A Simple Five-Dimensional Wave Equation for a Dirac Particle

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A first-order relativistic wave equation is constructed in five dimensions. Its solutions are eight-component spinors, which are interpreted as single-particle fermion wave functions in four-dimensional spacetime. Use of a “cylinder condition” (the removal of explicit dependence on the fifth coordinate) reduces each eight-component solution to a pair of degenerate four-component spinors obeying the Dirac equation. This five-dimensional method is used to obtain solutions for a free particle and for a particle moving in the Coulomb potential. It is shown that, under the cylinder condition, the results are the same as those from the Dirac equation. Without the cylinder condition, on the other hand, the equation predicts some interesting new phenomena. It implies the existence of a scalar potential, and for zero-mass particles it leads to a four-dimensional fermionic equation analogous to Maxwell’s equation with sources.

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I Introduction

The Dirac equation correctly describes the behaviour of a single relativistic fermion. However, it is hard to solve and also hard to visualise. In this paper, we study a mathematically somewhat simpler (but five-dimensional) first-order linear equation, and show that it is entirely equivalent to Dirac's.

Several previous authors, beginning with Dirac himself [1], have considered five-dimensional generalisations of the one-body Dirac equation. In most cases (e.g. [2-6]), these authors have used a spacelike fifth dimension. This choice of metric has been the usual one for higher dimensional physical models since the time of Kaluza, in whose unified theory of gravity and electromagnetism it is in fact a necessary assumption if Maxwell's equations are to have the correct sign [7].

If we avoid Kaluza's interpretation of the fifth component of momentum as charge, we are free to use a second time-dimension rather than a fourth space-dimension. This has been done in classical general relativity by numerous authors (e.g. Kocinski [8]). In the “induced matter” version of Kaluza theory developed by Wesson [9-12], Ponce de Leon [13, 14], and others, ordinary massive particles in four-dimensional space-time are treated as massless neutrinos in a curved, non-compact five-dimensional space equivalent to the “bulk” space of membrane theory. Induced matter has been discussed in terms of both one-time and two-time metrics.

Also relevant is the work of Bars and his collaborators [15-17], who have proposed that the Standard Model is simply a gauge-fixed form of some two-time (and four-space) theory. Their six-dimensional approach, although developed independently, in some ways closely parallels an earlier one originating with Dirac [18-21]. For Dirac, the goal was to explain four-dimensional physics in terms of conformal geometry; for Bars, it is to reveal different four-dimensional dynamical systems as “holographic views” of the same six-dimensional system differing in gauge choice.

Like Dirac and Bars, we choose the new dimension to be timelike, but retain only the conventional three space dimensions, thus abandoning the conformal and holographic interpretations of the higher-dimensional manifold. In our model, the wave function of a single fermion is represented by a spinor with eight complex components. A first order linear wave equation in five-dimensional spacetime governs the behaviour of the wave function. We then introduce a constraint in the spirit of Kaluza's “cylinder condition” which prevents the fifth dimension from appearing explicitly in the final results. With this constraint in place, the eight-component spinor wave function reduces to two coupled four-component spinors, both of which obey the ordinary four-dimensional Dirac equation. When the cylinder condition is not imposed, the five-dimensional wave equation may be viewed as a pair of coupled four-dimensional equations, with possible new physical effects implied by the coupling. For example, we will find that when one of the two coupled spinors is held constant over a four-dimensional region, the other behaves like a conventional Dirac wave function in the presence of a scalar potential.
Eight-component spinors, or at least pairs of four-component spinors, and double copies of the Dirac equation, have occasionally been used (e.g. by Joyce [22]) even without any reference to higher dimensional space. The earliest example seems to be the work of Lanczos [23, 24], who in 1929 proposed two coupled four-dimensional quaternionic “wave equations” which he showed were equivalent to two independent copies of the Dirac equation. We will find that the somewhat cumbersome equations discovered by Lanczos are a limiting case of our simpler five-dimensional equation.

The paper is organised as follows:

In Section II, we review the standard geometric algebra approach to the four-dimensional Dirac equation. In Section III, we extend this approach to five dimensions, presenting our new wave equation and showing that it reduces to the standard four-dimensional free particle Dirac equation when a “cylinder condition” is used to eliminate explicit dependence on the second time dimension. In Section IV, we find plane-wave solutions of this equation which correspond to Dirac free particles. In Section V, as an example of our approach, we solve the new wave equation with the cylinder condition for the case of a Coulomb potential, obtaining the standard hydrogen atom spectrum. Finally, in Section VI, we relax the cylinder condition and find that the five-dimensional equation predicts several interesting new effects, notably the existence of a scalar field.

