Entanglement purification for high dimensional multipartite systems

Yong Wook Cheong,1 Seung-Woo Lee,1 Jinhyoung Lee,1,2 and Hai-Woong Lee3
1Quantum Photonic Science Research Center, Hanyang University, Seoul 133-791, Korea
2Department of Physics, Hanyang University, Seoul 133-791, Korea
3Department of Physics, Korea Advanced Institute of Science and Technology, Taejon 305-701, Korea
(Dated: December 23, 2005)

If entangled states are transmitted through noisy quantum channel, then the correlation properties of the states can be changed. This fact can be usefully employed to error detection, which is closely linked to entanglement purification protocols (EPPs). We propose new EPPs which extract a generalized GHZ state from ensemble of mixed entangled state with a framework of error detection.

I. INTRODUCTION

Entanglement plays an important role in quantum information theory. Using a maximally entangled state, we can teleport an unknown quantum state perfectly [1], or share a secret key [2]. When the pure entangled state is transmitted through a noisy quantum channel, however, the state becomes mixed. Then, the mixedness makes the fieldity of teleportation degraded and the key unsecure in the sense that the authorized people can not decide whether the disagreement of the shared key originates from eavesdropping or noises in the quantum channel. It is thus the most crucial to share an entangled state without any impurity for the reliable information processes. For the purpose, various entanglement purification protocols (EPPs) have been proposed to obtain almost pure entangled states from the ensembles of mixed states [3, 4, 5, 6, 7, 8, 9, 10, 11].

Multipartite entangled states have intensively been investigated, based on qubits, as quantum information theory is being established. The achievements in these efforts include Greenberger-Horne-Zeilinger (GHZ) nonlocality [12], quantum secret sharing [13], cluster-state computation [14], and quantum error correcting codes (QECCs) [15, 16, 17, 18]. On the other hand, as most of physical systems live in higher-dimension Hilbert spaces, it is desired to examine whether such achievements are reproduced for high d-dimensional systems (so-called qudits) and whether to acquire any advantages over qubits. It was in fact shown that quantum key distribution is more secure in qudits than qubits from the eavesdropping [19]. The generalizations to high dimensions include GHZ nonlocality [20], Mermin-Ardehali Bell inequalities [21], entanglement swapping [22], and QECCs [17, 18].

Here we consider EPPs in high dimension so as to purify multiparticle entangled states of qudits for advantageous high-dimensional processing. EPPs were proposed to distill Bell states from given ensembles of mixed states for two qubits [3, 4] and they were further developed to distill multipartite GHZ states [5], and two-colorable graph state [6, 7]. Further they were generalized for two qudits to distill their maximally entangled states [8]. An EPP for continuous variable was also proposed [9]. Recently an EPP to distill W-state [23] was proposed [10]. The experimental realization of EPPs is also reported [11]. The EPPs proposed so far can be classified into recursive, hash, and breed methods [4]. The recursive method, that will be discussed in this paper, is known less efficient than the hash/breed ones, but it enables a rather simple manipulation and it works very well even for the systems with initially low purities.

If a quantum channel is of substantially weak noise, QECCs may be employed to share a highly pure entangled state at a remote distance [24]. As EPP and QECC both are intended to cure the quantum coherence from quantum errors, one may conjecture a link between them. QECC can in fact be constructed from one-way EPP in which only one directional classical communication is allowed and vice versa [4]. This result was applied to prove the security of quantum key distribution [25]. Two-way EPPs in which bidirectional classical communication is permitted can be obtained from quantum error detecting codes (QEDCs) [26, 27, 28]. In fact, most of recursive two-way EPPs [4, 5, 6, 7, 8], including the one that will be discussed in this paper, can be constructed and analyzed in the framework of QEDCs.

In this paper, we first discuss the correlation properties of (generalized) GHZ states. Then we investigate an error model for high dimensional systems which we are concerned with. Then, by analyzing qualitatively the established EPPs, we propose two new EPPs that purify maximally entangled GHZ states of many qudits and compare their performance.

