approx E/c$.

3. Analytic solution to the diffusion equation

We shall introduce the Fourier transformation

$$n(t, \vec{x}, E) = \frac{1}{(2\pi)^3} \int d\vec{\omega} f_\omega(E, t)e^{i\vec{\omega}(\vec{x} - \vec{x}_g)}.$$  \hspace{1cm} (19)

Using the Fourier expansion of the $\delta$ function

$$\delta^3(\vec{x} - \vec{x}_g) = \frac{1}{(2\pi)^3} \int d\vec{\omega} e^{i\vec{\omega}(\vec{x} - \vec{x}_g)},$$  \hspace{1cm} (20)

we obtain from Eq. (18) equation for the Fourier components $f_\omega(E, t)$:

$$\frac{\partial f_\omega(E, t)}{\partial t} - b(E, t)\frac{\partial f_\omega(E, t)}{\partial E} + \left[3H(t) - \frac{\partial b(E, t)}{\partial E} + \vec{\omega}^2\frac{D(E, t)}{a^2(t)}\right] f_\omega(E, t) = \frac{Q(E, t)}{a^3(t)}.$$  \hspace{1cm} (21)

The characteristic equation for Eq. (21) is

$$\frac{dE}{dt} = -b(E, t),$$  \hspace{1cm} (22)

with the solution $E'(t) = E'(E, t, t')$ being identical with the generation-energy trajectory $E_g(E, t, t')$ used in (Berezinsky & Grigorieva 1988) and (Berezinsky et al. 2002), where $E_g$
is the energy with which a particle must be generated at time \( t' \) in order to have at \( t \) the observed energy \( E \). Here and everywhere below we shall use the energy \( E \) at the time \( t \) of observation to mark a characteristic trajectory \( \mathcal{E}' = E'(E, t, t') \). Sometimes for brevity we shall omit \( t \).

The solution to Eq. (21) with energies taken on characteristics is given by

\[
f_{\omega}(E, t) = \int_{t_g}^{t} \frac{Q(\mathcal{E}', t')}{a^2(t')} \exp \left\{ - \int_{t'}^{t} \frac{Q(\mathcal{E}', t'')}{a^2(t'')} \left[ 3H(t'') - \frac{\partial b(\mathcal{E}'', t'')}{\partial \mathcal{E}''} + \bar{\omega}^2 D(\mathcal{E}'', t'') \right] \right\}
\]

(23)

where \( t_g \) is the generation time.

We introduce now \( \lambda(E, t, t') \), the analogue of the Syrovatsky variable given by Eq. (3)

\[
\lambda(E, t') = \int_{t'}^{t} \frac{D(\mathcal{E}'', t'')}{a^2(t'')},
\]

(24)

where \( \mathcal{E}'' = E''(E, t, t'') \) is the characteristic trajectory.

The exponent in Eq. (23) can be simplified using the notation

\[
\alpha_{\omega}(E, t, t') = \int_{t'}^{t} \frac{D(\mathcal{E}'', t'')}{a^2(t'')} \exp \left\{ - \int_{t'}^{t} \frac{Q(\mathcal{E}', t'')}{a^2(t'')} \left[ 3H(t'') - \frac{\partial b(\mathcal{E}'', t'')}{\partial \mathcal{E}''} + \bar{\omega}^2 D(\mathcal{E}'', t'') \right] \right\}
\]

(25)

Thus,

\[
e^{-\alpha_{\omega}(E, t, t')} = \left( \frac{1 + z'}{1 + z} \right)^3 e^{-\bar{\omega}^2 \lambda(E, t')} \exp \int_{t'}^{t} \frac{D(\mathcal{E}'', t'')}{a^2(\mathcal{E}'')} \exp \left\{ - \int_{t'}^{t} \frac{Q(\mathcal{E}', t'')}{a^2(t'')} \left[ 3H(t'') - \frac{\partial b(\mathcal{E}'', t'')}{\partial \mathcal{E}''} + \bar{\omega}^2 D(\mathcal{E}'', t'') \right] \right\}.
\]

(26)

Coming back from the Fourier component \( f_{\omega} \) to density \( n \), using identity

\[
i\bar{\omega}(\vec{x} - \vec{x}_g) - \bar{\omega}^2 \lambda = -\lambda \left[ \bar{\omega} - i \frac{\vec{x} - \vec{x}_g}{2\lambda} \right]^2 - \frac{(\vec{x} - \vec{x}_g)^2}{4\lambda},
\]

and performing integration

\[
\int d\bar{\omega} \exp \left[ -\lambda \left( \bar{\omega} - i \frac{\vec{x} - \vec{x}_g}{2\lambda} \right)^2 \right] = (\pi/\lambda)^{3/2},
\]