II Geometric Approach to the Dirac Equation

Throughout this paper, we use Clifford (geometric) algebra techniques to handle vectors and spinors; geometric algebra and its applications to physics are exhaustively reviewed in [25-30]. The Clifford algebra describing a flat spacetime with \( m \) positive-norm and \( n \) negative-norm unit vectors is called \( \mathcal{C}\ell(m, n) \); for example, the Clifford algebra associated with Minkowski space is \( \mathcal{C}\ell(3, 1) \). The appropriate Clifford algebra for our five-dimensional case is \( \mathcal{C}\ell(3, 2) \).

Note that we are using the “++−−” convention for the metric of flat spacetime:

\[
dx^A dx_A = dx^\mu dx_\mu - (dx^4)^2 = dx^i dx_i - (dx^0)^2 - (dx^4)^2 = ds^2
\]

The point is significant, because \( \mathcal{C}\ell(n, m) \) and \( \mathcal{C}\ell(m, n) \) are not generally isomorphic. (In Eq. (1) Latin lower-case indices run over the space coordinates from 1 to 3. Greek lower-case indices run from 0 to 3, where \( x^0 \) is the ordinary time. Latin upper-case indices run from 0 to 4, where \( x^4 \) is the new, “extraordinary” time dimension. We employ “natural” units in which \( c \) and \( \hbar \) are 1.)

In four dimensions, the traditional covariant matrix formulation of the Dirac equation with the ++−− metric [31]

\[
\gamma^\mu \partial_\mu |\Psi> = -m |\Psi>
\]  

(where the \( \{\gamma^\mu\} \) are the Dirac gamma matrices, \( |\Psi> \) is the state vector, and \( m \) is mass) can be translated into a matrix-free expression in the language of
geometric algebra by a two-step process. First one places the real and imaginary parts of the four components of the column-spinor $|\Psi>$ in one-to-one correspondence with the eight real components of $\Psi$, a general even-grade multivector of $\mathbb{C}\ell(3,1)$. Then one directly computes the effect of matrix multiplication by $\gamma^\mu$ on $|\Psi>$ in some given representation and seeks a Clifford algebra operator having the same effect on the multivector $\Psi$. In the Pauli-Dirac representation of the gamma matrices, it can be readily verified that the effect of the $\gamma^\mu$ on a column matrix are the same as that of the operator which first left-multiplies the multivector $\Psi$ by $e^\mu$ and then right-multiplies it by $e^0e_1e_2$. (Here the $\{e_\mu\}$ are unit vectors in Minkowski spacetime.) Replacing the gamma matrices in the conventional Dirac equation by their Clifford algebra equivalents leads immediately to the so-called “Hestenes form” of the Dirac equation [30]:

$$e_\mu \partial^\mu \Psi = m\Psi e_0 e_1 e_2$$  \hspace{1cm} (3)

Despite its appearance, this representation of the Dirac equation is covariant and completely interchangeable with the matrix version, as discussed in [32-36].

(We note in passing that, as shown by Lounesto [34], Eq. (3) also holds in the $- - + +$ metric, the lack of isomorphism between $\mathbb{C}\ell(3,1)$ and $\mathbb{C}\ell(1,3)$ notwithstanding. The reason for this is that the appropriate Clifford equivalent of $\gamma^\mu |\Psi>$ in this metric turns out to be $e^\mu \Psi e^0$. Because the Dirac equation in the $- - + +$ metric has a factor of $i$ in the right-hand side of Eq. (2) and $i|\Psi>$ corresponds to $\Psi e_1 e_2$ in either metric [26], the final result is once again Eq. (3). Nearly all authors who have previously studied Eq. (3) have worked in the $- - + +$ metric.)

Eq. (3) involves both right- and left-multiplication, somewhat complicating its solution. The main advantage of Eq. (3) over Eq. (2) is not so much computational as conceptual. In particular, Eq. (3) highlights the often-overlooked correlation between the dimensionality of spacetime and the number of components possessed by a spinor.

That such a correlation exists is suggested by the fact that Dirac spinors, appropriate to four-dimensional spacetime, have four complex components; non-relativistic Pauli spinors, appropriate to three-dimensional space, have two; and the simple Schrödinger wave function without spin has only one. In the matrix approach this trend is without obvious explanation, but it follows naturally from the geometric algebra approach to quantum theory. In the general $(m+n)$ dimensional case, a complex column-vector with $2^{m+n-2}$ components can always be placed in one-to-one correspondence with an even real element of $\mathbb{C}\ell(m,n)$. Thus, if we assume that a spinor is by definition an even element of the algebra, the dimensionality of its column-vector representation follows automatically [37].

