Relation between CKM and MNS matrices induced by
Bi-Maximal Rotations in the Seesaw Mechanism

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Abstract

It is found that the seesaw mechanism not only explains the smallness of neutrino masses but
also accounts for the large mixing angles simultaneously, even if the unification of the neutrino
Dirac mass matrix with that of up-type quark sector is realized. In this mechanism, we show that
the mixing matrix of the Dirac-type mass matrix gets extra rotations from the diagonalization of
Majorana mass matrix. We thus find that provided the mixing angle around y-axis to diagonalize
the Majorana mass matrix vanishes, we can explain the large mixing angles of leptonic sector found
in atmospheric and long baseline reactor neutrino oscillation experiments. We also can derive
the information about the absolute values of neutrino masses and Majorana mass responsible for
the neutrinoless double beta decay experiment through the data set of neutrino experiments. In
the simplified case that there is no CP phase, we find that the neutrino masses are decided as
$m_1 : m_2 : m_3 \approx 1 : 2 : 8$ and that there are no solution which satisfy $m_3 < m_1 < m_2$ (inverted mass
spectrum). Then, including all CP phases, we reanalyze the absolute values of neutrino masses
and Majorana mass responsible for the neutrinoless double beta decay experiment.

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I. INTRODUCTION

Neutrino sector has many curious properties which are not shared by the quark and charged leptonic sectors. For example, neutrino masses are very small \([1]\) compared with those of quarks and charge leptons. The large mixing angles seen in the experiments of atmospheric neutrino oscillation and long baseline reactor neutrino oscillation experiments (related to solar neutrino deficit) \([2, 3, 4, 5]\) are also new features, not seen in the quark sector.

It is well known that the seesaw mechanism \([6, 7, 8]\) can explain the small mass scale of neutrinos naturally. In this mechanism, neutrino mass matrix which describes the low energy observables is given approximately by

\[
\mathcal{M}_\nu = -(\mathcal{M}_D)^T (\mathcal{M}_R)^{-1} (\mathcal{M}_D),
\]

where \(\mathcal{M}_D\) and \(\mathcal{M}_R\) are the Dirac and Majorana mass matrices of neutrino, respectively. The unpleasant overall minus sign can be absorbed by redefinition of field as \(\nu \rightarrow i\gamma^5\nu\), i.e.

\[
\mathcal{L}_{\text{mass}} = - (\bar{\nu}_L)^C \mathcal{M}_\nu \nu_L + \text{h.c.} = -\nu^T \mathcal{C} \left( \frac{1 - \gamma^5}{2} \right) \mathcal{M}_\nu \nu + \text{h.c.}
\]

\[
\rightarrow -\nu^T \mathcal{C} \left( \frac{1 - \gamma^5}{2} \right) \mathcal{M}_\nu \nu + \text{h.c.}
\]

If we require that the order of magnitude of \(\mathcal{M}_D\) is the weak scale and that of \(\mathcal{M}_R\) is the GUT scale, we can roughly obtain the desired order of magnitude of \(\mathcal{M}_\nu\).

In addition, this mechanism can also explain the large mixing angles in the leptonic sector inheriting the unification of lepton and quark sectors as is seen in \(SO(10)\) GUT. Especially, it has been pointed out in some articles (e.g. \([9]\)) that there exist interesting and amusing relations between CKM and MNS matrices:

\[
\bullet \ \theta_{\text{sol}} + \theta_{\text{Cabibbo}} \simeq 45^\circ \quad (1.2)
\]

\[
\bullet \ \theta_{\text{atm}} + \theta_{23}^{\text{CKM}} \simeq 45^\circ. \quad (1.3)
\]

These relations may imply that there exist some nontrivial relations between CKM and MNS matrices and that the seesaw mechanism has a comparatively simple structures as seen below.