II. CORRELATION PROPERTY

We start out by discussing the correlation properties of (generalized) GHZ states. Here we consider a composite system of three subsystems, since the generalization to the n-subsystems case is straightforward. Three-qubit GHZ
states are given by

\[ |\phi_{ijk}\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0jk\rangle + (-1)^i|1jk\rangle), \]

where \(|0\rangle = \sigma_z|0\rangle, \ -|1\rangle = \sigma_z|1\rangle\), \(i,j,k \in \{0,1\}\), \(\bar{i},\bar{j},\bar{k}\) are complements of \(j,k\), and \(A, B, C\) represent three parties, Alice, Bob, and Charles. All the GHZ states are joint eigenstates of the correlation operators; \(X^A \otimes X^B \otimes X^C\), \(Z^A \otimes Z^B\), and \(Z^A \otimes Z^C\). Here \(X\) and \(Z\) are Pauli operators \(\sigma_x\) and \(\sigma_z\), respectively. The superscript, \(A, B,\) and \(C\) will be omitted unless any confusions arise. We can also suppose without loss of generality that Alice, Bob, and Charles. All the GHZ states are joint eigenstates of the correlation operators;

where \(|\mathcal{E}\rangle\) is what Alice, Bob and Charles intend to share. Note that \(|\mathcal{E}\rangle\) is a (+1)-eigenstate of \(X^A \otimes X^B \otimes X^C\), \(Z^A \otimes Z^B\), and \(Z^A \otimes Z^C\). The other GHZ states have \((-1)\)-eigenvalue for at least one of \(X^A \otimes X^B \otimes X^C\), \(Z^A \otimes Z^B\), or \(Z^A \otimes Z^C\).

On the other hand, generalized GHZ states are given by

\[ |\psi_{e,f,g}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{ek}|k, k+f, k+g\rangle, \]

where \(e,f,g \in \{0,1,\ldots,d-1\}\) and \(\omega = e^{i2\pi/d}\). Generalized GHZ states are joint eigenstates of the correlation operators; \(X_d \otimes X_d \otimes X_d\), \(Z_d^A \otimes Z_d^B\), and \(Z_d^A \otimes Z_d^C\). Here \(X_d\) and \(Z_d\), \(d\)-dimensional extensions of \(X\) and \(Z\) are given by

\[ X_d = \sum_{k=0}^{d-1} |k\rangle \langle k-1 \ (\text{mod} \ d)|, \]

\[ Z_d = \sum_{k=0}^{d-1} \omega^k |k\rangle \langle k|. \]

The operator \(Z_d^\dagger\) is adjoint to \(Z_d\). Of the generalized GHZ states, \(|\psi_{000}\rangle\) is one that Alice, Bob, and Charles intend to share and it is the only (+1)-eigenstate of \(X_d \otimes X_d \otimes X_d\), \(Z_d^A \otimes Z_d^B\), and \(Z_d^A \otimes Z_d^C\).

\section{Error Model}

Now we discuss the error model we consider. If a quantum state \(\rho\) is transmitted through noisy quantum channel, then the state changes, and the final state \(\rho'\) can be described with Kraus representation as follows

\[ \rho' = \sum_i K_i \rho K_i^\dagger. \]

Here \(K_i\) can be represented as a linear combination of some basis. The most commonly used basis of error operators on a qubit system is given by

\[ \varepsilon_2 = \{I, X, Y (= \sigma_y), Z\}. \]

Here \(X\) and \(Z\) are respectively called a bit flip error and a phase flip error when they are regarded as errors, and they generate the basis. General unitary errors that may occur during the transmission of a qubit can be represented with the error basis. However, they could jump to one of the \(I, X, Y\), and \(Z\), when proper syndrome measurement is performed \cite{12}. Thus, the QECCs (or QEDCs) which overcome bit flip error and phase flip error may overcome more general errors.

A well known error basis on a qubit is given by

\[ \varepsilon_d = \{E_{i,j} = Z_d^j X_d^j |i, j \in \{0, 1, \ldots, d-1\}\}. \]

If \(i \neq 0(j \neq 0)\), we call the error \(E_{i,j}\) as a phase(level) shift error, which is a generalization of phase(bit) flip error. General error occurred on a qubit could be collapsed to one of \(E_{i,j}\) by proper syndrome measurement, thus a QECC which overcome all the error in \(\varepsilon_d\) may overcome general errors.