It follows that in five dimensions we expect to write the wave function as an eight-component complex-valued spinor (or, equivalently, as a sixteen-component real-valued even element of the algebra). If this wave function is to represent a single particle, we must either introduce a restriction which eliminates half of the components or else accept the existence of two distinct classes
of fermions. We will return to this issue later.

The appropriate five-dimensional generalisation of the various terms in Eq. (3) depends on the value of $k$ in $C\ell(5-k,k)$, that is, on the number of timelike dimensions. In Section III below we will see that for $k = 2$ it is possible to find a wave equation which, at least in the free particle case, involves only left-multiplication, thus avoiding one of the disadvantages of Eq. (3).

### III Five-Dimensional Wave Equation for a Free Particle

Let $E$ be the pseudoscalar “volume” element of $C\ell(3,2)$:

$$E = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

where $\{e_A\}$ are unit vectors. Note that $E^2 = 1$. Because $E$ is pseudoscalar, it commutes with every even-grade element of the algebra; because the dimension of the spacetime happens to be odd, it also commutes with every odd-grade element. This contrasts sharply with the four-dimensional case, in which there exists no non-scalar element commuting with every multivector.

Consider the first-order wave equation:

$$e_A \partial^A \Phi = -Em\Phi$$

where $\Phi$ is an even element of $C\ell(3,2)$, corresponding to an eight component complex spinor in the more usual matrix notation. The relativistic invariance of Equation (5) is evident. Only left-multiplication is used, although because $E$ and $\Phi$ commute, we could also have written the right side of (5) as $-m\Phi E$. Eq. (5) is our proposed five-dimensional field equation for the free particle, which we wish to prove equivalent to Dirac’s.

Note that if we left-multiply both sides of Eq. (5) by $e_B \partial^B$ we obtain the five-dimensional Klein-Gordon equation. Here we see the significance of using two times instead of four space-coordinates: if we had chosen $C\ell(4,1)$, $E^2$ would be $-1$ and the Klein-Gordon equation would have the wrong sign.

We will now demand that

$$\partial^4 \Phi = 0$$

that is, that there be no explicit dependence of $\Phi$ on the new “extraordinary” time coördinate. (This requirement is of course reminiscent of the “cylinder condition” of Kaluza’s unified theory [7], and we will call it by that name for convenience; note however that Kaluza’s fifth dimension was space-like.) We claim that, with this restriction, the pseudoscalar field equation decomposes into two independent copies of the usual Dirac equation.

To show this, we will project Eq. (5) onto ordinary Minkowki space. Consider the operation $(\cdot)_\pm$, defined by
This operation separates those Clifford blades in \( \Phi \) which contain factors of \( e^4 \) from those which do not. Similar "rejection/projection" operators are used routinely in Clifford algebra, and their properties are well-known [30]. In our case, since \( \Phi \) is an even-grade element of \( \text{Cl}(3,2) \), it is easy to show that \( \Phi^+ \) contains no terms of the form \( e_\mu \wedge e_4 \), while \( \Phi^- \) contains only such terms – hence the names "rejection" and "projection", meaning of \( \Phi \) onto \( e_4 \).

From the definition (7) and the fact that
\[
e^4 \wedge e_\mu = -e_\mu \wedge e^4
\]
we see that
\[
(e_\mu \Phi)_{\pm} = e_\mu \Phi_{\mp}
\]
Applying the \((\cdot)_{\pm}\) operator to Eq. (5) and using Eq. (9), we find (after recalling that the pseudoscalar \( E \) commutes with every spinor)
\[
e_4 \partial^4 \Phi_{\pm} + e_\mu \partial^\mu \Phi_{\mp} = -m \Phi_{\pm} E
\]
which can be viewed as a pair of coupled equations relating \( \Phi^+ \) to \( \Phi^- \).

In fact, if we were to drop the first term (as we eventually will when we invoke the cylinder condition) and replace \( E \) by \( \pm i \) we would have the two coupled field equations suggested by Lanczos [23] in 1929. Lanczos of course made no reference to the fifth dimension, and therefore had no simple underlying equation like Eq. (5). He interpreted his two fields, equivalent to our \( \Phi^+ \) and \( \Phi^- \), as independent four-component spinors.

We now define
\[
\Xi \equiv \Phi(1 - e_3 e_4)
\]
It will prove significant that \( \frac{1}{2}(1 - e_3 e_4) \) is idempotent; Lanczos also used idempotent multipliers in the process of going from his field equations to the conventional Dirac equation.