To clarify our procedures, we use the following notations, i.e. \(\mathcal{M}_D\) and \(\mathcal{M}_R\) are diagonalized as

\[
\mathcal{V}_R^\dagger \mathcal{M}_D \mathcal{V}_L = \mathcal{M}_D, \quad \mathcal{U}^T \mathcal{M}_R \mathcal{U} = \mathcal{M}_R,
\]
where $\hat{M}_D$ and $\hat{M}_R$ are diagonalized Dirac and Majorana mass matrices, respectively and $V_{L,R}$ and $U$ are unitary matrices. Using these notations in Eq. (1.1), $M_\nu$ can be written as

\begin{equation}
M_\nu = V_L^\dagger (\hat{M}_D) V_R^T \cdot U (\hat{M}_R)^{-1} U^T \cdot V_R (\hat{M}_D) V_L^\dagger,
\end{equation}

(1.4)

where we define a unitary matrix, $U_R = V_R^T U$. In SO(10) GUT, there are some nontrivial relations between quark and leptonic sectors and furthermore between $V_L$ and $V_R$ above the symmetry breaking scale, which we adopt in this work. This is because SO(10) includes a subgroup, $G = SU(4)_{PS} \times SU(2)_L \times SU(2)_R$. $SU(4)_{PS}$ symmetry leads to the relations between quark and lepton Yukawa coupling matrices, i.e.

\begin{equation}
Y_u = Y_\nu, \quad Y_d = Y_e,
\end{equation}

(1.5)

where the indices of $u, d, \nu, e$ correspond to up-type quark, down-type quark, neutrino, charged-lepton, respectively. In addition, $SU(2)_L \times SU(2)_R$ symmetry leads to left-right symmetry. Since the two indices to denote the matrix elements of Dirac-type mass matrix correspond to left- and right-handed neutrinos, this symmetry reduces the degrees of freedom of the matrix, i.e. it should be a symmetric matrix, and this leads to a relation

\begin{equation}
V_R = V_R^T.
\end{equation}

(1.6)

For simplicity of the argument, we assume that these relations hold approximately at low energies. Adopting a certain basis in which the down-type quark mass matrix is diagonalized, the former relation in Eq. (1.5) leads to

\begin{equation}
V_R^\dagger = V_{CKM}.
\end{equation}

(1.7)

Then, we can rewrite Eq. (1.4) as

\begin{equation}
M_\nu = V_{CKM}^T (\hat{M}_D) U_R (\hat{M}_R)^{-1} U_R^T (\hat{M}_D) V_{CKM}.
\end{equation}

(1.8)

The r.h.s. of Eq. (1.8) is furthermore diagonalized as

\begin{equation}
M_\nu = V_{CKM}^T O (\hat{M}_\nu) O^\dagger V_{CKM},
\end{equation}

(1.9)
where $\hat{M}_\nu$ is the diagonalized neutrino mass matrix with mass eigenvalues $\mu_1, \mu_2$ and $\mu_3$. The matrix $V_{CKM}^\dagger \times O$ is what we call MNS matrix of leptonic sector

$$
\begin{bmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{bmatrix} =
\begin{bmatrix}
V_{MNS} \\
\vphantom{V_{MNS}}
\end{bmatrix}
\begin{bmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{bmatrix}.
$$

(1.10)

In this way, the mixing matrix, $V_{CKM}$, is accompanied by extra rotations by $O$, so that we can explain the disagreement between CKM and MNS matrices and the large mixing angles of leptonic sector once $O$ contains large (maximal) mixing angles.

In this manuscript, we especially concentrate our attention on the relations found in Eqs. (1.2), (1.3). As we sketch right below, these relations are realized provided the extra rotations due to $O$ are bi-maximal rotations around $x$- and $z$-axes. This may be natural since these relations are concerning $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$ generation mixings.

According to an approximation proposed by Wolfenstein [10], we can parametrize $V_{CKM}$ as

$$
V_{CKM} = \begin{bmatrix}
1 - \frac{A^2}{2} & \lambda & A\lambda^3 \rho (1 - i\eta) \\
-\lambda & 1 - \frac{A^2}{2} & A\lambda^2 \\
A\lambda^3 (1 - \rho - i\eta) & -A\lambda^2 & 1
\end{bmatrix},
$$

where $\lambda \simeq \sin \theta_C$ and $A, \rho$ and $\eta$ are quantities of the order of unity. Roughly speaking, since the bi-maximal extra rotations shifts these angles to

$$
-\lambda \rightarrow -\lambda + 45^\circ, \\
-A\lambda^2 \rightarrow -A\lambda^2 + 45^\circ,
$$

we can obtain the desired relations in Eq. (1.2) and Eq. (1.3).

