Quantum information in loop quantum gravity

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A coarse-graining of spin networks is expressed in terms of partial tracing, thus allowing to use tools of quantum information theory. This is illustrated by the analysis of a simple black hole model, where the logarithmic correction of the Bekenstein-Hawking entropy is shown to be equal to the total amount of correlations on the horizon. Finally other applications of entanglement to quantum gravity are briefly discussed.

I. INTRODUCTION

Loop Quantum Gravity (LQG) is a canonical quantization of General Relativity, which relies on a 3+1 decomposition of space-time [1,2]. It describes the states of 3d geometry and their evolution in time (through the implementation of a Hamiltonian constraint). The states of the canonical hypersurface are the spin networks, which represent polymeric excitations of the gravitational field. Spin networks are also used to describe the states in the path integral approach to the quantization of gravity — the boundary state in quantum geometry [1,2,3].

A spin network is a graph \( \Gamma \) with vertices \( v \) and oriented edges \( e \). The spin network state is the assignment of a SU(2) representation \( V^e \) to each edge \( e \) and a SU(2)-invariant linear map (an intertwiner) \( \mathcal{I}_v : \bigotimes_e \text{ingoing} V^e \to \bigotimes_e \text{outgoing} V^e \) to each vertex \( v \). Denote the Hilbert space of intertwiners at the vertex \( v \) as \( \mathcal{H}_v = \text{Int}(\bigotimes_e \text{ingoing} V^e \to \bigotimes_e \text{outgoing} V^e) \).

A spin network state \( |\Gamma,j,I\rangle \) defines a function of the holonomies along the graph edges \( T_{\Gamma,j,I}[g] \). For a fixed graph and a fixed assignment of the representations we omit their labels and denote a basis state in \( \mathcal{H} \equiv \bigotimes_v \mathcal{H}_v \) as \( |I_1 \cdots I_V\rangle \equiv |\vec{I}\rangle \), where each \( I_v \) enumerates the intertwiners at the vertex \( v \). The corresponding function \( T_{\vec{I}}[g] \) equals to the tensor contraction of the matrix represenations of the group elements \( \bigotimes_e D^e(g_e) \) with the intertwiner \( \bigotimes_v \mathcal{I}^e_v \),

\[
T_{\vec{I}}[g] \equiv \langle g|\vec{I}_1 \cdots \vec{I}_V\rangle \equiv \text{tr} \bigotimes_{v=1}^V \bigotimes_{e=1}^E D^{e,v}(g_e) \cdot \mathcal{I}_v, \tag{1}
\]

where \( E \) is a number of edges and \( V \) a number of vertices of the graph. It is a gauge invariant function, its value is preserved under the (residual) action of the SU(2) gauge group at the graph’s vertices. Such gauge invariant functions, called gauge invariant cylindrical functions, are the wave functions of quantum geometry [1,2,3].

A note on normalization: the intertwiners are normalized as \( ||\mathcal{I}_v|| = 1 \), and the spin network states are normalized to one \( \langle \vec{I} | \vec{J} \rangle = \delta_{\vec{I},\vec{J}} \) by absorbing the factors \( \prod_e 1/\sqrt{d_e} \). An arbitrary pure state is given by \( |\Psi\rangle = \sum_{\vec{I}} c_{\vec{I}} |\vec{I}\rangle \), with \( \sum_{\vec{I}} |c_{\vec{I}}|^2 = 1 \).

Consider a closed connected spin network based on the oriented graph \( \Gamma \) and a bounded connected region \( B \) of this spin network. The interior \( \text{int}(B) \) of \( B \) consists of the set of vertices \( v \in B \) and the edges between them. The exterior \( \text{ext}(B) \) of \( B \) consists of all other vertices. Its boundary \( \partial B \) consists of the set of edges \( e \) such that one of its end vertices is inside \( B \) and the other is outside. The state of \( B \) is the tensor product of all the intertwiners attached to the vertices \( v \in \text{int}(B) \):

\[
\mathcal{H}_B = \mathcal{H}_{\text{int}(B)} \equiv \bigotimes_{v \in \text{B}} \mathcal{H}_v.
\]

The Hilbert space of boundary states \( \mathcal{H}_{\partial B} \) is the space of intertwiners between the representations \( j_e \) attached to the edges crossing the boundary \( \partial B \).