Eq. (11) can be rewritten as
\[
\Xi_{\pm} \equiv \Phi_{\pm} - \Phi_{\mp} e_3 e_4
\]
Inserting Eq. (12) into Eq. (10) and right-multiplying by \( e_3 e_4 \) produces:
\[
e_3 \partial^4 \Xi_{\pm} + e_\mu \partial^\mu \Xi_{\mp} = m \Xi_{\mp} E e_3 e_4
\]
Now assume the “cylinder” condition Eq. (6) holds: \( \Phi \) and therefore also \( \Xi_{\pm} \), does not depend explicitly on \( x^4 \). We can now drop the first term in Eq. (13), which represents the coupling between \( \Xi_{\pm} \) and \( \Xi_{\mp} \). Thus, (writing out \( e_3 e_4 E \) in full and setting \( \Xi_{\pm} = \Psi \)), we are left with two copies of the equation:
\[
e_\mu \partial^\mu \Psi = m \Psi e_0 e_1 e_2
\]
which will be recognised as (3), the free-particle Dirac equation in Hestenes’ form.

We note that Eq. (13) somewhat resembles the spin one-half wave equation of Dirac’s six-dimensional theory [18] mentioned above, in which the higher dimensions are interpreted conformally. In the Clifford approach to conformal geometry [38], idempotents like those in Eq. (11) play an important role. This suggests that the symmetries of the conformal group may underlie the seemingly rather arbitrary relationship Eq. (12) required for our theory to have the correct limit.

We have seen that Eq. (5) is equivalent to the standard free-particle Dirac equation (3). We would now like to find a similar five-dimensional equation which is equivalent to the Dirac equation in the presence of an external vector potential.

To accomplish this, we assume that the equation we seek has the form

\[ e_A \partial^A \Phi = -m \Phi E + q A \Phi \Gamma \]  

(15)

where \( q \) is a scalar charge, \( A \) is a (five-dimensional) vector potential, and \( \Gamma \) is a blade to be determined. We see that \( \Gamma \) must be of even grade, since the grade of each term in the sum must be odd. We will try to find a \( \Gamma \) which, after applying the cylinder condition and setting \( A_4 = 0 \), makes (15) identical to the standard Hestenes-Dirac equation for this case [36]:

\[ e^\mu \partial_\mu \Psi = m \Psi e_3 e_4 E + q A \Psi e_1 e_2 \]  

(16)

Using the rejection/projection operators and assuming the cylinder condition, we obtain from (15)

\[ e_\mu \partial^\mu \Psi = m \Psi e_3 e_4 E + q A [(\Phi \Gamma)_+ - (\Phi \Gamma)_- e_3 e_4] \]  

(17)

For this to agree with Eq. (16), we must have

\[ \Phi_+ - \Phi_- e_3 e_4 = (\Phi \Gamma)_- e_1 e_2 e_3 e_4 - (\Phi \Gamma)_+ e_1 e_2 \]  

(18)

There are two possible solutions, depending on whether \( \Gamma \) does or does not contain a factor of \( e_4 \), viz. \( \Gamma = e_1 e_2 \) and \( \Gamma = -e_1 e_2 e_3 e_4 = e_0 E \)

### IV Plane Wave Representation

It may be seen by substitution that Equation (5) has plane wave solutions of the form

\[ \Phi = \phi (\cos(k^A x_A) + \Gamma \sin(k^A x_A)) \]  

(19)

where \( \phi \) is a constant spinor amplitude and \( \Gamma \) is any blade such that \( \Gamma \Gamma = -1 \) and

\[ k^A e_A \Phi \Gamma = -m E \Phi \]  

(20)
We want these waves to represent the free Dirac particle, i.e. for the \( \Psi \) derived from Eq. (19) to be the same (after the cylinder condition has been applied) as that for the ordinary Dirac plane wave [39]

\[
\Psi = \psi(\cos(k^\mu x_\mu) + e_1 e_2 \sin(k^\mu x_\mu))
\]

(21)

Using the rejection/projection operations on Eq. (19) and noting that by the cylinder condition \( k^4 = 0 \), we find the following two criteria

\[
\psi = \phi_+ - \phi_- e_3 e_4
\]

(22)

and

\[
\psi e_1 e_2 = (\phi \Gamma)_+ - (\phi \Gamma)_- e_3 e_4
\]

(23)

that is:

\[(\phi_+ - \phi_- e_3 e_4)e_1 e_2 = (\phi \Gamma)_+ - (\phi \Gamma)_- e_3 e_4 \]

(24)

Now it is evident that \( e_4 \) can occur at most once in the spinor (that is, in the even grade blade) \( \Gamma \). Let us first assume that \( e_4 \) is a factor of \( \Gamma \), so that there is an odd number of factors of the type \( e_\mu \). Then by Eq. (9):

\[
(\phi \Gamma)_\pm = \phi_\mp \Gamma
\]

(25)

Inserting this into Eq. (15), we find that

\[
\Gamma = -e_1 e_2 e_3 e_4 = e_0 E
\]