We can easily understand the relation between CKM and MNS matrices geometrically. The explicit forms of $V_{CKM}$ and $V_{MNS}^\dagger$ with a standard parametrization are given by

$$
V_{CKM} \approx \begin{bmatrix}
0.9745 & 0.2243 & 0.0037 \\
-0.2243 & 0.9737 & 0.0413 \\
0.0057 & -0.0411 & 0.9991
\end{bmatrix},
V_{MNS}^\dagger \approx \begin{bmatrix}
0.8482 & -0.3746 & 0.3746 \\
0.5297 & 0.5998 & -0.5998 \\
0 & 0.7071 & 0.7071
\end{bmatrix},
$$

where we set $\theta_{12}^{MNS} = \theta_{sol} = 32^\circ, \theta_{23}^{MNS} = \theta_{atm} = 45^\circ, \theta_{13}^{MNS} = \theta_{CHOOZ} = 0^\circ$ and ignore CP phases tentatively. Decomposing these matrices to vector representations

$$
V_{CKM} = \begin{bmatrix}
\bar{v}_1 \\
\bar{v}_2 \\
\bar{v}_3
\end{bmatrix},
V_{MNS}^\dagger = \begin{bmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\bar{u}_3
\end{bmatrix},
$$

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we can express these in vector space as shown in Fig. 1. We parametrize the orthogonal matrix $O$ as $O = O_x(\Theta_1) \cdot O_z(\Theta_3)$. Then, from the relation between CKM and MNS matrices, $V_{MNS}^\dagger = O^\dagger \times V_{CKM}$, we can easily see that $\vec{v}_i$ is rotated around x-axis by $\Theta_1$ first and around z-axis by $\Theta_3$ next to get $\vec{u}_i$. Thus, we can roughly achieve the above relations, once $\Theta_1$ and $\Theta_3$ are (almost) maximal.

In what follows, we parametrize the unitary matrices $U_R$ by three mixing angles and five CP phases after absorbing one overall CP phase by the rephasing of fields, i.e.

$$
U_R = P_1 \times V_R \times P_2
$$

$$
V_R = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1^R & \sin \theta_1^R \\
0 & -\sin \theta_1^R & \cos \theta_1^R
\end{bmatrix}
\begin{bmatrix}
\cos \theta_2^R & 0 & e^{-i\delta^R} \sin \theta_2^R \\
0 & 1 & 0 \\
-e^{i\delta^R} \sin \theta_2^R & 0 & \cos \theta_2^R
\end{bmatrix}
\begin{bmatrix}
\cos \theta_3^R & \sin \theta_3^R & 0 \\
-\sin \theta_3^R & \cos \theta_3^R & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

where $P_i \overset{\text{def}}{=} \text{diag}(e^{i\epsilon_1}, 1, e^{i\kappa_i})$, in which three mixing angles, $\theta_1^R, \theta_2^R$ and $\theta_3^R$, and one CP phase, $\delta^R$, are embedded.

This paper is organized as follows. In Sec. II we discuss the extra rotations, sketched above, more carefully and emphasize that we can not only realize the relation (1.2), (1.3),...
but also derive the absolute values of neutrino masses, by comparing with the existing experimental data of neutrino oscillations. The effects of five CP phases which embedded in $U_R$, i.e. $\epsilon_i, \kappa_i$ and $\delta^R$, are discussed in Sec.III.

II. BI-MAXIMAL EXTRA ROTATIONS AND ESTIMATION OF ABSOLUTE VALUES OF NEUTRINO MASSES

In this section, we investigate neutrino mass matrix by switching off the five CP phases for simplicity. Recalling Eq.(1.8),

$$M_\nu = V^T_{CKM}(\hat{M}_D)V_R(\hat{M}_R)^{-1}V^T_R(\hat{M}_D)V^*_{CKM},$$

we express the diagonalized mass matrices as

$$\hat{M}_D = \text{diag}(m_1, m_2, m_3) \quad \hat{M}_R = \text{diag}(M_1, M_2, M_3)$$

and parametrize the matrix $V_R$ in a specific form by two rotations around y- and z-axes as

$$V_R = \begin{bmatrix} \cos \theta^R_2 & 0 & \sin \theta^R_2 \\ 0 & 1 & 0 \\ -\sin \theta^R_2 & 0 & \cos \theta^R_2 \end{bmatrix} \begin{bmatrix} \cos \theta^R_3 & \sin \theta^R_3 & 0 \\ -\sin \theta^R_3 & \cos \theta^R_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