In the simplest coarse-graining procedure one contracts the intertwiners attached to each internal vertex and thus obtains an intertwiner between the edges crossing the boundary \( \partial B \). One can glue these same intertwiners using a non-trivial parallel transport between the internal vertices, i.e., using non-trivial group elements \( g_e \) on each internal edge \( e \in \text{int}(B) \). For an arbitrary set of group elements \( \{g_e, e \in E_B\} \), the parallel-transport dependent boundary state in \( \mathcal{H}_{\partial B} \) is [1,2].
\[ T_B[g_{\epsilon e}^{(0)}] = \int_{SU(2)} dg \, \text{tr}_{e \in B} \left[ \bigotimes_{e \in \partial B} D^{j_e}(g)^{e_e} \bigotimes_{v \in B} D^{j_v}(g_v^{(0)}) \cdot \bigotimes_{I} T_v \right], \]  

(2)

where \( e, \epsilon \in E_B \) is a sign \( \pm \) depending on whether the edge \( e \) is ingoing \((s(e) \notin B)\) or outgoing \((s(e) \in B)\). The trace is taken over all the SU(2) representations \( V_{\gamma} \), labelling the edges that are linked to the vertices of \( B \). The integration over SU(2) ensures that the resulting tensor is SU(2) invariant, thus an intertwiner \([12]\).

Due to the global SU(2) invariance and to the SU(2) invariance of the intertwiners \( T_v \), not every distinct set \( \{g_{\epsilon e} \in E_B\} \in SU(2)^E_B \) leads to a distinct boundary state. To get distinct states one has to quotient by the gauge invariance. The simplest orbit is the one-point orbit defined by \( g_{\epsilon e} = 1 \), \( \forall e \in E_B \), which corresponds to the contraction of the internal intertwiners \([12]\).

Recall now the standard definition of a reduced density operator \([4, 11]\). Consider two subsystems \( A \) and \( B \), \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \), with the direct product basis \( |m\rangle_A \otimes |n\rangle_B \). The corresponding wave functions are \( \langle x|m \rangle = \psi_m(x) \) for \( A \) and \( \langle y|n \rangle = \phi_n(y) \) for \( B \). A generic state \( |\Psi\rangle \in \mathcal{H} \) is given by a linear combination \( |\Psi\rangle = c_{mn} |m\rangle_A \otimes |n\rangle_B \). If the matrix elements of the operator \( O \) are \( \langle mn|O|m'n\rangle = o_{mn} \delta_{m'n} \), then

\[ \langle \Psi|O|\Psi\rangle = \sum_{m,m',n} o_{mn} c_{m'n} = \text{tr} \left( \rho^A_{\Psi} \right), \]

(3)

where the reduced density operator \( \rho^A_{\Psi} \equiv \text{tr}_B \rho_{\Psi} \) is obtained by tracing out the subsystem \( B \), \( \rho_{mn'm'} = \rho_{mn,m'n} \). In the coordinate basis the operator \( O \) is given by

\[ O(x, y; x', y') = o(x, x')\delta(y - y'), \]

(4)

which is the orthonormality of the functions \( \phi_n \) the reduced density operator is

\[ \rho^A_{\Psi}(x, x') = \int dydy' \overline{\Psi}(x, y)\overline{\Psi}(x', y')\delta(y - y') = \sum_{m,m',n} c_{mn} \delta_{m'n} \psi_m(x)\overline{\psi}_{m'}(x'), \]

(5)

with \( \langle \Psi|O|\Psi\rangle = \int dx dx'o(x, x')\rho^A_{\Psi}(x, x') \).