(26)

while Eq. (20) becomes

\[
k^A e_A \Phi = m \Phi e_0
\]

(27)

But this is not the only solution! We could also have assumed that \( \Gamma \) is composed entirely of vectors orthogonal to \( e_4 \). In that case, Eq. (25) must be replaced by

\[
(\phi \Gamma)_\pm = \phi_\pm \Gamma
\]

(28)

and we see that

\[
\Gamma = e_1 e_2
\]

(29)

Similarly Eq. (26) becomes

\[
k^A e_A \Phi = -m \Phi e_0 e_3 e_4
\]

(30)

If we insert a completely general spinor \( \Gamma \) into Eq. (15), we find that

\[
\Gamma_+ - \Gamma_- e_3 e_4 = e_1 e_2
\]

(31)

Evidently, when \( \Gamma_- = 0 \), we are left with \( \Gamma_+ = e_1 e_2 \), and when \( \Gamma_+ = 0 \), we are left with \( \Gamma_- = e_0 E \). We must therefore in general set \( \Gamma \) equal to a linear
combination of the two values given in Eqs. (26) and (29). Inserting such a superposition into Eq. (31) reveals that the two scalar coefficients add up to unity, and thus that we may think of them as the squared sine and cosine of some phase angle \( \theta \):

\[
\Gamma = e_1 e_2 (\cos^2 \theta - e_3 e_4 \sin^2 \theta)
\]  

(32)

Such a \( \Gamma \) satisfies Eq. (21) as expected. However, it will be recalled that \( \Gamma \Gamma = -1 \). This condition and Eq. (32) can hold simultaneously only if \( \theta \) is an integer multiple of \( \pi/2 \), that is to say, if \( \Gamma \) takes one of the two values specified in Eqs. (26) and (29).

Thus, we have two separate classes of plane waves, both of which, thanks to the cylinder condition, correspond to the same ordinary Dirac plane waves. It will be noted that the values of \( \Gamma \) are the same which arose in the treatment of the vector potential above.

V  The Coulomb Potential

Consider an electron moving in a spherically symmetric external potential

\[
qA = -\frac{\lambda}{|r|} e_0
\]  

(33)

where \( \lambda \) is a scalar constant. Then, with the cylinder condition, Eq. (15) becomes:

\[
e_\mu \partial^\mu \Phi = -m \Phi E - \frac{\lambda}{|r|} e_0 \Phi \Gamma
\]  

(34)

The only specific properties of \( \Gamma \) we will need in this section are that \( \Gamma \Gamma = -1 \) and \( \Gamma e_0 = e_0 \Gamma \). Eq. (34) may be rewritten as:

\[
\nabla \Phi - e_0 \partial_t \Phi = -m \Phi E + \frac{\lambda}{|r|} \zeta \Phi e_0 \Gamma
\]  

(35)

where

\[
\zeta F = e_0 F e_0
\]  

(36)

for any multivector \( F \). Note that \( \zeta \zeta F = F \).

As in the usual four-dimensional Dirac Coulomb problem, we will assume that the energy \( \varepsilon \) is related to the time-derivative of the state function. In non-relativistic quantum mechanics, we know that:

\[
\varepsilon \Psi = i \partial_t \Psi
\]  

(37)

(with \( \hbar = 1 \)). In the Dirac theory as formulated by Hestenes in the language of geometric algebra [32-36], the imaginary scalar \( \iota \) is replaced by the real bivector \( e_1 e_2 \) (acting on \( \Psi \) from the right); clearly, this squares to \(-1\) as expected. There is no obvious prescription specifying uniquely what substitution to make in the five-dimensional theory, but an obvious choice is \( \Gamma \). This choice of course
coincides with the Dirac-Hestenes choice in the case that $\Gamma$ is given by $e_1 e_2$, but even when $\Gamma$ is given by $e_0 E$ the proposed substitution will give the same final results as the conventional Dirac equation after the cylinder condition is applied. We therefore write:

$$\varepsilon \Phi = \partial_t \Phi \Gamma$$  \hspace{1cm} (38)

Substituting this into Eq. (35) and right-multiplying by $\Gamma e_0$, we get

$$\eta \nabla \Phi - \varepsilon \zeta \Phi = -m \eta \Phi E + \frac{\lambda}{|r|} \zeta \Phi$$  \hspace{1cm} (39)

where

$$\eta F \equiv F \Gamma e_0$$  \hspace{1cm} (40)

for any multivector $F$. Note that $\eta \eta F = F$ and $\eta \zeta F = \zeta \eta F$.

To solve Eq. (40), we employ a geometric method analogous but not identical to that used by Temple and Eddington [36, 40, 41] to solve the conventional four-dimensional Coulomb problem.

