Black hole entropy in 3D gravity with torsion

M. Blagojević and B. Cvetković*
Institute of Physics, P.O.Box 57, 11001 Belgrade, Serbia

Abstract

The role of torsion in three-dimensional quantum gravity is investigated by studying the partition function of the Euclidean theory in Riemann-Cartan spacetime. The entropy of the black hole with torsion is found to differ from the standard Bekenstein-Hawking result, but its form is in complete agreement with the first law of black hole thermodynamics.

I. INTRODUCTION

In our attempts to properly understand basic features of the gravitational dynamics at both classical and quantum level, black holes are often used as an arena for testing new ideas. In the early 1970s, Bekenstein [1] and Hawking [2] discovered that black holes are thermodynamic objects, with characteristic temperatures and entropies. An intensive study of these concepts has led us to conclude that they are closely related to the quantum nature of gravity. In this regard, the discovery of the BTZ black hole in three-dimensional (3D) gravity was of particular importance, as it allowed us to investigate these issues in a substantially simpler context [3].

Following the traditional approach based on general relativity (GR), 3D gravity has been studied mainly in the realm of Riemannian geometry, leading to a number of outstanding results [4–10]. However, it has, already from the 1960s, been well-known that there is a more natural, gauge-theoretic conception of gravity based on Riemann-Cartan geometry, which contains both the curvature and the torsion of spacetime as basic elements of the gravitational dynamics (see, e.g. [11,12]). The application of these ideas to 3D gravity started in the 1990s by Mielke, Baekler and Hehl [13], see also [14]. Recent developments in this direction led to several interesting conclusions: (a) the Mielke-Baekler model of 3D gravity with torsion possesses the black hole solution, (b) it can be formulated as a Chern-Simons gauge theory, and (c) suitable asymptotic conditions generate the asymptotic conformal symmetry, described by two independent Virasoro algebras with different central charges [15–18]. In the present paper, we continue our study of 3D gravity with torsion by investigating the important concept of black hole entropy. Using the Euclidean formulation of the Mielke-Baekler model, we found a new expression for the black hole entropy, and examined its consistency with the first law of black hole thermodynamics.

*Email addresses: mb@phy.bg.ac.yu, cbranislav@phy.bg.ac.yu
The paper is organized as follows. In Sect. II, we discuss basic aspects of the Euclidean 3D gravity with torsion, defined by the Mielke-Baekler action \[13\]. In Sect. III, we recover the related black hole solution, which is of particular importance for thermodynamic considerations. In Sect. IV, we derive the canonical expressions for energy and angular momentum of the Euclidean black hole with torsion. Sect. V contains basic results of the paper. First, assuming that the black hole manifold contains an inner boundary at the horizon (as explained in subsection V.A), we derive a new expression for the entropy of the black hole with torsion. Beside the Bekenstein-Hawking term, it contains an additional contribution, which depends on both the strength of torsion and the position of “inner horizon”. For vanishing torsion, the entropy reduces to the form found earlier by Solodukhin in the context of Riemannian geometry, but with a Chern-Simons term in the action \[19\]. Then, using our results for the black hole energy, angular momentum and entropy, we prove the validity of the first law of black hole thermodynamics. Thus, torsion is shown to be in complete agreement with the first law. Finally, Sect. VI is devoted to concluding remarks, while appendices contain some technical details on Euclidean continuation and the hyperbolic geometry in 3D.

In our conventions, Minkowskian 3D gravity with torsion is defined by an action \[I_M\] in Riemann-Cartan spacetime with signature \[\eta^M = (+, -, -)\] \[18\]. The analytic continuation of the theory is determined by another action \[\bar{I}_M\], such that \[iI_M \mapsto \bar{I}_M\] in spacetime with \[\bar{\eta} = (-, -, -)\]. Although \[\bar{I}_M\] is the object of our prime interest, technically, we base our exposition on the standard Euclidean formalism defined by \[iI_M \mapsto -I_E\] and \[\eta^E = (+, +, +)\]. Our conventions are given by the following rules: index \(E\) is omitted for simplicity, the Greek indices refer to the coordinate frame, the Latin indices refer to the tangent frame; the middle alphabet letters \((i, j, k, \ldots; \mu, \nu, \lambda, \ldots)\) run over 0, 1, 2, while the first letters of the Greek alphabet \((\alpha, \beta, \gamma, \ldots)\) run over 1, 2; \(\eta_{ij} = \text{diag}(1, 1, 1)\) are the tangent frame components of the metric; totally antisymmetric tensor \(\varepsilon^{ijk}\) and the related tensor density \(\varepsilon^{\mu\nu\rho}\) are both normalized by \(\varepsilon^{012} = +1\).

II. EUCLIDEAN GRAVITY WITH TORSION

Following the analogy with Poincaré gauge theory [11,12], Euclidean gravity with torsion in 3D can be formulated as a gauge theory of the Euclidean group \(E(3) = ISO(3)\) (EGT for short), the analytic continuation of the Poincaré group \(P(3) = ISO(1, 2)\). The underlying geometric structure is described by Riemann-Cartan space.

EGT in brief. Basic gravitational variables in EGT are the triad field \(b^i\) and the spin connection \(A^{ij} = -A^{ji}\) (1-forms). The corresponding field strengths are the torsion and the curvature: \(T^i = db^i + A^i m \wedge b^m, R^{ij} = dA^{ij} + A^i m \wedge A^m j\) (2-forms). Gauge symmetries of the theory are local translations and local rotations, parametrized by \(\xi^\mu\) and \(\varepsilon^{ij}\).