In the language of the intertwiners, a partial tracing does not present any new features. Taking an intertwiner \( |\tilde{I}\rangle = \otimes_{v \in \Gamma[I_v]} |I_v\rangle \) and tracing out \( B \) produces a basis state of \( A \), \( |\tilde{I}_A\rangle = \otimes_{v \in A} |I_v\rangle \in \mathcal{H}_A^B \), with a corresponding expressions for the reduced density matrices of general states.

Expectation values of the operators that depend only on the region \( A \) are calculated according to Eq. \([4]\). For example, the volume operator \([12]\) is a sum of the vertex operators. The isomorphism between the spin network states and zero-angular momentum states of the corresponding abstract spin system leads to an expectation value of the form of Eq. \([3]\). In particular, if \( A \) contains a single vertex \( v = 1 \) and the operator’s matrix elements are \( \langle O_1|\tilde{I}_{\tilde{J}} = o(I_1, J_1) \sum_{I_2 = -2}^{2} \delta_{I_2,j_1} \rangle \), then for a generic pure state \( |\Psi\rangle = \sum_{\tilde{I}} s_{\tilde{I}} |\tilde{I}\rangle \in \mathcal{H}_{1} \) the expectation is

\[ \langle O_1 |\Psi\rangle = \sum_{\tilde{I}, J_1} o(I_1, J_1) s_{\tilde{I}}, s_{\tilde{J}}, s_{\tilde{J},..,s_{\tilde{J},..}}. \]

(6)

On the other hand, the total coarse-graining of \( B \) results in

\[ T_B \equiv \text{tr}_{v \in B} \bigotimes_{v \in \text{int}(B)} D^{j_v}(1) \cdot \bigotimes_{v \in B} \mathcal{H}_{B}, \]

(7)

where \( \text{tr}_{v \in B} \) denotes the summation over pairs of indices pertaining to the vertices in the region \( B \). It turns a normalized basis state \( |\tilde{I}_A\rangle \otimes |\tilde{I}_B\rangle \in \mathcal{H}_A^0 \otimes \mathcal{H}_B^0 \) into a non-normalized state (see \([12]\)) \( |\tilde{I}_A\rangle \otimes |\tilde{I}_B\rangle \in \mathcal{H}_A^0 \otimes \mathcal{H}_B^0 \). It can be decomposed as \( |\tilde{I}_B\rangle = \sum_{v \in B} c_{\tilde{v},o} |\tilde{I}_v\rangle \), where the intertwiners \( T_B,B, o = 1,..,\dim \mathcal{H}_B \), form the orthonormal basis of \( \mathcal{H}_{B} \). A general coarse-graining procedure leads to the same equation, but with the coefficients depending on the holonomies \( \{g_{\epsilon e} \} \in \text{int}(B) \). A pure state \( |\Psi\rangle = \sum_{\tilde{I}} s_{\tilde{I}} |\tilde{I}\rangle \in \mathcal{H}_1^0 \) becomes a pure state

\[ |\Psi_{(B)}\rangle = \sum_{\tilde{I}_B, \tilde{I}_A,a} s_{\tilde{I}_B} \tilde{I}_A^{a} \tilde{I}_B^{a} |\tilde{I}_A^{a} |\tilde{I}_B^{a} \rangle / \| |\tilde{I}_B^{a} \|, \]

(8)

in \( \mathcal{H}_{B} \otimes \mathcal{H}_A^0 \).

Cylindrical functions do not allow a natural separation of variables in \( O_1(g, g') \) that is analogous to Eq. \([4]\), since there is no relation between the number of the intertwiners \( I_v, v = 1,..,|V| \), that define the tensor product structure of \( \mathcal{H}_A^0 \), and the number of edges, that define the structure of \( H_T = L^2(SU(2)^E/SU(2)^V) \). To obtain a wave functional that corresponds to \( |\tilde{I}_A\rangle \) one needs to contract with the representations \( D_{j} \) that corresponds to all the edges of the coarse-grained graph \( \Gamma[B] \). Hence the coarse-graining procedures that were described above allow to introduce the reduced subsystems in the language of cylindrical functions. From the definition of a total coarse-graining of a spin network state \( \phi_{\tilde{I}}(g) = \langle g|\tilde{I}_A|\tilde{I}_B \rangle \) over a region \( B \) results in \( \phi_{\tilde{I}[B]}(g) \equiv \langle g|\tilde{I}_A \tilde{I}_B \rangle \). It is given explicitly by

\[ \phi_{\tilde{I}[B]}(g) = \text{tr} \bigotimes_{v \in \text{int}(B)} D^{j_v}(g_v) \cdot \mathcal{M}, \]