In this paper, we do not consider general error, rather than we suppose that only errors which are contained in \(\varepsilon_2(\varepsilon_d)\) can occur stochastically. Then initial (generalized) GHZ state \(|\phi_{000}\rangle\langle \psi_{000}|\) can be transformed to one of another (generalized) GHZ states related to the error occurred. Thus the shared state through the noisy channel, \(\rho'\)
becomes a classical mixture of (generalized) GHZ states, that is, $\rho'$ can be diagonalized with the basis of (generalized) GHZ states.

Our restriction about the error is only for the convenience of discussion and it reduce greatly the complexity of calculation which have to be performed. If we suppose general errors, then shared state, $\rho'$ may not be diagonalized with the (generalized) GHZ state basis. However, for all the EPPs considered here, the off-diagonal elements of $\rho'$ are independent of the diagonal ones in our choice of basis. When an EPP works well, the transformed state $\rho''$ must be closed to pure $|\phi_{000}\rangle (|\psi_{000}\rangle)$ state, therefore, one diagonal element of $\rho''$ becomes closed to one. Since off-diagonal elements do not affect diagonal elements, they do not have any influence on the performance of EPPs at all. Therefore, if an EPP works well under our error restriction, then it will do also under the general errors. We note that this kind of error restriction is relevant to the description of EPPs $[3, 5]$.  

IV. EPP FOR GHZ STATE

Keeping correlation properties of the (generalized) GHZ state in mind, we can construct the EPPs which purify the (generalized) GHZ state. Here we discuss the structure of EPPs for GHZ state, then give a simple EPP $[5]$. In the next section we extend the simple EPP to the generalized GHZ state case.

Now consider the transmission of GHZ state, $|\phi_{000}\rangle$. If bit flip errors ($X$) or phase flip errors ($Z$) are occurred during the transmission, the shared entangled state becomes one of another GHZ states. Then the shared state has $(-1)$-eigenvalue for at least one of $X \otimes X \otimes X$, $Z^A \otimes Z^B$, and $Z^A \otimes Z^C$. For example, if a bit flip error occurs on $B$, then the reduced state is $X^B |\phi_{000}\rangle (= |\phi_{000}\rangle)$, which is a $(-1)$-eigenstate of $Z^A \otimes Z^B$. Thus the occurrence of bit-flip error can be detected by the measurements of $Z^A$, $Z^B$, and $Z^C$. If no bit flip error occurs, then all the measurement outcomes must be the same. On the other hand, if a phase flip error occurs on $B$, then the reduced state is $Z^B |\phi_{000}\rangle (= |\phi_{000}\rangle)$, which is a $(-1)$-eigenstate of $X \otimes X \otimes X$. Thus the measurement outcomes of $X^A, X^B,$ and $X^C$ can detect the occurrence of phase flip error. If no phase flip error occurs, then the numbers of the $|1\rangle_z$-outcomes must be even. If $|1\rangle_z$-outcome is even, of the measurement result. If a phase flip error occurred on one qubit of two GHZ states, then the shared state has destroyed after the measurement, other GHZ states, however, could be remained undisturbed.

Now Alice, Bob, and Charles share a set of GHZ states which are prepared as $|\phi_{000}\rangle$ initially (See Fig.1). They perform some measurement to detect unwanted error. If measurement outcomes are affirmative, then they keep some GHZ state from the set. Kept state are purified one, if initial purity is higher than a certain threshold value. Mathematically, EPP for GHZ state can be described using a series of operators $M^A \otimes M^B \otimes M^C$, where each $M_i$ is a direct product of $X, Y, Z,$ and $I$ such as $X^{A_1} \otimes X^{A_2}$. Alice, Bob, and Charles individually measure $M^A = M^B = M^C = M_i$ for the set of GHZ states and announce their measurement outcomes. Some GHZ states are destroyed after the measurement, other GHZ states, however, could be remained undisturbed.