In 3D, we can simplify the notation by introducing

\[
A^{ij} = -\varepsilon^{ijk} \omega_k, \quad R^{ij} = -\varepsilon^{ijk} R_k, \quad \varepsilon^{ij} = -\varepsilon^{ijk} \theta_k.
\]

In local coordinates \(x^\mu\), we can write \(b^i = b^i \mu dx^\mu, \omega^i = \omega^i \mu dx^\mu\), the field strengths are

\[
T^i = db^i + \varepsilon^{i,jk} \omega^j \wedge b^k = Db^i, \\
R^i = d\omega^i + \frac{1}{2} \varepsilon^{i,jk} \omega^j \wedge \omega^k,
\] (2.1)
and gauge transformations take the form
\[
\delta_0 b^i_\mu = -\varepsilon^{ijk}\xi^\rho b^j_\mu \theta^k - (\partial_\mu \xi^\rho) b^i_\rho - \xi^\rho \partial_\rho b^i_\mu ,
\]
\[
\delta_0 \omega^i_\mu = -\nabla_\mu \theta^i - (\partial_\mu \xi^\rho) \omega^j_\rho - \xi^\rho \partial_\rho \omega^i_\mu ,
\]
(2.2)
where \( \nabla_\mu \theta^i = \partial_\mu \theta^i + \varepsilon^{ijk}\xi^\rho \theta^k \). The covariant derivative \( \nabla \equiv dx^\mu \nabla_\mu \) acts on a general tangent-frame spinor in accordance with its spinorial structure, while \( DX = \nabla \wedge X \) is the covariant exterior derivative of a form.

The metric structure of EGT is defined by
\[
g = \eta_{ij} b^i \otimes b^j \equiv g_{\mu\nu} dx^\mu \otimes dx^\nu , \quad \eta_{ij} = \text{diag}(1,1,1) .
\]
Although metric and connection on an arbitrary manifold can be specified as independent fields, in EGT they are related to each other by the metricity condition: \( \nabla g = 0 \). Consequently, the geometric structure of EGT corresponds to Riemann-Cartan geometry.

We display here a useful EGT identity:
\[
\omega^i \equiv \tilde{\omega}^i + K^i ,
\]
(2.3)
where \( \tilde{\omega}^i \) is the Levi-Civita (Riemannian) connection, and \( K^i \) is the contortion 1-form, defined implicitly by \( T^i = \varepsilon^i_{\ mn} K^m \wedge b^n \).

**Topological action.** General gravitational dynamics is determined by Lagrangians which are at most quadratic in field strengths. Omitting the quadratic terms, we get the topological model for 3D gravity, proposed by Mielke and Baekler [13]:
\[
I = a I_1 + \Lambda I_2 + \alpha_3 I_3 + \alpha_4 I_4 + I_M ,
\]
(2.4a)
where \( I_M \) is a matter contribution, and
\[
I_1 = 2 \int b^i \wedge R_i ,
\]
\[
I_2 = -\frac{1}{3} \int \varepsilon^{ijk} b^j \wedge b^i \wedge b^k ,
\]
\[
I_3 = \int \left( \omega^i \wedge d\omega_i + \frac{1}{3} \varepsilon^{ijk} \omega^j \wedge \omega^i \wedge \omega^k \right) ,
\]
\[
I_4 = \int b^i \wedge T_i .
\]
(2.4b)
The first term, with \( a = 1/16\pi G \), is the usual Einstein-Cartan action, the second term is a cosmological term, \( I_3 \) is the Chern-Simons action for the spin connection, and \( I_4 \) is a torsion counterpart of \( I_1 \). The Mielke-Baekler model is a natural generalization of GR with a cosmological constant (GR\( \Lambda \)).

**The vacuum field equations.** Variation of the action with respect to \( b^i \) and \( \omega^i \) yields the gravitational field equations. Dynamical properties in the region outside the gravitational sources are determined by the field equations in vacuum:
\[
2a R_i + 2\alpha_4 T_i - \Lambda \varepsilon^{ijk} b^j \wedge b^k = 0 ,
\]
\[
2a T_i + 2\alpha_3 R_i + \alpha_4 \varepsilon^{ijk} b^j \wedge b^k = 0 .
\]
In the sector $\alpha_3 \alpha_4 - a^2 \neq 0$, these equations take the simple form [*]

$$2T^i = pe^i_{jk} b^j \wedge b^k , \quad 2R^i = qe^i_{jk} b^j \wedge b^k , \quad (2.5)$$

with

$$p = \frac{\alpha_3 \Lambda + \alpha_4 a}{\alpha_3 \alpha_4 - a^2} , \quad q = -\frac{(\alpha_4)^2 + a \Lambda}{\alpha_3 \alpha_4 - a^2} .$$

Thus, vacuum solutions are characterized by constant torsion and constant curvature. For $p = 0$, the vacuum geometry is Riemannian, while for $q = 0$, it becomes teleparallel. Note that $p$ and $q$ satisfy the following identities:

$$aq + \alpha_4 p - \Lambda = 0 , \quad ap + \alpha_3 q + \alpha_4 = 0 .$$

In Riemann-Cartan spacetime, one can use the identity (2.3) to express the curvature $R^i = R^i(\omega)$ in terms of its Riemannian piece $\tilde{R}^i = \tilde{R}^i(\tilde{\omega})$ and the contortion:

$$R^i = \tilde{R}^i + DK^i - \frac{1}{2} \varepsilon^{imn} K_m \wedge K_n .$$

This identity, combined with the relation $K^i = p b^i / 2$, which follows from the field equations (2.5), leads to

$$2\tilde{R}^i = \Lambda_{\text{eff}} \varepsilon^i_{jk} b^j \wedge b^k , \quad \Lambda_{\text{eff}} \equiv q - \frac{1}{4} p^2 , \quad (2.6a)$$

or equivalently:

$$\tilde{R}^{ij} = -\Lambda_{\text{eff}} b^i \wedge b^j , \quad (2.6b)$$

where $\Lambda_{\text{eff}}$ is the effective cosmological constant. Using the Riemannian terminology, we can say that our spacetime is maximally symmetric, in the sense that its metric has maximal number of Killing vectors. In what follows, our attention will be focused on the model (2.4) with $\alpha_3 \alpha_4 - a^2 \neq 0$, and with positive $\Lambda_{\text{eff}}$ (Euclidean anti-de Sitter sector):

$$\Lambda_{\text{eff}} \equiv \frac{1}{\ell^2} > 0 . \quad (2.7)$$

The corresponding Riemannian scalar curvature is negative: $\tilde{R} = -6\Lambda_{\text{eff}}$.