(9)
with
\[ M = \text{tr}_{v \in B} \bigotimes_{e \in \text{int}(B)} D^{j_e}(1) \bigotimes_{e \in \Gamma} \mathcal{I}_e / \sqrt{||I_B||}. \] (10)

Consider for simplicity a pure state \(|\Psi\rangle \in \mathcal{H}_r^0\). Mixed states are treated analogously. From the above facts it follows \([12]\) that the coarse-grained states \(\phi_{\mathcal{J}}(g)(g)\) play a role of the basis wave functions \(\varphi_m(g)\) in the standard partial tracing. In particular, a “coordinate” expression of the reduced density matrix \(\rho^A_{\mathcal{I}_A,\mathcal{J}_A} = \sum_{\mathcal{I}_B} s_{\mathcal{I}_A,\mathcal{I}_B} \bar{s}_{\mathcal{J}_A,\mathcal{J}_B} \) is given by

\[ \rho^A_{\mathcal{I}_A,\mathcal{J}_A} = \sum_{\mathcal{I}_A,\mathcal{J}_A} s_{\mathcal{I}_A,\mathcal{I}_B} \bar{s}_{\mathcal{J}_A,\mathcal{J}_B} \phi_{\mathcal{I}B}(g)\phi_{\mathcal{J}B}(g') / ||I_B||, \] (11)

which is equivalent to

\[ \rho^A_{\mathcal{I}_A,\mathcal{J}_A}(g, g') = \sum_{\mathcal{I}_A,\mathcal{J}_A} s_{\mathcal{I}_A,\mathcal{I}_B} \bar{s}_{\mathcal{J}_A,\mathcal{J}_B} \phi_{\mathcal{I}B}(g)\phi_{\mathcal{J}B}(g'), \] (12)

while the reduced functional matrix elements of \((O_1)\) are

\[ O^A_1(g, g') = \sum_{\mathcal{I}_A,\mathcal{J}_A} \phi_{\mathcal{I}B}(g)O(\mathcal{I}_1, \mathcal{J}_1)\phi_{\mathcal{J}B}(g'), \] (13)

where \(|\mathcal{J}\rangle \equiv |\mathcal{J}_1\mathcal{J}_2\ldots\mathcal{J}_\mathcal{V}\rangle\).

### III. BLACK HOLE ENTROPY

A generic surface on a spin network background is thus described as a set of patches, each punctured by a unique link of the spin network. The spin network defines how the patches, and therefore the whole surface, is embedded in the surrounding 3d space and describes how the surface folds. For a closed surface, the region of the spin network which is inside the surface defines an intertwiner between the patches of the surface.

An important remark is that any spin \(j\) representation \(V^j\) can be decomposed as a symmetrized tensor product of \(2j\) spin-\(\frac{1}{2}\) representations \(V^{1/2}\). Therefore, one can interpret that a fundamental patch or elementary surface is a spin-\(\frac{1}{2}\) representation. All higher spin patches can be constructed from such elementary patches. For example, considering two spin-\(\frac{1}{2}\) patches, they can form a spin 0 representation or a spin 1 representation: in one case, the two patches are folded one on another and cancel each other, while in the later case they add coherently to form a bigger patch of spin 1. Considering an arbitrary surface, one can then look at it at the fundamental level, decomposing it into spin-\(\frac{1}{2}\) patches, or one can look at it at a coarse-grained level decomposing the same surface into bigger patches of spin \(s > \frac{1}{2}\). From this point of view, the size of the patches used to study a surface is like the choice of a ruler of fixed size used by the observer to analyze the properties of the object. Thus one can study the coarse-graining or renormalisation of these quantities when one observes the surface at a bigger scale, using bigger patches to characterize the surface \([8]\).