In general, we should parametrize $V_R$ by three mixing angles as seen in Eq.(1.11). We, however, can achieve bi-maximal rotations with a minimal set of mixing angles as is seen in Eq.(2.2) and this can be understood as follows. Defining

$$\mathcal{M} = \frac{M_1}{m_1^2} \times (\hat{M}_D)V_R(\hat{M}_R)^{-1}V^T_R(\hat{M}_D)$$

$$= \begin{bmatrix} 1 \\ \frac{m_2}{m_1} \\ \frac{m_3}{m_1} \end{bmatrix} V_R \begin{bmatrix} 1 \\ \frac{M_1}{M_2} \\ \frac{M_1}{M_3} \end{bmatrix} V^T_R \begin{bmatrix} 1 \\ \frac{m_2}{m_1} \\ \frac{m_3}{m_1} \end{bmatrix},$$

we can estimate diagonal terms as $\mathcal{M}_{ii} \sim \frac{M_i}{M_i'} \left(\frac{m_i}{m_i'}\right)^2$ and off-diagonal terms as

$$\mathcal{M}_{12} = \frac{m_2}{m_1} \theta^R_3 \quad \mathcal{M}_{13} = \frac{m_3}{m_1} \theta^R_2 \quad \mathcal{M}_{23} = \frac{m_2 m_3}{m_1^2 M_2} M_1 \theta^R_1,$$

where we assume that these angles are extremely small. Requiring that the order of magnitude of these quantities are unity, we find

$$\frac{M_1}{M_i} = \mathcal{O} \left(\frac{m_1^2}{m_i^2}\right) \quad \theta^R_1 = \mathcal{O} \left(\frac{m_2}{m_3}\right) \quad \theta^R_2 = \mathcal{O} \left(\frac{m_1}{m_3}\right) \quad \theta^R_3 = \mathcal{O} \left(\frac{m_1}{m_2}\right).$$
Using these naive estimations, we can easily find that the choice of $V_R$ in Eq. (2.2) just leads to the conditions that $\mathcal{M}_{22} \simeq \mathcal{M}_{33}$ and $\mathcal{M}'_{22} \simeq \mathcal{M}_{11}$ (where $\mathcal{M}'_{22}$ denotes a matrix element after rotating by $\Theta_1$), once we expand the allowed region of $M_1$ to negative region ($\epsilon_2 = 0$ or $\pi/2$). Using Eq. (2.2), we can express $\mathcal{M}$ as

$$\mathcal{M} = c_3^2 \times \begin{bmatrix}
    c_2^2 \left(1 + \frac{M_1}{M_2} t_3^2\right) + \frac{M_1}{M_3} s_3^2 \\
    -\frac{m_2}{m_1} c_2 t_3 \left(1 - \frac{M_1}{M_2}\right) \\
    -\frac{m_3}{m_1} s_2 c_2 \left(1 + \frac{M_1}{M_2} t_3^2 + \frac{M_1}{M_3} c_3^2\right) \\
    \end{bmatrix} \begin{bmatrix}
    1 - \frac{M_1}{M_2} t_3^2 \\
    \frac{m_2}{m_1} c_2 t_3 \left(1 - \frac{M_1}{M_2}\right) \\
    \frac{m_3}{m_1} s_2 c_2 \left(1 + \frac{M_1}{M_2} t_3^2 + \frac{M_1}{M_3} c_3^2\right) \\
    \end{bmatrix},$$

(2.5)

where we define $s_2 = \sin \theta_2^R$ etc. Referring Eq. (2.4), we can approximate $\mathcal{M}$ up to leading order as

$$\mathcal{M} \simeq \begin{bmatrix}
    1 \\
    -\frac{m_2}{m_1} \theta_3^R \\
    -\frac{m_3}{m_1} \theta_2^R \\
    \end{bmatrix},$$

(2.6)

Requiring that Eq. (2.7) can be diagonalized by $O = O_x(\Theta_1) \cdot O_z(\Theta_3)$ with two rotations around x- and z-axes, we can immediately diagonalize $\mathcal{M}_\nu$ as

$$\mathcal{M}_\nu = \frac{m_1^2}{M_1} \times V_{CKM}^T (O_x(\Theta_1)O_z(\Theta_3)) \begin{bmatrix}
    \rho_1 \\
    \rho_2 \\
    \rho_3 \\
    \end{bmatrix} (O_x(\Theta_1)O_z(\Theta_3))^T V_{CKM},$$