**III. THE BLACK HOLE SOLUTION**

For positive $\Lambda_{\text{eff}}$, the field equation (2.6) has a well-known solution for the metric—the Euclidean BTZ black hole. In spite of its dynamical complexity, this solution enables a simple approach to the gravitational thermodynamics, based on the observation that the Euclidean action at the black hole contains non-trivial thermodynamic information [8,9,20,22]. The Euclidean black hole metric in Schwarzschild-like coordinates reads

$$ds^2 = N^2 dt^2 + N^{-2} dr^2 + r^2 (d\varphi + N_\varphi dt)^2 , \quad (3.1)$$

$$N^2 = \left( -8GM + \frac{r^2}{\ell^2} - \frac{16G^2 J^2}{r^2} \right) , \quad N_\varphi = -\frac{4GJ}{r^2} .$$
Since the Riemannian curvature of the solution is negative, \( \tilde{R} = -6/\ell^2 \), \( \Lambda_{\text{eff}} \) is positive. The zeros of \( N^2 \), \( r_+ \) and \( r_- \equiv -i\rho_- \), are related to the black hole parameters by relations
\[
8G\ell^2 m = r_+^2 - \rho_-^2, \quad 4G\ell J = r_+\rho_-.
\]
The Euclidean metric (3.1) is obtained from the corresponding Minkowskian expression [18] by the process of analytic continuation, described in Appendix A. In the Euclidean sector, both \( \varphi \) and \( t \) are taken to be periodic (Appendix B),
\[
0 \leq \varphi < 2\pi, \quad 0 \leq t < \beta, \quad \beta \equiv \frac{2\pi\ell^2}{r_+^2 + \rho_-^2},
\]
while the radial coordinate \( r \) is in the range \( r_+ \leq r < \infty \). Topologically, any surface with constant \( r \) is an ordinary 2-torus, parametrized by \( \varphi \) and \( t \), and the whole black hole manifold \( \mathcal{M} \) is a solid torus. The black hole horizon \( r = r_+ \) is a circle at the core of the solid torus, so that the manifold does not contain the region \( r < r_+ \), corresponding to the inner part of the Minkowskian black hole. It is clear that \( \mathcal{M} \) has the (asymptotic) boundary located at spatial infinity \( r \to \infty \). Later, in subsection V.A, we shall reconsider arguments in favor of the assumption that the horizon should be regarded as an additional, inner boundary of \( \mathcal{M} \) [8,21,22].

The horizon is a one-dimensional subspace \( r = r_+ \) with the metric \( ds^2 = \ell^2 d\psi^2 \), where \( \psi = r_+ \varphi / \ell - \rho_- t / \ell^2 \). The “area” of the horizon is \( 2\pi r_+ \). For later convenience, we introduce the quantity
\[
\Omega = N_\varphi(r_+) = -\frac{\rho_-}{\ell r_+},
\]
which defines the black hole angular velocity.

Starting with the BTZ metric (3.1), we construct the black hole with torsion in the following two steps. First, we choose \( b^i \) to have the simple, “diagonal” form:
\[
\begin{align*}
 b^0 &= N dt, \\
 b^1 &= N^{-1} dr, \\
 b^2 &= r (d\varphi + N_\varphi dt).
\end{align*}
\]
Then, we combine the field equation \( K^i = pb^i / 2 \) with the identity (2.3), and obtain the connection:
\[
\omega^i = \bar{\omega}^i + \frac{p}{2} b^i,
\]
where the Levi-Civita connection \( \bar{\omega}^i \) is determined by the condition \( d\bar{\omega}^i + \varepsilon^i_{jk} \bar{\omega}^j b^k = 0 \):
\[
\begin{align*}
 \bar{\omega}^0 &= Nd\varphi, \\
 \bar{\omega}^1 &= -N^{-1} N_\varphi dr, \\
 \bar{\omega}^2 &= -\frac{r}{\ell^2} dt + r N_\varphi d\varphi.
\end{align*}
\]
The pair \( (b^i, \omega^i) \) in (3.2) defines the Euclidean black hole in Riemann-Cartan spacetime.

Let us also display here the Euclidean AdS\(_3 \) metric, which is formally obtained from (3.1) by the replacements \( 8Gm = -1, J = 0, \) and \( \varphi = \phi \):
\[
ds^2 = f^2 dt^2 + f^{-2} dr^2 + r^2 d\phi^2, \quad f^2 \equiv 1 + \frac{r^2}{\ell^2},
\]
The same replacement in (3.2) yields AdS\(_3 \) with torsion. Using \( \phi \) instead of \( \varphi \), we wish to stress the difference in topological properties between the black hole and AdS\(_3 \), as discussed in Appendix B.
IV. ENERGY AND ANGULAR MOMENTUM

For isolated macroscopic systems, energy and angular momentum are dynamical quantities of fundamental importance for their thermodynamic behavior. With the inclusion of gravity, these quantities can be expressed as certain surface integrals over the asymptotic values of dynamical variables. As a first step in our approach to the thermodynamics of black holes with torsion, we use the standard canonical formalism to calculate energy and angular momentum as the asymptotic charges of the Euclidean black hole (3.2).

A. Asymptotic conditions

For any gauge theory, asymptotic conditions and their symmetries are of essential importance for the physical content of the theory, as they give rise to the conserved charges, which characterize the dynamical behavior of the system. General asymptotic structure of the Minkowskian 3D gravity with torsion and its relation to conformal symmetry is well understood [18]. Here, we wish to find global charges of the Euclidean black hole, which is a much simpler problem.

The canonical procedure for calculating global charges is well established. We start by choosing the asymptotic conditions at spatial infinity so that the fields $b^i$ and $\omega^i$ are restricted to the family of black hole configurations (3.2), parametrized by $m$ and $J$. In other words, $b^i$ and $\omega^i$ have the following behavior as $r \to \infty$:

$$b^i_\mu \sim \begin{pmatrix} r - \frac{4Gm\ell}{r} & 0 & 0 \\ -\frac{4G}{r} & 0 & 0 \\ -\frac{r}{\ell^2} & 0 & -\frac{4GJ}{r} \end{pmatrix}, \quad (4.1a)$$

$$\omega^i_\mu \sim \begin{pmatrix} 0 & 0 & r - \frac{4Gm\ell}{r} \\ 0 & r - \frac{4GJ\ell}{r} & 0 \\ -\frac{r}{\ell^2} & 0 & -\frac{4GJ}{r} \end{pmatrix} + \frac{p}{2} b^i_\mu. \quad (4.1b)$$

Since these conditions represent a restricted version of the general anti-de Sitter asymptotics, they are sufficient to define only a restricted set of the conformal charges—energy and angular momentum [5,18].