We make the assumption (justified by the cylinder condition and the requirement that the final result reduce to Dirac’s) that, like $\Psi$ in the four-dimensional case, $\Phi$ is an eigenfunction of the relativistic angular momentum operator with eigenvalue $\kappa$. Therefore [36]:

$$r \nabla \Phi = (r \mathbf{\cdot} \nabla + 1 - \kappa \zeta) \Phi$$  \hspace{1cm} (41)

Using the properties of the operators $\zeta$ and $\eta$, we find:

$$\eta r \mathbf{\cdot} \nabla \Phi + \eta \Phi - \kappa \eta \zeta \Phi - \varepsilon r \zeta \Phi = -E m \eta \Phi + \lambda e_r \zeta \Phi$$  \hspace{1cm} (42)

where $e_r \equiv r / |r|$.

To eliminate the second term, we change to a new variable

$$u \equiv |r| \Phi$$  \hspace{1cm} (43)

and introduce two new operators $S$ and $T$ by:

$$SF \equiv (\kappa + \lambda \eta e_r) \zeta F$$  \hspace{1cm} (44)

and

$$TF \equiv E e_r (m - E \varepsilon \zeta) F$$  \hspace{1cm} (45)

for any multivector $F$. These greatly simplifies the appearance of Eq. (41), which becomes just:

$$\partial_r u = \frac{1}{|r|} Su - Tu$$  \hspace{1cm} (46)

We now expand $u$ in powers of $|r|$, and proceed exactly as in the Temple-Eddington method of solving the four-dimensional Coulomb problem:

$$u = \sum_{p=0}^{\infty} |r|^{p+q} e^{(p+q)|r|} C_p$$  \hspace{1cm} (47)
where \( \xi \) is an integer greater than zero, \( q \) and \( \beta \) are scalars to be determined, and \( C_p \) is some constant spinor. Equating coefficients we eventually obtain:

\[
\sqrt{m^2 - \varepsilon^2(\xi + \sqrt{\kappa^2 - \lambda^2})} = \varepsilon \lambda
\]  

(48)

Eq. (48) is standard in four-dimensional Dirac theory, and is easily solved for \( \varepsilon \) to give the usual relativistic energy spectrum [42-44] first derived by Sommerfeld [45]:

\[
\varepsilon = m/\sqrt{1 + \frac{\lambda^2}{(\xi + \sqrt{\kappa^2 - \lambda^2})^2}}
\]  

(49)

Here \( \kappa \) and \( \xi \) are related to the non-relativistic quantum numbers \( n \) and \( j \) by

\[
n = |\kappa| + \xi
\]  

(50)

and

\[
j = |\kappa| - \frac{1}{2}
\]  

(51)

The energy levels of a bound particle in a Coulomb potential are calculated from Eq. (49) and are of course, identical to those obtained from the Dirac equation. Interpreting \( m \) in Eq. (49) as the mass of the electron times \( c^2 \) and \( \lambda \) as the atomic number times the fine structure constant, we may evaluate Eq. (49) numerically for hydrogen-like atoms. Subtracting off the rest mass gives the bound state energy levels, which for hydrogen are easily found to be:

\[
\begin{align*}
1s_{1/2} : \kappa &= -1 : \xi = 0 : \varepsilon = -13.06 \text{eV} \\
2s_{1/2} : \kappa &= -1 : \xi = 1 : \varepsilon = -3.402 \text{eV} \\
2p_{1/2} : \kappa &= 1 : \xi = 1 : \varepsilon = -3.402 \text{eV} \\
2p_{3/2} : \kappa &= -2 : \xi = 0 : \varepsilon = -3.401 \text{eV}
\end{align*}
\]

and so on as found in almost every textbook of relativistic quantum mechanics. However, we have obtained these standard results directly from the five-dimensional Eq. (15), not from the four-dimensional Dirac equation (16).

VI Wave Equation without Cylinder Condition

The method presented here can be viewed in two quite different ways: as a technique for discovering solutions of the conventional Dirac equation, or as a wave equation actually obeyed by fermions, and containing the Dirac equation as a special case.

If one chooses to adopt the first viewpoint, the interest of our approach lies in its use of the pseudoscalar operator \( E \), which commutes with every multivector and thus reduces somewhat the awkwardness of standard four-dimensional Clifford algebra approach. The fifth dimension itself, from this point of view, is
simply part of the formal apparatus; because of the cylinder condition, it plays no role in the final results, which are identical to those of the conventional Dirac equation. Thus, Eq. (15) can be regarded as an auxiliary equation, and solving it as simply a technique for solving the Dirac equation, in which all the actual physics resides. An analogy might be made to the use of complex vectors, the imaginary parts of which are eventually set to zero, to simplify calculations in electromagnetism.