If $M_i$ is a tensor product of $X$ and $I$, then $M_i$ is an observable which detect phase flip errors $[10]$. A set of $|\phi_{000}\rangle$ is a $(-1)$-eigenstate of $M^A \otimes M^B \otimes M^C$. Thus if $M^A$, $M^B$, $M^C$ are measured for a set of $|\phi_{000}\rangle$, then the number of $|1\rangle_z$-outcome in the measurement must be even, if no phase flip error occurred. For example, Alice, Bob, and Charles individually perform the measurement of box P1 in Fig.1 on their two qubit which are in two GHZ states. By measuring target qubits of XOR-gates (M1 in Fig.1) with a basis of $\{|0\rangle, |1\rangle\}$, they can measure $M_1 = X^1 \otimes X^2$ where the superscripts 1, 2 indicate two GHZ states they share. They keep the source(control bits) when the number of $|1\rangle$ outcome is even, of the measurement result. If a phase flip error occurred on one qubit of two $|\phi_{000}\rangle$ states, then the number of $|1\rangle$ outcome is odd. Conversely, odd number of $|1\rangle$ outcome imply phase flip error occurrence, thus source is discarded in the case.

On the other hand, if $M_i$ is a tensor product of $Z$ and $I$, then $M_i$ is an observable which detect bit flip errors. A set of $|\phi_{000}\rangle$ is a $(-1)$-eigenstate of $M^A \otimes M^B \otimes M^C$. Thus if $M^A$, $M^B$, $M^C$ are measured for a set of $|\phi_{000}\rangle$, then all the measurement outcomes must be the same. Using this, bit flip error on $|\phi_{000}\rangle$ can be detected. For example, Alice, Bob, and Charles individually perform measurement of P2 on their two qubits which are in two GHZ states. By measuring target qubits (M2 in Fig.1) with a basis of $\{|0\rangle, |1\rangle\}$, they can measure $M_2 = Z^1 \otimes Z^2$. They keep the source when all the measurement outcomes are the same. If bit flip errors occurred on the two GHZ states are different, then at least one measurement outcome is different from others. Thus source is discarded when the different outcomes are measured.

By keeping source selectively as the measurement result, we can decrease the error fraction of the kept source state, however, the process of bit flip error detection increase phase flip error fraction and vice versa. Therefore both of P1 and P2 must be performed. Generally, the observable $M_i$ can be deduced from QEDCs. In fact, $M_1$ and $M_2$ described in Fig.1 are related to the QEDC proposed by Vaidman et al. $[28]$.  

they share only classical mixture of the generalized GHZ states given by $\langle \psi_{e'f'g'} | \psi_{e'f'g'} \rangle$. If not, they discard the source. In P2, each parties measure $M_2$ with the same basis. If all the measurement outcomes $i$ are replaced by $F$(quantum Fourier transform), $F^\dagger$ and GXOR respectively.

FIG. 1: An EPP for multipartite system. Alice, Bob, and Charles share a set of GHZ states. They perform local measurement and announce the outcomes to detect unwanted error. In dashed box P1 $M_1$-measurement detect phase flip(shift) error, and in dashed box P2 $M_2$-measurement detect bit flip(level shift) error. $H$ is a Hadamard operator. In case of qudit based multipartite system, $H$, $H^\dagger$ and XOR are replaced by $F$(quantum Fourier transform), $F^\dagger$ and GXOR respectively.

V. EPP FOR GENERALIZED GHZ STATE

Now Alice, Bob and Charles intend to share $|\psi_{000}\rangle$. However error $E_{i,j}$ can occur stochastically on each site, thus they share only classical mixture of the generalized GHZ states given by

$$\rho = \sum_{e,f,g=0}^{d-1} p_{e'f'g} |\psi_{e'f'g}\rangle \langle \psi_{e'f'g}|.$$  \hspace{1cm} (7)

Here $p_{000}$, the purity of $|\psi_{000}\rangle$ is the fraction of $|\psi_{000}\rangle$ in the $\rho$. Suppose that they share many copies of mixed entangled states of Eq. (7). An EPP for this situation can be constructed by modifying the EPP for GHZ-state. For this, $H$, $H^\dagger$ and XOR in Fig.1. are replaced by $F$, $F^\dagger$, GXOR respectively. Here $F$ is a Fourier transform, GXOR is an extension of XOR to 2-qudits system, and their mathematical forms are given by

$$F|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{nk} |k\rangle,$$

$$GXOR|\psi\rangle = |i\rangle |i \oplus j\rangle,$$ \hspace{1cm} (8)

where $|i \oplus j\rangle$ denotes $|i - j\rangle (\text{mod } d)$.