Having chosen the asymptotic conditions, we now wish to find the subset of gauge transformations (2.2) that respect these conditions and define the asymptotic symmetry. They are determined by restricting the original gauge parameters in accordance with (4.1), which yields

$$\xi^\mu = (\ell T_0, 0, S_0), \quad \theta^i = (0, 0, 0), \quad (4.2)$$

where $T_0$ and $S_0$ are arbitrary constants. In other words, the asymptotic symmetry is described by two Killing vectors, $\partial/\partial t$ and $\partial/\partial \varphi$, as could have been concluded directly from
the form of the black hole solution (3.2). The corresponding asymptotic symmetry group is 
\( SO(2) \times SO(2) \), a subgroup of the conformal group in two dimensions, in accordance with 
our choice of the asymptotic conditions.

**B. Asymptotic charges**

As we shall see, the asymptotic conditions (4.1) are acceptable at the canonical level, since 
the related asymptotic symmetry has well-defined canonical generators. The construction 
of the improved generator and the corresponding asymptotic charges follows the standard 
canonical procedure \([23,24,18]\).

**Hamiltonian and constraints.** Introducing the canonical momenta \((\pi^\mu_i, \Pi^\mu_i)\), cor-
responding to the Lagrangian variables \((b^\mu_i, \omega^\mu_i)\), the action (2.4) leads to the following 
primary constraints:

\[
\phi_i^0 \equiv \pi_i^0 \approx 0, \quad \phi_i^\alpha \equiv \pi_i^\alpha - \alpha_4 \varepsilon^{0\alpha\beta} b_{i\beta} \approx 0, \\
\Phi_i^0 \equiv \Pi_i^0 \approx 0, \quad \Phi_i^\alpha \equiv \Pi_i^\alpha - \varepsilon^{0\alpha\beta}(2ab_{i\beta} + \alpha_3 \omega_{i\beta}) \approx 0.
\]

The canonical Hamiltonian is linear in unphysical variables:

\[
\mathcal{H}_c = b_i^0 \mathcal{H}_i + \omega_i^0 \mathcal{K}_i + \partial_\alpha D^\alpha, \\
\mathcal{H}_i = -\varepsilon^{0\alpha\beta}(aR_{\alpha\beta} + \alpha_4 T_{\alpha\beta} - \Lambda \varepsilon_{ijk} b_{i\alpha} b_{j\beta}), \\
\mathcal{K}_i = -\varepsilon^{0\alpha\beta}(aT_{\alpha\beta} + \alpha_3 R_{\alpha\beta} + \alpha_4 \varepsilon_{imn} b_{i\alpha} b_{m\beta}), \\
D^\alpha = \varepsilon^{0\alpha\beta} \left[ \omega_i^0(2ab_{i\beta} + \alpha_3 \omega_{i\beta}) + \alpha_4 b_i^0 b_{i\beta} \right].
\]

In gauge theories, general dynamical evolution is governed by the total Hamiltonian, which is obtained from \(\mathcal{H}_c\) by adding a linear combination of the primary constraints. Thus, \(\mathcal{H}_T = \mathcal{H}_c + u_{i\mu}^i \phi_{i\mu} + v_{i\mu}^i \Phi_i^\mu\), where \(u_{i\mu}^i\) and \(v_{i\mu}^i\) are arbitrary Hamiltonian multipliers. The consistency conditions on the constraints lead to the determination of \(u_{i\alpha}^i\) and \(v_{i\alpha}^i\), whereupon \(\mathcal{H}_T\) takes its final form:

\[
\mathcal{H}_T = \hat{\mathcal{H}}_T + \partial_\alpha \hat{D}^\alpha, \\
\hat{\mathcal{H}}_T = b_i^0 \hat{\mathcal{H}}_i + \omega_i^0 \hat{\mathcal{K}}_i + u_i^0 \pi_i^0 + v_i^0 \Pi_i^0, \\
\hat{D}^\alpha = D^\alpha + b_i^0 \phi_i^\alpha + \omega_i^0 \Phi_i^\alpha = b_i^0 \pi_i^\alpha + \omega_i^0 \Pi_i^\alpha.
\]

The constraints \((\pi_i^0, \Pi_i^0, \mathcal{H}_i, \mathcal{K}_i)\) are first class, while \((\phi_i^\alpha, \Phi_i^\alpha)\) are second class.

**Canonical generator.** Applying the general Castellani’s algorithm \([24]\), the canonical 
gauge generator is expressed in terms of the first class constraints as follows:
\[
G = -G_1 - G_2 ,
\]
\[
G_1 \equiv \dot{\xi}^\rho \left( b^i \rho i^0 + \omega^i \rho_i^0 \right) + \xi^\rho \left[ b^i \rho H_i + \omega^i \rho K_i + (\partial_\rho b^i_0) \pi_i^0 + (\partial_\rho \omega^i_0) \Pi_i^0 \right] ,
\]
\[
G_2 \equiv \dot{\theta}^i \Pi_i^0 + \theta^i \left[ \bar{K}_i - \varepsilon_{ijk} \left( b^j_0 \pi^0 + \omega^j_0 \Pi^0 \right) \right] .
\]

Here, the time derivatives \( \dot{b}_0^i \) and \( \dot{\omega}_0^i \) are shorts for \( u_0^i \) and \( v_0^i \), respectively, and the integration symbol \( \int d^2 x \) is omitted for simplicity. The transformation law of the fields, \( \delta_0 \phi \equiv \{ \phi, G \} \), is in complete agreement with the gauge transformations \( (2.2) \) on shell.

The behaviour of the momentum variables at large distances is determined by the following general principle: the expressions that vanish on-shell should have an arbitrarily fast asymptotic decrease, as no solution of the field equations is thereby lost. Applying this principle to the primary constraints, we find the asymptotic behavior of \( \pi^\mu_i \) and \( \Pi_i^\mu \).