Considering the horizon as a closed surface the interior of the black hole is described by (a superposition of) spin networks with their edges puncturing the horizon and defining the patches of the horizon surface. For an external observer only the horizon information is relevant. For him the bulk spin network is fully coarse-grained and the state of the horizon surface is the intertwiner (invariant tensors) between the representations \(V^0\) invariant states on the space of \(2\) qubits (spin-\(\frac{1}{2}\) states) with the horizon area \(A = a_1/a_2\). The ignorance of a particular microstate makes the statistical state under consideration be the maximally mixed state \(\rho\) on the space of intertwiners \(\mathcal{H}_0\). Its orthogonal decomposition is simply

\[ \rho = \frac{1}{N} \sum_{i} |\mathcal{I}_i^r\rangle \langle \mathcal{I}_i^r|, \] (14)

where \(|\mathcal{I}_i^r\rangle\) form a basis of \(\mathcal{H}_0\) and \(N \equiv \dim \mathcal{H}_0\). A straightforward calculation results in the Bekenstein-Hawking entropy and its logarithmic correction:

\[ S = -\text{tr} \rho \log \rho = \log N - 2n \log 2 - \frac{3}{2} \log n. \] (15)
In a coarse-grained model of a black hole one considers horizon states to be given by intertwiners between $m$ representations of a fixed spin $s$, so the horizon area is $A = a_s m$. The space $\mathcal{H} = (V^s)^\otimes m$ decomposes as

$$
\bigotimes_{j=0}^{m} C^s \cong \bigoplus_{j=0}^{m} \mathcal{H}^j \equiv \bigoplus_{j} V^j \otimes \sigma^s_{m,j},
$$

where $V^j$ is the irreducible spin-$j$ representation of $SU(2)$, and $\sigma^s_{m,j}$ is the degeneracy subspace. From the asymptotic form of the multiplicities $c^s_{m,j} = \dim \sigma^s_{m,j}$ it follows that

$$
S \sim m \log s - \frac{3}{2} \log m.
$$

### IV. ENTANGLEMENT AND CORRELATIONS

Entanglement can be loosely defined as an exhibition of stronger-than-classical correlations between the subsystems. Recently it became one of the main resources of quantum information theory [3, 10]. We are interested to find how much entanglement is contained in arbitrary bipartite splittings of a horizon state. For a pure state $|\Psi\rangle$ there is a unique measure of entanglement—the degree of entanglement which is the entropy of either of the reduced density matrices $\rho^{A,B}$. I use here only one of the measures of the mixed state entanglement, namely the entanglements of formation. It is possible to show that for the states under consideration all measures of entanglement coincide [3]. The entanglement of formation is defined as follows: A state $\rho$ can be decomposed as a convex combination of pure states, $\rho = \sum_{\alpha} w_{\alpha} |\Psi_{\alpha}\rangle \langle \Psi_{\alpha}|$. The entanglement of formation is the averaged degree of entanglement of the pure states $|\Psi_{\alpha}\rangle$ (the von Neumann entropy of their reduced density matrices) minimized over all possible decompositions

$$
E_F(\rho) = \inf_{\{\Psi_{\alpha}\}} \sum_{\alpha} w_{\alpha} S(\rho_{\alpha}).
$$

To simplify the notation consider again the qubit model of a black hole. Let the $2n$ qubits be divided into the groups of $2k \leq n$ and $2n - 2k$ qubits. The corresponding Hilbert spaces are $\mathcal{H}_A \equiv (\mathbb{C}^2)^{\otimes 2k}$ and $\mathcal{H}_B \equiv (\mathbb{C}^2)^{\otimes 2n - 2k}$, respectively. Using the decomposition of Eq. 10 twice, the intertwiner space can be decomposed as follows:

$$
\mathcal{H}^0 = V^0 \otimes \sigma_{n,0} = \bigoplus_{j=0}^{k} V_{(j)}^0 \otimes (\sigma_{k,j} \otimes \sigma_{n-k,j}),
$$

where $V_{(j)}^0$ is the singlet state in $V^j \otimes V^j$. Hence the dimensionality of $\mathcal{H}^0$ is related to the multiplicities of the degeneracy subspaces through $N = c_{2n,0} = \sum_{j=0}^{k} c_{2k,j} c_{2n-2k,j}$.