(2.7)

where we define

$$\rho_1 = 1 + \left(\frac{\tan \Theta_3}{\cos \Theta_1}\right) x \quad \rho_2 = 1 - \left(\frac{\cot \Theta_3}{\cos \Theta_1}\right) x \quad \rho_3 = \frac{m_3^2}{M_1} \frac{M_1}{M_2},$$

(2.8)

$$\theta_2^R = -(\tan \Theta_1) \frac{m_2}{m_1} \theta_3^R \quad \frac{M_2}{M_3} = \frac{m_2}{m_3} \quad \tan 2\Theta_3 = \frac{-2 x}{(\cos \Theta_1)^2 + \rho_3 - 1},$$

(2.9)

$$x \overset{\text{def}}{=} \frac{m_2}{m_1} \theta_3^R.$$  

(2.10)

In what follows, we set $\Theta_1$ equals to $45^\circ$ as mentioned in the previous section and leave the degree of freedom of $\Theta_3$, because as seen in the following the former rotation around x-axis reduce the magnitude of $\lambda$ as $\lambda \rightarrow \lambda/\sqrt{2}$ and $\Theta_3$ cannot be taken to be maximal.
Eventually, we find the relation between CKM and MNS matrices up to $\mathcal{O}(\lambda^2)$ as follows.

$$V_{MNS} = V_{CKM}^\dagger \times O_2(45^\circ) \times O_2(\Theta_3)$$

\begin{align*}
&\sim \begin{bmatrix}
1 - \frac{\lambda^2}{2} & -\lambda & 0 \\
\lambda & 1 - \frac{\lambda^2}{2} - A\lambda^2 & 0 \\
0 & A\lambda^2 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix} \begin{bmatrix}
\cos \Theta_3 & \sin \Theta_3 & 0 \\
-\sin \Theta_3 & \cos \Theta_3 & 0 \\
0 & 0 & 1
\end{bmatrix} \\
&= \begin{bmatrix}
(1 - \frac{\lambda^2}{2}) \cos \Theta_3 + \frac{\lambda}{\sqrt{2}} \sin \Theta_3 & (1 - \frac{\lambda^2}{2}) \sin \Theta_3 - \frac{\lambda}{\sqrt{2}} \cos \Theta_3 \\
\lambda \cos \Theta_3 - \frac{\lambda}{\sqrt{2}} (1 + \left(A - \frac{1}{2}\right) \lambda^2) \sin \Theta_3 & \lambda \sin \Theta_3 + \frac{\lambda}{\sqrt{2}} (1 + \left(A - \frac{1}{2}\right) \lambda^2) \cos \Theta_3 \\
\frac{1}{\sqrt{2}} (1 - A\lambda^2) \sin \Theta_3 & -\frac{1}{\sqrt{2}} (1 - A\lambda^2) \cos \Theta_3
\end{bmatrix} \\
&\quad \begin{bmatrix}
\frac{1}{\sqrt{2}} (1 - \left(A + \frac{1}{2}\right) \lambda^2) \\
\frac{1}{\sqrt{2}} (1 + A\lambda^2)
\end{bmatrix}
\end{align*}

(2.12)

Then, comparing with standard parametrization of $V_{MNS}$,

$$V_{MNS} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{23} & \sin \theta_{23} \\
0 & -\sin \theta_{23} & \cos \theta_{23}
\end{bmatrix} \begin{bmatrix}
\cos \theta_{13} & 0 & e^{-i\delta} \sin \theta_{13} \\
0 & 1 & 0 \\
-e^{i\delta} \sin \theta_{13} & 0 & \cos \theta_{13}
\end{bmatrix} \begin{bmatrix}
\cos \theta_{12} & \sin \theta_{12} & 0 \\
-\sin \theta_{12} & \cos \theta_{12} & 0 \\
0 & 0 & 1
\end{bmatrix},$$

we can immediately find the relations between observed mixing angles, i.e. $\theta_{12}, \theta_{23}$ and $\theta_{13}$, and $\Theta_3, \lambda$ up to $\mathcal{O}(\lambda^2)$:

$$\theta_{12} = \Theta_3 - \frac{\lambda}{\sqrt{2}} \quad \theta_{23} = 45^\circ - \left(A + \frac{1}{4}\right) \lambda^2 \quad \theta_{13} = \frac{\lambda}{\sqrt{2}} \quad \delta = \pi.$$  