The canonical generator acts on functions of the phase-space variables via the Poisson bracket operation, which is defined in terms of functional derivatives. In general, \( G \) does not have well-defined functional derivatives, but this can be corrected by adding suitable surface terms. The improved canonical generator \( \tilde{G} \) is found to have the following form:

\[
\tilde{G} = G + \Gamma , \quad \Gamma = -\int_0^{2\pi} d\varphi \left( \xi^0 \mathcal{E}^1 + \xi^2 \mathcal{M}^1 \right) ,
\]

\[
\mathcal{E}^\alpha \equiv 2 \varepsilon^{\alpha 0 \beta} \left[ \left( a + \frac{\alpha_3 p}{2} \right) \omega_0^\beta + \left( \alpha_4 + \frac{ap}{2} \right) b_0^\beta + \frac{a}{\ell} \omega_0^\beta \right] b_0^\alpha ,
\]

\[
\mathcal{M}^\alpha \equiv 2 \varepsilon^{\alpha 0 \beta} \left[ \left( a + \frac{\alpha_3 p}{2} \right) \omega_2^\beta + \left( \alpha_4 + \frac{ap}{2} \right) b_2^\beta + \frac{a}{\ell} \omega_2^\beta \right] b_2^\alpha ,
\]

where \( \xi^\mu \) are the asymptotic parameters \( (4.2) \), i.e. constants. The adopted asymptotic conditions guarantee differentiability and finiteness of \( \tilde{G} \); moreover, \( \tilde{G} \) is also conserved.

**Canonical charges.** The value of the improved generator \( \tilde{G} \) defines the asymptotic charges. Since \( \tilde{G} \approx \Gamma \), the charges are completely determined by the boundary term \( \Gamma \). Canonical expressions for the energy and angular momentum are defined as the values of the surface term \( -\Gamma \), calculated for \( \xi^0 = 1 \) and \( \xi^2 = 1 \), respectively. However, what we really need are the charges corresponding to the analytically continued action \( \tilde{I} = -I_E \), which introduces an additional minus sign:

\[
E = -\int_0^{2\pi} d\varphi \mathcal{E}^1 , \quad M = -\int_0^{2\pi} d\varphi \mathcal{M}^1 .
\]

Consequently, energy and angular momentum of the black hole are given by

\[
E = m + \frac{\alpha_3}{a} \left( \frac{pm}{2} - \frac{J}{\ell^2} \right) , \quad M = J + \frac{\alpha_3}{a} \left( \frac{pJ}{2} + m \right) .
\]

Thus, the conserved charges are linear combinations of \( m \) and \( J \). The expressions \( (4.7) \) generalize the well-known results for the conserved charges in GR\(_A\) (where \( E = m \) and \( M = J \)), and give us a new physical interpretation of the parameters \( m \) and \( J \). Note also that transition to Riemannian theory \( (p = 0) \) still yields a non-trivial modification of the GR\(_A\) result \([19]\).
V. THE BLACK HOLE ENTROPY

Thermodynamic properties of black holes are closely related to the quantum nature of gravity. In this section, we shall examine the role of torsion in quantum 3D gravity by studying the partition function of the Euclidean 3D gravity with torsion.

**Partition function.** Let us consider the functional integral (in units $\hbar = 1$)

$$Z[\beta, \Omega] = \int DbD\omega \exp \left( -\tilde{I}[b, \omega, \beta, \Omega] \right),$$

(5.1a)

where $\beta$ and $\Omega$ are the Euclidean time period and the angular velocity of the black hole, respectively, the fields $(b^i, \omega^i)$ satisfy certain boundary conditions, and $\tilde{I}$ is the Euclidean action (2.4), corrected by suitable boundary terms. The Lagrangian boundary conditions define a set of the allowed field configurations $C_L$, such that:

(i) $C_L$ contains black holes with $(m, J)$ belonging to a small region around some fixed $(m, J)_0$,

(ii) $\beta$ and $\Omega$ remain constant on the boundary, and

(iii) one can construct the boundary terms which make the improved action $\tilde{I}$ differentiable.

If we recall the definition of $\beta$ and $\Omega$ from section III, we can see that they are determined in terms of $r_+$ and $\rho_-$ (functions of $m$ and $J$), so that the conditions (i) and (ii) may become incompatible. However, this is not a sincere problem. Indeed, it can be resolved by treating $\beta$ and $\Omega$ as independent parameters, determined in terms of $r_+$ and $\rho_-$ only “on shell”. Geometrically, an “off-shell” extension of $\beta$ and $\Omega$ leads to conical singularities, but at the end, after imposing the field equations, they disappear [8,9]. Interpreted in this way, the above boundary conditions correspond to the grand canonical ensemble.

Although the partition function (5.1a) cannot be calculated exactly, the semiclassical approximation around the black hole solution (3.2) gives very interesting insights into the role of torsion in quantum dynamics. Indeed, starting with the semiclassical expansion $\tilde{I} = \tilde{I}_{bh} + O(\hbar)$, where $\tilde{I}_{bh}$ is the value of the classical action $\tilde{I}$ at the black hole configuration, one finds, to the lowest order in $\hbar$, that the partition function is given by

$$\ln Z[\beta, \Omega] \approx -\tilde{I}_{bh}.$$ (5.1c)

The canonical action. Instead of working directly with the covariant action (2.4), we shall rather use its canonical form, as in [3]:

$$I_c = \int dt \int d^2 x \left( \pi_i^\mu \dot{b}_i^\mu + \Pi_i^\mu \dot{\omega}_i^\mu - \hat{H}_T \right),$$

(5.2a)
where $\hat{H}_T$ is the total Hamiltonian (4.3). The Lagrangian boundary conditions (i), (ii) and (iii), can be easily extended to the Hamiltonian boundary conditions, defined by the related phase-space configurations $C_H$. The action $I_c$ does not have well-defined functional derivatives, since its variation on $C_H$ produces not only the field equations, but also some boundary terms. The improved action has the general form

$$\bar{I}_c = I_c + B,$$

where the boundary term $B$ is chosen to make $\bar{I}_c$ differentiable. The partition function is now given as the functional integral over $C_H$.