(27)

we obtain the solution as

\[
n(t, \vec{x}, E) = \frac{\pi^{3/2}}{(2\pi)^3} \int_{t_g}^{t} \frac{Q(\mathcal{E}', t')}{(1 + z)^3} \frac{\exp \left[ - \frac{(\vec{x} - \vec{x}_g)^2}{4\lambda(E, t')} \right]}{\lambda(E, t')^{3/2}} \exp \left[ \int_{t'}^{t} \frac{D(\mathcal{E}'', t'')}{a^2(\mathcal{E}'')} \right]
\]

(28)
Now we shall find the solution for $t = t_0$ ($z = 0$) changing the variables $t' \rightarrow z$, $t'' \rightarrow z'$ and presenting the energy loss as the sum of adiabatic and collisional (interaction) energy losses:

$$b(E, t) = H(t)E + b_{int}(E, t).$$  \hspace{1cm} (29)

It gives

$$n(t_0, \vec{x}, E) = \int_0^{z_g} dz \left| \frac{dt}{dz} \right| (1 + z)Q(E_g, z) \exp \left[ \int_0^z dz' \left| \frac{dt'}{dz'} \right| \frac{\partial b_{int}(E', z')}{\partial E'} \right] \times \frac{\exp\left[-(\vec{x} - \vec{x}_g)^2/4\lambda(E, z)\right]}{[4\pi \lambda(E, z)]^{3/2}},$$ \hspace{1cm} (30)

where $E_g = E_g(E, z)$ is the generation energy in the source taken on the characteristic.

In this expression one can easily distinguish the term $dE_g/dE$, given for the case of protons interacting with CMB in (Berezinsky & Grigorieva 1988) and (Berezinsky et al. 2002). In our case it equals to

$$\frac{dE_g}{dE} = (1 + z) \exp \left( \int_0^x dz' \left| \frac{dt'}{dz'} \right| \frac{\partial b_{int}(E', z')}{\partial E'} \right).$$ \hspace{1cm} (31)

Now the solution for the case of arbitrary energy losses $b_{int}(E, z)$ can be written in the compact form

$$n(t_0, \vec{x}, E) = \int_0^{z_g} dz \left| \frac{dt}{dz} \right| Q(E_g(E, z), z) \exp\left[-(\vec{x} - \vec{x}_g)^2/4\lambda(E, z)\right] \frac{dE_g}{dE},$$ \hspace{1cm} (32)

where $n(t_0, \vec{x}, E)$ actually depends on $\vec{x} - \vec{x}_g$ as was foreseen above.

In the case of ultra-high energy protons interacting with CMB, $\partial b_{int}/\partial E$ is given according to Berezinsky & Grigorieva (1988) and Berezinsky et al. (2002) by

$$\frac{\partial b_{int}(E, t)}{\partial E} = (1 + z)^3 \left[ \frac{db_0(E')}{dE'} \right]_{E'=(1+z)E_g(E, z)}$$ \hspace{1cm} (33)

and

$$\frac{dE_g}{dE} = (1 + z) \exp \left[ \frac{1}{H_0} \int_0^x dz \frac{(1 + z)^2}{\sqrt{\Omega_m(1 + z)^3 + \Lambda}} \left( \frac{db_0(E')}{dE'} \right)_{E'=(1+z)E_g(E, z)} \right],$$ \hspace{1cm} (34)

where $b_0(E)$ is the interaction energy loss at $z = 0$.

Eq. (32) presents our final result in the form of the particle density $n(t_0, \vec{x}, E)$ in terms of comoving distances $\vec{x}$ for arbitrary energy loss $b_{int}(E, t)$ and diffusion coefficient $D(E, t)$. 


The main difference with the Syrovatsky solution is due to $\lambda(E,t)$ given by Eq. (24) and $dE_g/dE$ given by Eq. (31). Lemoine (2005) has heuristically written a solution to the diffusion equation in expanding universe, including only adiabatic energy losses. He used the Syrovatsky solution for the diffusion equation in terms of conformal time $\eta$. However, the Syrovatsky solution is not valid there, because both diffusion coefficient and the Hubble parameter depend on conformal time $\eta$. Besides, our equation (18) differs from the Lemoine’s equation by extra $a(t)$ in the diffusion term.