However, the successes of string theory and brane theory [e.g. 46] have made higher-dimensional approaches in which the additional coordinates are physical rather than merely abstract increasingly popular, and thus it is worthwhile to consider the second viewpoint as well: the possibility that Eq. (15) without the cylinder condition is the correct one-body description of a fermion.

The main difficulty with such a position is the discrepancy between the eight components of a spinor in $\text{Cl}(3,2)$ and the observed number of fermionic degrees of freedom. This problem was already encountered in 1929 by Lanczos [23], whose two coupled quaternionic wave equations we have seen to be closely related to Eq. (10). Gürsy [47] much later suggested that the Lanczos equations predict isospin doublets, an interpretation strongly endorsed by Gsponer and Hurni [24]. Each of the two four-component spinors in Lanczos’ theory, according to this explanation, represents a Dirac particle/antiparticle state with isospin up or down. Clearly a similar interpretation could be applied to the two halves of the single eight-component spinor in the present work: recall that the projected parts $\Phi^+$ and $\Phi^-$ reduce to Lanczos’ spinors when the cylinder condition is applied.

Indeed, one would expect either a new class of particles or a new quantum number to arise in going from four spinor components to eight, just as antiparticles arise in going from two components to four and spin in going from one component to two. If Eq. (15) rather than Eq. (16) is the correct description of a fermion, the degeneracy between $\Xi^+$ and $\Xi^-$ is lifted. Then it seems quite possible that each represents a different type of particle (in the way that the “large” and “small” parts of the ordinary Dirac spinor represent particles and antiparticles), or else a previously unrecognised quantum state (in the way that the upper and lower components of the Pauli spinor represent spin-up and spin-down states). Of course, with our cylinder condition enforced, the degeneracy becomes purely formal, and both $\Xi^\pm$ are equivalent.

Similar considerations arise from the fact that two quite different approaches to the vector potential, given by the two choices of $\Gamma$ (26) and (29), correspond to the same four-dimensional result. This is unproblematic if the present method is treated as merely a technique for solving the Dirac equation, but if the fifth dimension is an actual part of physical spacetime, the ambiguity introduces yet another degeneracy. This is true even in the case of unbound particles, because of our freedom to use either Eq. (26) or (29) in the phase. Thus, dropping the cylinder condition, or treating it as only an approximation, would suggest new physics beyond the Dirac equation.

One important prediction of the five-dimensional wave equation (without the cylinder condition) is the existence of a new scalar potential, which arises
in any four-dimensional region over which the $\Xi_-$ part of the wave function is constant. To show this, we first maintain the upper sign in Eq. (13) and impose the constraint

$$\partial^\mu \Xi_- << 1$$

For non-zero mass, we obtain

$$\Xi_- = \frac{1}{m} e_4 \partial^4 \Xi_+ e_0 e_1 e_2$$

Inserting Eq. (53) now into Eq. (13) with the lower sign, we find that

$$-\frac{1}{m} \partial^4 \partial^4 \Xi_+ e_0 e_1 e_2 + e_\mu \partial^\mu \Xi_+ = m \Xi_+ e_0 e_1 e_2$$

The standard four-dimensional Hestenes-Dirac equation with a scalar potential $s$ is given by:

$$-s \Psi e_0 e_1 e_2 + e_\mu \partial^\mu \Psi = m \Psi e_0 e_1 e_2$$

Comparing Eq. (54) to Eq. (55), we observe that the two are identical if we replace $\Xi_+$ by $\Psi$ as we did in Section III and identify $ms$ with the eigenvalue of the second partial derivative of $\Psi$:

$$\partial^4 \partial^4 \Psi = ms \Psi.$$  

Thus the dependence of $\Psi$ on the fifth dimension will manifest itself in four dimensions as a scalar potential. The scalar potential $s$ could perhaps be identified with the Higgs field, or with one of the scalar potentials giving rise to inflationary processes in cosmology. Often the existence of such physically important scalars is merely postulated, whereas in this approach, a scalar potential arises directly and necessarily from the wave equation itself!

This procedure works only in that case of non-zero mass. For $m = 0$, Eq. (13) with the restraint (52) becomes for the upper sign

$$e_4 \partial^4 \Xi_+ = 0$$

i.e., for a massless particle, requiring $\Xi_-$ to be constant over a space-time region is the same as requiring the cylinder condition to hold for $\Xi_+$ in the region.