For a thermodynamic system in equilibrium, the ensemble must be time independent. If one calculates the value of $\bar{I}_c$ on the set of static triads and connections, one finds that $I_c$ vanishes on shell ($\hat{H}_T \approx 0$), so that the only term that remains is the boundary term $B$. This term contains the complete information about the black hole thermodynamics.

### A. Boundary terms

The boundary term $B$ is constructed by demanding $\delta I_c + \delta B \approx 0$, where $\approx$ denotes an equality when the Hamiltonian constraints hold (on-shell or weak quality). In other words, $B$ should cancel the unwanted boundary terms in $\delta I_c$, arising from integrations by parts.

Using the relation $\delta \hat{H}_T \approx b^i_0 \delta \hat{H}_i + \omega^i_0 \delta \hat{K}_i$, we find that the general variation of $I_c$ at fixed $r$ has the form

$$\delta I_c \bigg|_r = - \int dt \int d^2 x \delta \hat{H}_T \bigg|_r$$

$$\approx 2 \int dt d\phi \left[ b^i_0 (a \delta \omega^i_2 + \alpha_3 \delta b^i_2) + \omega^i_0 (a \delta b^i_2 + \alpha_3 \delta \omega^i_2) \right]_r.$$ 

We can now restrict the phase space by the requirement

$$\omega^i = \tilde{\omega}^i + \frac{P}{2} b^i + \hat{O},$$

where $\hat{O}$ is arbitrarily small at the boundary, as no solution of the field equations is thereby lost. Consequently,

$$\delta I_c \bigg|_r \approx 2 \int dt d\phi \left[ - \frac{\alpha_3}{\ell^2} b^i_0 \delta b^i_2 + a (b^i_0 \delta \tilde{\omega}^i_2 + \tilde{\omega}^i_0 \delta b^i_2) 
+ \alpha_3 \tilde{\omega}^i_0 \delta \tilde{\omega}^i_2 + \alpha_3 \frac{P}{2} (b^i_0 \delta \tilde{\omega}^i_2 + \tilde{\omega}^i_0 \delta b^i_2) \right]_r.$$ 

The boundary term $B$ contains the contributions from infinity and from the horizon, which are determined by the requirement

$$\delta I_c \bigg|^{r \to \infty}_{r} - \delta I_c \bigg|^{r+}_{r} + \delta (B^{\infty} + B^{r+}) \approx 0.$$ 

**Spatial infinity.** The boundary term stemming from infinity has been already calculated in the construction of the improved Hamiltonian. It has the form
\[\delta I_c |_{r \to \infty} = -\int_0^\beta dt \delta \int d^2 x \hat{H}_T |_{r \to \infty} \approx -\delta B^\infty, \]

\[B^\infty = \beta E. \tag{5.4}\]

Here, \(E\) is the canonical energy (4.7), and the time period \(\beta\) is kept fixed, since we are in the grand canonical ensemble.

**The horizon.** If we consider Minkowskian black hole as a macroscopic object, it seems quite natural to treat only its “outer” part \(r > r_+\) as physical. The consistency of this idea leads to certain boundary conditions at \(r = r_+\), which give rise to additional boundary terms [8,22] (see also [25]). In the Euclidean formalism, in spite of the fact that the “inner” region \(r < r_+\) is absent from the black hole manifold, there are convincing arguments that one still needs boundary terms at the horizon [8,21,22]. These arguments are based on the observation that the Killing vector field \(\partial_t\) is not well defined at \(r = r_+\). Motivated by these considerations, we assume that the line \(r = r_+\) is removed from the black hole manifold, which modifies the topology: the horizon becomes an additional (inner) boundary of the manifold.

After accepting such an assumption, we have to introduce appropriate boundary conditions at the horizon, in order to further improve the differentiability of the action. Let us observe that the values of the triad field and Riemannian connection satisfy the following relations on the horizon:

\[b^a_0 - \Omega b^a_2 = 0, \quad \tilde{\omega}^a_0 - \Omega \tilde{\omega}^a_2 = -\frac{2\pi}{\beta} \delta^a_2, \tag{5.5}\]

for \(a = 0, 2\). Having in mind that the boundary conditions should correspond to the grand canonical ensemble, we promote these “on-shell” relations into the “off-shell” boundary conditions at the horizon, with \(\Omega\) and \(\beta\) as independent parameters (compare with [8,18]).

Now, we are ready to calculate the corresponding boundary term. Using the general relation (5.3) and the boundary conditions (5.5), we find

\[\delta I_c |_{r_+} \approx \delta B|r_+, \]

\[B|r_+ = 2 \int_0^\beta dt \int_0^{2\pi} d\phi \left\{ \Omega \left[ -\frac{\alpha_3}{2\ell^2} b^i_2 b^i_2 \right. \right. \]

\[+ \left. \left. \left( a + \frac{\alpha_3 p}{2} \right) b^i_2 \tilde{\omega}^i_2 + \frac{\alpha_3}{2} \tilde{\omega}^i_2 \tilde{\omega}^i_2 \right] - \frac{2\pi}{\beta} \left[ \left( a + \frac{\alpha_3 p}{2} \right) b^i_2 \omega^i_2 + \alpha_3 \tilde{\omega}^i_2 \tilde{\omega}^i_2 \right] \right\}. \]

In order to calculate \(B|r_+\) at the black hole configuration, we use the relations

\[b^i_2 b^i_2 |_{r_+} = r_+^2, \quad b^i_2 \tilde{\omega}^i_2 |_{r_+} = -4GJ, \quad \tilde{\omega}^i_2 \tilde{\omega}^i_2 |_{r_+} = \frac{\rho_2^2}{\ell^2}, \]

which yield

\[B|r_+ = -\beta \Omega M - \left[ \frac{2\pi r_+}{4G} + 4\pi^2 \alpha_3 \left( pr_+ - 2\frac{\rho_2}{\ell} \right) \right]. \tag{5.6}\]
B. Entropy

The value of the improved action at the black hole configuration is equal to the sum of boundary terms (5.4) and (5.6):
\[
\tilde{I}_{bh} = B^\infty + B^r + \beta (E - \Omega M) - \left[ \frac{2\pi r_+}{4G} + 4\pi^2 \alpha_3 \left( pr_+ - 2\frac{\rho_-}{\ell} \right) \right].
\]

(5.7)

The thermodynamic interpretation of this result, based on Eq. (5.1c), tells us that \( \beta \) is the inverse temperature, \( \Omega \) is the chemical potential corresponding to \( M \), and

\[
S = \frac{2\pi r_+}{4G} + 4\pi^2 \alpha_3 \left( pr_+ - 2\frac{\rho_-}{\ell} \right)
\]

(5.8)
is the entropy of the black hole with torsion.