Now P1(P2) is a step for phase(level) shift error detection. After P1(P2), phase(level) shift error fraction decreased, however level(phase) shift error fraction increased. Thus they perform P1 and P2 alternatively. In P1, they measure $M_1$ with the basis of $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$. If the sum of their measurement outcomes is a multiple of $d$, they keep the source. If not, they discard the source. In P2, each parties measure $M_2$ with the same basis. If all the measurement outcomes are the same, they keep the source, otherwise they discard the source.

To see the performance of P1 and P2 more closely, suppose that they share $|\psi_{e'f'g'}\rangle$ as a source and $|\psi_{e'f'g'}\rangle$ as a target. In P1, after Fourier transform, inverse Fourier transform and GXOR, the state shared by them is transformed to...
\(|\psi_{e,f,g}\rangle|\psi_{e',f',g'}\rangle \rightarrow \sum_{m+n+l+e \equiv 0 \pmod{d}} \sum_{m'+n'+l'+e' \equiv 0 \pmod{d}} \omega^{(fn+gl+f'n'+g'l')}|m,n,l\rangle|m'\otimes n',l\otimes l'\rangle. \) (9)

From Eq.(9), the sum of each M1-measurement outcomes is the form of 
\((m\otimes m')+(n\otimes n')+(l\otimes l')\) where 
\(m+n+l+e \equiv 0 \pmod{d}\) and 
\(m'+n'+l'+e' \equiv 0 \pmod{d}\). Therefore the sum is a multiple of \(d\), only in case of 
\(e = e'\). i.e. in case that same phase shift errors occur on the source and the target. Therefore the source is kept only if \(e = e'\). The state of the kept source is transformed to 
\(|\psi_{e,f,g}⟩\) after inverse Fourier transform on the source in P1. Then the state of kept source are transformed to \(\rho' = \sum_{e,f,g} p'_{efg} |\psi_{efg}\rangle \langle \psi_{efg}|\), thus new classical mixture of GHZ states is obtained. We obtain the recurrence relations between 
\(p_{efg}\) and \(p'_{efg}\), one of which is given by

\[ p'_{000} = \frac{\sum_{f,g=0}^{d-1} p^2_{0fg}}{\sum_{e,f,g,f',g'=0}^{d-1} p_{efg} p_{e'f'g'}}, \] (10)

If \(p'_{000} > p_{000}\), then we can say that P1 purify \(\rho\).

In P2, all the measurement outcomes are same in case of \(f = f'\) and \(g = g'\), i.e. in case that same level shift errors occur on the source. Therefore source is kept only if \(f = f'\) and \(g = g'\), and the state of kept source is 
\(|\psi_{e+e',f,g}\rangle\) in P2. The recurrence relation between 
\(p'_{000}\) and \(p_{efg}\) in the kept source after P2 is given by

\[ p'_{000} = \frac{\sum_{e=0}^{d-1} p(e) p(d-e,0,0)}{\sum_{e,e',f,g=0}^{d-1} p_{efg} p_{e'f'g}}. \] (11)

If \(p'_{000} > p_{000}\), then P2 purify \(\rho\). If the initial purity is high enough, we can obtain highly pure generalized GHZ state from the ensemble of mixed state by repetition of P1 and P2. With the help of recurrence relations including Eq. (10) and (11), we can decide numerically whether our EPP extracts \(|\psi⟩\) successfully from the given mixture of Eq. (7) or not.

Now we present an indirect EPP which purifies generalized GHZ state via generalized Bell state purification. An indirect EPP which purify GHZ state via Bell state purification has been proposed \(^2\), and generalized Bell state purification has also been presented \(^3\). We get an indirect EPP which purify generalized GHZ state easily by combining those two scheme as described below.