Eq. (32) becomes the Syrovatsky solution (2), when $D(E,t)$ and $b(E,t)$ do not depend on time and $a(t) = 1$. In this case $dE_g/dE = b(E_g)/b(E)$ (Berezinsky & Grigorieva 1988), and using $b(E_g)dt = dE_g$ one obtains Eq. (2) with

$$\lambda = \int dt' D(E',t') = \int_E^{E_g} \frac{dE'}{b(E')} D(E'),$$

being the Syrovatsky variable (3). Proper transition to the Syrovatsky solution is one of the tests of our solution (32).

4. Tests of the method

We want to test our method of equation solution for some cases, when solutions are known. One of them is given by the case of time-independent energy loss and diffusion coefficient, which must result for $a(t) = 1$ in the Syrovatsky solution. In Section 3 it has been already demonstrated that our solution passes this test.

The second test consists in the convergence to the universal spectrum, and the third, most important one, examines the cosmological part of our solution: The solution of equation for the rectilinear propagation is obtained by the same method as for diffusion and result coincides with well known formula. Below we shall describe these tests.

4.1. Convergence to the universal spectrum

According to the propagation theorem (Aloisio & Berezinsky 2004) in the case when distances between sources ($d$) become smaller than propagation and interaction lengths (e.g. when $d \to 0$), the diffuse spectrum has an universal form, independent of the mode of propagation.

For the power-law generation spectrum with exponent $\gamma_g > 2$,

$$Q(E, z) = (\gamma_g - 2)L_0(1 + z)^\alpha E^{-\gamma_g},$$

(35)
where $L_0$ is the particle luminosity of a source at $z = 0$ and $\alpha$ takes into account a hypothetical cosmological evolution (in Eq. (35) all energies are measured in GeV, luminosity in GeV s$^{-1}$ and $E_{\text{min}} = 1$ GeV). In this case the universal spectrum for the diffuse flux $J(E)$ is given (Aloisio & Berezinsky 2004) as

$$J(E) = \frac{c}{4\pi} L_0 (\gamma_g - 2) \int_0^{z_{\text{max}}} dz \left| \frac{dt}{dz} \right| (1 + z)^m E_g^{-\gamma_g}(E, z) \frac{dE_g}{dE},$$

(36)

where $L_0 = L_0 n_s(0)$ is the emissivity at $z = 0$, $n_s(0)$ is density of the sources at $z = 0$, $n_s(z) = n_s(0)(1 + z)^\beta$ describes the hypothetical source evolution and $\alpha + \beta = m$.

Let us now calculate the diffuse flux from the density of the particles $n(t_0, \vec{x}, E)$ given by our general solution (32). According to the propagation theorem we must obtain the universal spectrum (36).

Assuming the homogeneous distribution of the sources $n_s(z)$ in the space (which provides the propagation theorem) we can find the diffuse flux multiplying $Q(E_g, z)$ in Eq. (32) by the density of the sources $n_s(z) = n_s(1 + z)^\beta$ and integrating over positions of the sources in the coordinate space. Using (35) we have

$$J_p(E) = \frac{c}{4\pi} (\gamma_g - 2) L_0 \int 4\pi \bar{x}_s^2 dx_s \int_0^{z_{\text{max}}} dz \left| \frac{dt}{dz} \right| (1 + z)^m E_g^{-\gamma_g}(E, z) \exp\left[ -\frac{x_s^2}{4\lambda(E, z)} \right] \frac{dE_g}{dE},$$

(37)

where $\bar{x}_s = \bar{x} - \vec{x}_g$. Changing the order of integration and using

$$\int_0^\infty dx \frac{4\pi x^2}{(4\pi \lambda)^{3/2}} \exp\left( -\frac{x^2}{4\lambda} \right) = 1,$

we arrive indeed at the universal spectrum (36), as it must be. The assumption of the power-law generation spectrum in Eq. (35) does not reduce the generality of the proof.

4.2. Rectilinear propagation

The third test which we shall study here is less trivial than other two and it examines the cosmological part of our solution. We consider the rectilinear propagation of ultra-relativistic particles. This case is well known, and is given by light propagation. We shall find a solution for this case, solving the propagation equation by the method employed in Section 3.