Microscopic interpretation of the black hole entropy is naturally related to the number of dynamical degrees of freedom located at the boundary of spacetime [4,8,9]. For \( \alpha_3 = 0 \), our result (5.3) reduces to the first term—the Bekenstein-Hawking value of \( S \). The additional term, proportional to \( \alpha_3 \), stems from the Chern-Simons contribution to the action (2.4). The first piece of this term, proportional to \( pr_+ \), can be interpreted as the contribution of the torsion degrees of freedom at the outer horizon, while the second piece is due to degrees of freedom at the “inner horizon” \( \rho_- \). Since the Euclidean black hole manifold does not contain the inner horizon (\( r_- = -i\rho_- \) is imaginary), the appearance of \( \rho_- \) in (5.8) could be understood as a consequence of the analytic structure of the theory, which is expected to contain relevant information on the Minkowskian sector. For vanishing torsion (\( p = 0 \)), the entropy reduces to the result obtained by Solodukhin, in his study of Riemannian GR \( \Lambda \) with a Chern-Simons term [19].

For a given black hole with fixed \( r_+ \) and \( \rho_- \), the condition \( S \geq 0 \) imposes certain bounds on the parameters \( \alpha_3 \), \( p \) and \( \ell \). It is an interesting question what happens with \( S \) at the absolute zero of temperature, \( T = (r_+^2 + \rho_-^2)/2\pi\ell^2 r_+ \to 0 \). Formally, the black hole at the absolute zero is in the ground state, defined by \( r_+ , \rho_- \to 0 \) (\( m,J \to 0 \)), and the entropy (5.8) of the ground state vanishes, in agreement with the third law of thermodynamics. However, this line of arguments is not acceptable, since in the ground state region, the semiclassical result (5.8) is outside of its domain of validity. Indeed, for \( S = 0 \), the generalized Smarr formula \( 2\beta (E - \Omega M) = S \) (obtained by direct calculation) implies that the whole \( \tilde{I}_{bh} \) vanishes; but if \( \tilde{I}_{bh} = 0 \), the 0-loop approximation of the semiclassical expansion is not reliable. Thus, the mathematical limitations of the result (5.8) do not allow us to have a true estimate of the black hole entropy in the extreme case of the black hole ground state.

Using the rules of Euclidean continuation described in Appendix A, the entropy can be easily expressed in terms of the corresponding Minkowskian parameters [18]:

\[
S = \frac{2\pi r_+}{4G} + 4\pi^2 \alpha_3 \left( pr_+ - 2\frac{r_-}{\ell} \right)
\]

(5.9)
C. The first law of thermodynamics

If the black hole solution is an extremum of the canonical action $I_c$ on the set of allowed phase-space configurations, $\delta I_c|_{bh} \approx -\delta(B^\infty + B^r)|_{bh} = 0$, we obtain the relation
\[
\delta S = \beta \delta E - \beta \Omega \delta M ,
\] (5.10)
which represents the first law of black hole thermodynamics (compare with arguments given in Ref. [25]). The result (5.10) is also confirmed by a direct calculation, using our expressions for $S, E$ and $M$. Thus, the existence of torsion in 3D is in complete agreement with the first law of black hole thermodynamics.

Using $\beta = 1/T$, the first law (5.10) can be written in the form
\[
T \delta S = \delta E - \Omega \delta M .
\] (5.11)

VI. CONCLUDING REMARKS

In this paper, we investigated the role of torsion in the black hole thermodynamics by studying the grand canonical partition function of the Euclidean black hole with torsion, in the lowest-order semiclassical approximation.

(1) The black hole entropy is obtained from the boundary term at the horizon.

(2) It differs from the Bekenstein-Hawking result by an additional term, which describes the torsion degrees of freedom at the outer horizon, and degrees of freedom at the “inner horizon”. For $p = 0$, we have a Riemannian theory with Chern-Simons term in the action, and our $S$ coincides with Solodukhin’s result [19].

(3) The existence of torsion is in complete agreement with the first law of black hole thermodynamics.

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APPENDIX A: EUCLIDEAN CONTINUATION

Euclidean continuation of Minkowskian black holes can be formally expressed as a mapping $f_E$ from Minkowskian to Euclidean variables, such that
\[
f_E : \ t \mapsto -it , \quad J \mapsto -iJ .
\] (A1)

As a consequence, the induced mapping of the triad field reads
\[
b^0 \mapsto -ib^0 , \quad b^1 \mapsto b^1 , \quad b^2 \mapsto b^2 .
\]
The analytic continuation maps $\eta^M = (1, -1, -1)$ into $\bar{\eta} = (-1, -1, -1)$. Note, however, that we define our Euclidean theory to have the positive-definite metric:

$$f_E : \quad \eta^M \mapsto \eta^E = (1, 1, 1).$$

Demanding that the torsion $T^i$ maps in the same way as $b^i$, we find

$$\omega^0 \mapsto -\omega^0, \quad \omega^1 \mapsto -i\omega^1, \quad \omega^2 \mapsto -i\omega^2,$$

which then defines the mapping of the curvature $R^i$. It is now easy to derive the mappings of different terms in the Mielke-Baekler action:

$$b^i R_i \mapsto -ib^i R_i, \quad b^0 b^1 b^2 \mapsto -ib^0 b^1 b^2,$$
$$\mathcal{L}_{cs}(\omega) \mapsto \mathcal{L}_{cs}(\omega), \quad b^i T_i \mapsto -b^i T_i,$$

where, on the right-hand sides, we are using not $\bar{\eta}$, but $\eta^E$. Now, the mapping of the complete action integral,

$$f_E : \quad iI \mapsto -I,$$

can be effectively expressed by the following mapping of parameters:

$$f'_E : \quad a \mapsto a, \quad A \mapsto -A,$$
$$\alpha_3 \mapsto i\alpha_3, \quad \alpha_4 \mapsto -i\alpha_4.$$  \hfill (A3)

In particular, using $-1/\ell^2 = A_{\text{eff}} \mapsto -A_{\text{eff}} = -1/\ell^2$, we see that $A_{\text{eff}}$ changes the sign, while $\ell$ remains unchanged.