The equation with the lower sign in the same case is interesting for a different reason: it may be thought of as a four-dimensional Dirac equation with “sources”. The ordinary four-dimensional Dirac-Hestenes equation for a massless particle (i.e. Eq. (3) with the right hand side set to zero) is formally identical to Maxwell’s equation in empty space. Both equations (Maxwell’s and Dirac’s) may be written as:

$$e_\mu \partial^\mu \Psi = 0$$

the only difference being that in the Dirac case $\Psi$ is a general even-grade multivector of the form:

$$\Psi \equiv (\alpha + e_0 e_1 e_2 e_3 \beta) + \frac{1}{2} e_\mu \wedge e_\nu F^{\mu\nu}$$
where \(\alpha, \beta\), and the six \(F^{\mu\nu}\) are the components of the wave function, whereas in the Maxwell case \(\Psi\) is restricted to grade two:

\[
\Psi \equiv \frac{1}{2} e_\mu \wedge e_\nu F^{\mu\nu}
\]

(60)

where \(F^{\mu\nu}\) is the electromagnetic field tensor. However, the Maxwell equation with source current \(J\) given by:

\[
e_\mu \partial^\mu \Psi = -4\pi J
\]

(61)

has no counterpart in the conventional Dirac theory. On the other hand, we have from Eq. (13) with the lower sign and zero mass, writing \(\Psi\) for \(\Xi_+\):

\[
e_\mu \partial^\mu \Psi = -e_4 \partial^4 \Xi_-
\]

(62)

so that formally Eqs. (61) and (62) are identical if

\[
J \equiv \frac{1}{4\pi} e_4 \partial^4 \Xi_-
\]

(63)

The constraint given by Eq. (52) guarantees that \(J\) is a conserved quantity in four dimensions.

It may be recalled that \(\Xi_-\) is the sum of a pseudoscalar term and various bivector terms all containing \(e_4\). Consequently \(e_4 \partial^4 \Xi_-\) (and therefore also \(J\)) is the sum of a pure space-time vector and a trivector term not containing \(e_4\). Thus, just as the four-dimensional massless Dirac equation differs from the vacuum Maxwell equation only by the presence of additional scalar and pseudoscalar terms in the field, so (for zero mass) Eq. (13) with the lower sign differs further from the Maxwell equation with sources only by the presence of an additional trivector term in the source current. It seems reasonable to suppose that in both cases these differences may be related to the contrast between the integer spin of the photon and the half-integer spin of the Dirac particle.

In the Maxwell case, the source currents represent an external charge distribution. Eq. (62), however, indicates that the source currents for Eq. (61) arise from the \(\Xi_-\) part of the wave function itself. This is somewhat reminiscent of de Broglie’s “double solution” approach to the Schrödinger equation [48], which has enjoyed a resurgence of interest in the last decade because of its relationship to Bohmian quantum mechanics [49]. Perhaps Eq. (61) provides a link between the solitons of de Broglie’s theory and the solitons in the five-dimensional induced matter theory of Wesson and Ponce de Leon [9, 13, 14].

VII Conclusion

We have shown that the simple five-dimensional wave equation Eq. (5) is a powerful tool for the study of fermions. When the cylinder condition is enforced, this equation is exactly equivalent to the conventional Dirac equation, but is in some respects more tractable: it thus provides a useful new technique for doing
relativistic quantum mechanics. The two solutions $\Xi_+$ and $\Xi_-$ of the projected five-dimensional equation are degenerate, and either one may be identified with the conventional Dirac wave function.

In the case of a free particle, we have discovered two families of plane-waves given by Eq. (19) and differing only in the two possible choices of $\Gamma$, Eqs. (26) and (29). As long as the cylinder condition holds, there is no way to distinguish these two families, and either one (but not both at once) may be taken as a representation of the free particle. Likewise, in the case of a bound particle, we found the same two choices of $\Gamma$ satisfy Eq. (15). Either selection, with the cylinder condition, splits Eq. (15) into two identical copies of the Dirac equation. Thus, solving (15) is the same as solving the Dirac equation, as long as the cylinder condition is in force, and we may choose that value of $\Gamma$ which is most convenient for a given potential.

When the cylinder condition is not imposed, i.e. when the dependence on the fifth dimension is retained, the number of spinor degrees of freedom is doubled, suggesting the possible existence of a new quantum number. In the limiting case where four of the spinor components are held constant and those remaining are identified with the four-dimensional wave function of a massive fermion, Eq. (5) predicts the existence of a scalar field which (per unit mass) is simply the eigenvalue of the operator $\partial^4\partial^4$. If the particle is instead assumed to be massless, Eq. (5) is formally identical to Maxwell’s equation with sources; such sources are absent in conventional fermionic quantum mechanics. Thus, in both cases, new physical effects arise from the relaxation of the cylinder condition.
VIII References


36. Reference [26], Chapter 8.