1) Alice, Bob and Charles divide an ensemble of the state for three qudits into equal amount of two subensembles.

2) Bob measures his qudit from one subensemble in the state 
\(|\xi_e⟩ = \frac{1}{\sqrt{d}} \sum_k \omega^k |k⟩\), and announces the result. Charles does the same thing for the other subensemble. If measurement outcome is \(|\xi_e⟩\), then Alice performs the \((Z_d)^e\) operation on her qudit. Then they have two reduced two-qudits state. (one is shared between Alice and Bob and the other is shared between Alice and Charles).

3) They perform the EPP which purify generalized Bell state from two-qudits state \(^3\). Then Alice and Bob(Charles) can obtain generalized Bell states.

4) Alice chooses two generalized Bell states purified from each subensemble and performs GXOR on her two qudits. She measures target qudit with a basis of \(|0⟩, |1⟩, \ldots, |d-1⟩\}. If the outcome is \(|m⟩\), then Charles performs \((X_d)^m\) on his qudit. Then they obtain \(|\psi_{000}⟩\).

Now we compare the performances of two EPPs, direct one and indirect one. Consider a mixed state given by

\[ x|\psi_{000}⟩⟨\psi_{000}| + \frac{(1-x)}{d^3} I \] (12)

We calculate the minimum initial purity needed for successful purification with the state, and obtain the result shown in the Table I. The result under the direct EPP(indirect EPP) is in the middle(right) column of the Table. We can see that direct EPP performs properly in lower initial purity than indirect EPP. Also direct one is more efficient than indirect one. For instance, consider the case of \(d = 6\) in Eq. (12). When initial purity is 0.5, about 48 ensemble are needed to extract a generalized GHZ state of purity 0.99 using direct EPP. On the other hand, about 192 ensemble are needed for the same purpose using indirect EPP. Our result shows that direct EPP is more resilient to noise and efficient than indirect one, which is relevant to the result of ref. \(^2\).

VI. DISCUSSIONS AND CONCLUSIONS

We discussed EPPs with a framework of error detection. QEDCs themselves have not been studied intensively contrary to QECCs, however, the effort on the QECCs could be applied to the QEDCs. Stabilizer formalism \(^10\),
for instance, developed in QECCs can be utilized in construction of QEDCs. The EPPs discussed here have two step P1 and P2. Each step detects only one type of error. Therefore Calderbank-Shor-Steane(CSS) codes \[15\] will be useful for the construction of EPPs for multipartite system, since the codes detect bit flip error and phase flip error separately. As Chen et al. noted, a GHZ state is one of CSS states, and EPPs for GHZ state can be generalized to the CSS state \[15\]. This generalization is related to the fact that two type of error can be detected separately in CSS code. On the other hand, purification of W-state which is a non-CSS state was also proposed \[10\], although it is non-trivial with a framework of error detection.

Study of QECCs/QEDCs in qudits system, is less active than that of qubits system. However there is a conjecture, that is useful for the construction of the qudit based QECCs: Qudit based QECCs could be constructed from substituting Hadamard gate as Fourier transform or inverse Fourier transform, and XOR gate as GXOR-gate to the qubit based QECCs \[18\]. The QEDC which is linked to our direct EPP can be constructed from the conjecture.

In summary we analyze the structure of EPPs which purify GHZ state, and propose two new EPPs which purify generalized GHZ state and we compare the performance of them. Our discussion is very systematic one due to the framework of error detection.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.4073</td>
<td>0.4167</td>
</tr>
<tr>
<td>3</td>
<td>0.2305</td>
<td>0.3000</td>
</tr>
<tr>
<td>4</td>
<td>0.1555</td>
<td>0.2227</td>
</tr>
<tr>
<td>5</td>
<td>0.1155</td>
<td>0.1780</td>
</tr>
<tr>
<td>6</td>
<td>0.0907</td>
<td>0.1489</td>
</tr>
</tbody>
</table>

TABLE I: Minimum threshold of initial purity needed for successful purification is given when the initial entangled state is given by Eq.(11). d is a dimension of qudit. Column A is for direct EPP and B is for indirect EPP.