The rectilinear propagation of particles with velocity $\vec{v}$ results in adding this velocity to the expansion velocity $H(t)\vec{r}$ in the rhs of Eq. (13) and in excluding the diffusion term from the equation. Introducing the unit vector $\vec{e}$ in the direction of propagation, and using
\[ \vec{dr} = a(t)\vec{dx} \] we obtain from Eqs. (13) and (18):

\[ \frac{\partial n}{\partial t} + \frac{c\vec{e}}{a(t)} \frac{\partial n}{\partial \vec{x}} - b(E, t) \frac{\partial n}{\partial E} + 3H(t)n - n \frac{\partial b}{\partial E} = \frac{Q(E, t)}{a^3(t)} \delta^3(\vec{x} - \vec{x}_g), \tag{38} \]

where \( n = n(t, \vec{x}, E) \).

Performing the Fourier transformations (19) and (20), and using

\[ \frac{c\vec{e}}{a(t)} \frac{\partial \tilde{n}(t, \vec{x}, E)}{\partial \vec{x}} = \frac{1}{(2\pi)^3} \int d\vec{\omega} \frac{i(\vec{\omega})c}{a(t)} f_\omega(E, t) e^{i\vec{\omega}(\vec{x} - \vec{x}_g)}, \]

we obtain from (38) the following equation for the Fourier transform \( f_\omega(E, t) \):

\[ \frac{\partial}{\partial t} f_\omega(E, t) - b(E, t) \frac{\partial}{\partial E} f_\omega(E, t) + \left[ 3H(t) - \frac{\partial b(E, t)}{\partial E} + i \frac{(\vec{e}\vec{\omega})c}{a(t)} \right] f_\omega(E, t) = \frac{Q(E, t)}{a^3(t)}. \tag{39} \]

With the help of the characteristic equation (22), which gives \( \mathcal{E}' = E'(E, t, t') \), the solution of Eq. (39) is found as

\[ f_\omega(E, t) = \int_{t_g}^t dt' \frac{Q[E', t']}{a^3(t')} e^{-\alpha_\omega(E,t,t')}, \tag{40} \]

with

\[ \alpha_\omega(E, t, t') = \int_{t'}^t dt'' \left[ 3H(t'') - \frac{\partial b(E'', t'')}{\partial E''} + i \frac{(\vec{e}\vec{\omega})c}{a(t'')} \right] . \tag{41} \]

To calculate \( \alpha_\omega(E, t, t') \) we use

\[ \int_{t'}^t dt'' H(t'') = \ln \frac{1 + z'}{1 + z}, \]

\[ i(\vec{e}\vec{\omega})c \int_{t'}^t dt'' = i\vec{\omega}[\vec{x}'(t) - \vec{x}'(t')], \]

and \( b(E, t) = H(t)E + b_{\text{int}}(E, t) \). Then it follows

\[ e^{-\alpha_\omega(E,t,t')} = \left( \frac{1 + z}{1 + z'} \right)^2 e^{-i\vec{\omega}(\vec{x} - \vec{x}')} \exp \left[ \int_{t}^{t'} dz'' \left| \frac{d\omega''}{dz''} \right| \frac{\partial b_{\text{int}}(E'', z'')}{\partial E''} \right]. \tag{42} \]

Coming back to \( n(t, \vec{x}, E) \) we have

\[ n(t, \vec{x}, E) = \int \frac{d\vec{\omega}}{(2\pi)^3} e^{i\vec{\omega}(\vec{x} - \vec{x}_g)} \int_{t_g}^t dt' \frac{Q(E', t')}{a^3(t')} \left( \frac{1 + z}{1 + z'} \right)^2 e^{i\vec{\omega}(\vec{x} - \vec{x})} \times \exp \left[ \int_{z}^{z'} dz'' \left| \frac{d\omega''}{dz''} \right| \frac{\partial b_{\text{int}}(E'', z'')}{\partial E''} \right]. \tag{43} \]
Using the Fourier expansion of $\delta$ function and its properties,

$$
\int \frac{d\vec{\omega}}{(2\pi)^3} e^{i\vec{\omega}(\vec{x}' - \vec{x}_g)} = \delta^3[\vec{x}(t') - \vec{x}_g] = \frac{1}{4\pi x_g^2} \delta[x(t') - x_g] = \frac{a(t_g)}{4\pi c x_g^2} \delta(t' - t_g),
$$
we get the solution at $t = t_0$ ($z=0$) and $\vec{x} = 0$

$$
n(t_0, \vec{x} = 0, E) = \int dt' Q(E', t') \frac{a(t_g)}{4\pi c x_g^2} \delta(t' - t_g) \exp \left[ \int_0^{z'} dz'' \left| \frac{dt''}{dz''} \right| \frac{\partial b_{int}(E'', z'')}{\partial E''} \right],
$$
where $a(t')(1 + z') = 1$ is used.