**APPENDIX B: HYPERBOLIC 3D SPACE**

Here, we review some facts about the Euclidean AdS$_3$, known also as the hyperbolic 3D space $H^3$ (see, e.g. [9,26]). Consider a hypersurface

$$(y^0)^2 + (y^1)^2 + (y^2)^2 - z^2 = -\ell^2, \quad \ell^2 > 0,$$

embedded in a four-dimensional Minkowski space $M_4$ with metric $\eta_{MN} = (1, 1, 1, -1)$. The hypersurface consists of two disjoint hyperboloids, with $z \geq \ell$ and $z \leq -\ell$, and $H^3$ can be visualized as one of these hyperboloids. Clearly, $H^3$ has the isometry group $SO(1,3)$, and can be thought of as the coset space $SO(1,3)/SO(3)$. The Riemannian scalar curvature of $H^3$ is negative, $\tilde{R} = -6/\ell^2$, and the signature is $(+, +, +)$.

Using the parametrization

$$y^0 = \ell \cosh \rho \sinh \psi, \quad y^1 = \ell \sinh \rho \cos \phi,$$
$$z = \ell \cos \rho \cosh \psi, \quad y^2 = \ell \sinh \rho \sin \phi,$$

the metric on $H^3 (z \geq \ell)$ takes the form

$$ds^2 = \ell^2 \left( d\rho^2 + \cosh^2 \rho \ d\psi^2 + \sinh^2 \rho \ d\phi^2 \right).$$  \hfill (B1)
The change of coordinates $\psi = t/\ell$, $r = \ell \sinh \rho$, shows that $H^3$ is isometric to the Euclidean version of AdS$_3$, equation (3.3).

Introducing $\cos \chi = 1 / \cosh \rho$, we find another useful form of the metric:

$$d s^2 = \frac{\ell^2}{\cos^2 \chi} \left( d \chi^2 + d \psi^2 + \sin^2 \chi d \phi^2 \right). \quad (B2)$$

The coordinate transformation

$$\phi = \frac{r_+}{\ell^2} t + \frac{\rho_-}{\ell} \varphi, \quad \psi = \frac{r_+}{\ell} \varphi - \frac{\rho_-}{\ell^2} t,$$

$$\cos \chi = \sqrt{\frac{r_+^2 + \rho_-^2}{r_+^2 + \rho_-^2}}, \quad (B3)$$
yields the black hole metric in Schwarzschild coordinates $(t, r, \phi)$. The BTZ metric (3.1) is obtained from the AdS$_3$ metric (B2) by making the following (isometric) identifications:

(i) $(\chi, \psi, \phi) \rightarrow (\chi, \psi, \phi + 2\pi)$, which eliminates the conical singularity at $\chi = 0$ (horizon) in (B2), and is equivalent to $(r, \varphi, t) \rightarrow (r, \varphi + \Phi, t + \beta)$, with

$$\Phi = \frac{2\pi \ell \rho_-}{r_+^2 + \rho_-^2}, \quad \beta = \frac{2\pi \ell^2 r_+}{r_+^2 + \rho_-^2}.$$

(ii) $(\chi, \psi, \phi) \rightarrow (\chi, \psi + 2\pi r_+/\ell, \phi + 2\pi \rho_-/\ell)$, which is equivalent to $\varphi \rightarrow \varphi + 2\pi$.

Thus, the Euclidean black hole may be described as the quotient of the hyperbolic space $H^3$ by the isometry (i)+(ii). The topology of the black hole manifold is a solid torus, $R^2 \times S^1$.

The identification (i) shows that $\varphi$ is not the usual Schwarzschild azimuthal angle $\varphi'$. The relation between them is $\varphi' = \varphi + \Omega t$, where $\Omega = -\rho_-/r_+\ell = N_\varphi(r_+)$. Indeed,

$$(\varphi', t) \rightarrow (\varphi' + \Phi + \Omega \beta, t + \beta) = (\varphi', t + \beta).$$

Note that $d \varphi + N_\varphi dt = d \varphi' + (N_\varphi - \Omega) dt$, so that $N'_\varphi = N_\varphi - \Omega = 0$ at the horizon.

The Poincaré upper half-space model for $H^3$ is given by the metric

$$d s^2 = \frac{1}{z^2} \left( dx^2 + dy^2 + dz^2 \right), \quad z > 0. \quad (B4)$$

It follows from (B2) by a simple coordinate transformation:

$$x = \exp \psi \sin \chi \cos \phi,$$
$$y = \exp \psi \sin \chi \sin \phi,$$
$$z = \exp \psi \cos \chi.$$

In the standard spherical coordinates with $R = \exp \psi$, we have:

$$d s^2 = \frac{\ell^2}{\cos^2 \chi} \left( \frac{d R^2}{R^2} + d \psi^2 + \sin^2 \chi d \phi^2 \right). \quad (B5)$$

The identification $\varphi \rightarrow \varphi + 2\pi$ is now described by $(\chi, R, \phi) \rightarrow (\chi, R e^{2\pi r_+/\ell}, \phi + 2\pi \rho_-/\ell)$. 

15
REFERENCES


* The field equations are algebraic equations for the field strengths \((T^i, R^i)\). In the degenerate case \(\alpha_3 \alpha_4 - \alpha^2 = 0\), the number of independent equations reduces from six to three, and the general solution for \((T^i, R^i)\) contains three arbitrary functions. For a particular choice of these functions, one still finds the black hole as an exact solution. Here, we restrict our attention to the non-degenerate case in order to avoid dynamically indeterminate situations.