The basis states of Alice and Bob are respectively labeled as $|j, m, a_j\rangle$ and $|j, m, b_j\rangle$. Here $0 \leq j \leq k (\leq n-k)$ and $-m \leq j \leq m$ have their usual meanings and the degeneracy labels, $a_j$ and $b_j$, enumerate the different subspaces $V^j$.

$$
|I^{a_j,b_j}\rangle \equiv \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-1)^{j-m} |j, -m, a_j\rangle \otimes |j, m, b_j\rangle
$$

It is possible to show that the entanglement of formation of the state $\rho$ is:

$$
E_F(\rho) = \frac{1}{N} \sum_{j=0}^{k} c_{2k,j} c_{2n-2k,j} \log(2j + 1),
$$

which is true for any spin $s \geq \frac{1}{2}$. In the large $n$ limit the case of Alice and Bob having $n$ qubits each is especially interesting. In this case

$$
E_F(\rho|n:n) \sim \frac{1}{2} \log n,
$$

which is again true for any $s \geq \frac{1}{2}$.

More generally, we find that for all bipartite splittings of the spin network with sufficiently large number of qubits comprising the smaller space, and for any $s \geq \frac{1}{2}$, it is possible to show that the quantum mutual information between the black hole horizon and its parts is three times the entanglement between the halves,

$$
I_{a}(A : B) \equiv S(\rho_A) + S(\rho_B) - S(\rho) \simeq 3S_F(\rho|A:B).
$$

In particular, if the ratio between the number of qubits is kept fixed while $n$ is arbitrary, the logarithmic correction $\frac{1}{2} \log n$ asymptotically equals to $I_{a}(A : B)$, so the deviation of the black hole entropy from its classical value equals to the total amount of correlations between the halves of spin networks that describe it [3]. Hence in a model where the black hole horizon would be constructed out of independent uncorrelated qubits, the entropy would scale linearly in the number of qubits $2n$. However, the requirement of invariance under $SU(2)$ creates correlations between the horizon qubits, which are revealed through the logarithmic correction $\frac{1}{2} \log n$ to the entropy law formula.

Returning to the qubit black hole it is interesting to note that a fraction of unentangled states in Eq. 20 when a pair of qubits is segregated from the rest is $(s_0^2)^{2n-2k} \sim \frac{1}{2} + \frac{3}{2n}$. It leads to an interesting coincidence with the evaporation model [13] and allows to speculate about corrections to it.

Moreover, using the relations between coarse-graining and partial tracing it is possible to investigate the entanglement in spin-networks [12] and its possible role in the emergence of classical geometry.

Finally, let me mention the “information loss paradox” [14]. While it is not obvious that the unitarity must persist in the process of creation and evaporation of black holes, consideration of the matter alone is not sufficient to convincingly preserve it [14]. Entanglement between
gravitational and matter degrees of freedom offers the way to restore it. In the simplest scenario initially the spacetime is approximately flat and the matter is in some state $\rho$. We describe it by a state $\Phi$ that corresponds to a classical nearly Minkowski metric. The evolution that ends in the black hole evaporation is unitary and is schematically described as $\Xi = U(\Phi \otimes \rho)U^\dagger$, where $\Xi$ is the final entangled state of matter and gravity. Reduced density operators give predictions for the gravitational background and the matter distribution on it. The evolution of matter is obtained by tracing out the gravitational degrees of freedom and is a completely positive non-unitary map [10]. If we assume that the initial states are pure, then the entropy of a reduced density operator is exactly the degree of entanglement between matter and gravity, $E(\Xi)$. Hence, the increase in the entropy of matter is not an expression of information loss, but a measure of the created entanglement, i.e. redistribution of information.

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