Performing integration over $t'$ and using Eq. (31) we finally have

$$
n(t_0, E) = \frac{Q(E_g, t_g)}{4\pi c x_g^2(1 + z_g)} \frac{dE_g}{dE},
$$
where $x_g$ is the comoving distance to a source.

One can compare $n(t_0, E)$ from Eq. (45) with energy flux $F$ (in erg cm$^{-2}$s$^{-1}$) of photons emitted by a source with luminosity $L$ at comoving distance $x_g$, as given by Eq. (2.42) in the book by Kolb & Turner (1990):

$$
F = \frac{L}{4\pi x_g^2(1 + z_g)^2},
$$
where one factor $(1 + z_g)$ arises from the time dilation and the other one from energy redshift.

In our case $L \to Q(E_g) dE_g$ and $F \to cn(E) dE$, with one factor $(1 + z_g)$ disappearing because we consider the number of particles instead of the luminosity and energy flux.

Therefore, our method passes this test, too.

5. Conclusions

We shall conclude giving the formulae in the form convenient for practical use.

Our basic diffusion equation (18) for ultra-relativistic particles propagating from a single source, reads

$$
\frac{\partial n}{\partial t} - b(E, t) \frac{\partial n}{\partial E} + 3H(t)n - n \frac{\partial b(E, t)}{\partial E} - \frac{D(E, t)}{a^2(t)} \nabla^2 n = \frac{Q(E, t)}{a^3(t)} \delta^3(\vec{x} - \vec{x}_g),
$$
where $n(t, \vec{x}, E)$ is the particle number density per unit energy in expanding volume of the universe, $\vec{x}$ is coordinate corresponding to the comoving distance, $dE/dt = -b(E, t)$ describes...
the total energy losses, which include adiabatic $H(t)E$ and interaction $b_{\text{int}}(E, t)$ energy losses, and $Q(E, t)$ is the generation function, given by the number of particles generated by a single source, located at coordinate $\vec{x}_g$, per unit energy and unit time.

Solution of Eq. (47) can be presented in the spherically-symmetric case as

$$n(x_g, E) = \int_0^{z_g} \frac{dz}{dz} \left| \frac{dt}{dz} \right| Q(E_g(E, z), z) \frac{\exp\left[-\frac{x_g^2}{4\lambda(E, z)}\right]}{[4\pi \lambda(E, z)]^{3/2}} \frac{dE_g}{dE},$$

(48)

where

$$\lambda(E, z) = \int_0^z \frac{dz'}{dz'} \left| \frac{dt'}{dz'} \right| \frac{D(\mathcal{E}', z')}{a^2(z')},$$

(49)

$$\frac{dE_g}{dE} = (1 + z) \exp \left[ \int_0^z \frac{dz'}{dz'} \left| \frac{dt'}{dz'} \right| \frac{\partial b_{\text{int}}(\mathcal{E}', z')}{\partial \mathcal{E}'} \right],$$

(50)

where $\mathcal{E}' = E'(E, z')$ is a characteristic trajectory, which gives energy $E'$ of a particle at epoch $z'$, if this energy is $E$ at $z = 0$; $E_g(E, z)$ has the same meaning. The upper limit $z_g$ in the integral of Eq. (48) is provided by maximum energy of acceleration as $E_g(E, z_g) = E_{\text{max}}$, or by $z_{\text{max}}$, what is smaller.

The solution (48) is intentionally presented in the form similar to the Syrovatsky solution with $\lambda(E, z)$ being an analogue of the Syrovatsky variable. However, this solution cannot be obtained neither from the Syrovatsky solution (2) nor by the Syrovatsky method of solution, which is based on time-independent quantities $\lambda$ and $\tau$ as new variables.

Eq. (48) is convenient for various applications.

In case the sources are distributed homogeneously with density $n_s(x_g)$ the total density of particles $n_{\text{tot}}(E)$ can be found by integration of Eq. (48) over $4\pi x_g^2 dx_g$, with the diffuse flux given by $J(E) = (c/4\pi)n_{\text{tot}}(E)$. In case of discrete distribution of the sources, the density $n_{\text{tot}}(E)$ must be found by summation over sources, like it is done in (Aloisio & Berezinsky 2004). For non-stationary source, being switched on at redshift $z_1$ and switched off at redshift $z_2$, the limits of integration in Eq. (48) must be taken as $z_1$ and $z_2$ with additional acceleration cut on $z_{\text{max}}$ given by equation $E_g(E, z_g) \leq E_{\text{max}}$ (see also Berezinsky et al. (1990b)).

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