Note that this model deduces the order of magnitude of $\theta_{13}$. Though this value is not so small for $\lambda = \sin \theta_C$, it is still not conflict with the experimental data from CHOOZ experiment, i.e.

$$\sin^2 \theta_{13} \leq 0.041 \quad (3\sigma \text{ C.L.}).$$

We can fix $\Theta_3$ using the experimental data of $\theta_{sol}(\simeq \theta_{12})$. Combining this result with a constraint of the ratio on mass-squared differences from experimental data

$$\frac{\rho_2^2 - \rho_1^2}{\rho_3^2 - \rho_2^2} = \frac{\Delta m_{sol}^2}{\Delta m_{atm}^2} = \Delta,$$

we can finally fix the remaining dimensionless parameter, $x$ in Eqs. (2.9), (2.10), i.e.

$$\rho_1 = 1 + \sqrt{2}(\tan \Theta_3)x \quad \rho_2 = 1 - \sqrt{2}(\cot \Theta_3)x \quad \rho_3 = 1 - 2\sqrt{2}(\cot 2\Theta_3)x - 2x^2.$$
Note that there are possibilities that $\rho_i$'s take negative values. We, however, can always define $\rho_i$ to positive, thanks to the Majorana phases. Using the best fit values [16]:

$$\tan^2 \theta_{sol} = 0.39$$

$$\Delta m^2_{sol} = 8.2 \times 10^{-5} \text{eV}^2 \quad \mid \Delta m^2_{atm} \mid = 2.2 \times 10^{-3} \text{eV}^2,$$

$\Theta_3$ is fixed as $\Theta_3 = 41.15^\circ$ and we eventually find

$$x = -4.410$$

$$\rho_1 = -4.450 \quad \rho_2 = 8.137 \quad \rho_3 = -36.21$$

$$\mid \rho_2/\rho_1 \mid = 1.828 \quad \mid \rho_3/\rho_1 \mid = 8.137.$$

Note that $\rho_3$ is larger than $O(1)$. We, however, confirm the validity of the approximation in Eq.(2.7) since we expect that the order of magnitude of $m_1/m_2$ is similar to $m_u/m_c$ in up-quark sector from quark-lepton symmetry in $SO(10)$ GUT. We cannot find any solutions in case of $\Delta m^2_{atm} < 0$, i.e. inverted mass spectrum case. In general, there are two possible cases reflecting the uncertainty of the sign of mass-squared difference in atmospheric neutrino oscillation experiment, normal or inverted mass spectrum, i.e. $\Delta m^2_{atm} > 0$ or $\Delta m^2_{atm} < 0$, respectively. There are some proposals to fix the sign of atmospheric neutrino mass squared difference, i.e. discrimination between normal mass spectrum and inverted one by utilizing the difference of matter effect of the earth between electron neutrino and electron anti-neutrino at Neutrino Factory [17, 18].

Then, we can find absolute values of neutrino masses by using the following equation,

$$\mu_i = \frac{|\rho_i|}{\sqrt{\rho_2^2 - \rho_1^2}} \times \sqrt{\Delta m^2_{sol}}.$$

These equations lead to

$$\mu_1 = 0.5916 \times 10^{-2} \text{ (eV)} \quad \mu_2 = 1.082 \times 10^{-2} \text{ (eV)} \quad \mu_3 = 4.814 \times 10^{-2} \text{ (eV)}.$$

III. THE EFFECTS OF CP PHASES AND THE ESTIMATION OF THE MAJORANA MASS RESPONSIBLE FOR THE NEUTRINOLESS DOUBLE BETA DECAY

In previous section, we neglect five CP phases, $\epsilon_i, \kappa_i$ and $\delta^R$. In this section, we reanalyze neutrino mass matrix including all CP phases. At first, embedding $\delta^R$ into $V_R$ corresponds
to

\[ V_R = \begin{bmatrix}
\cos \theta_R^2 & 0 & e^{-i \delta_R} \sin \theta_R^2 \\
0 & 1 & 0 \\
-e^{i \delta_R} \sin \theta_R^2 & 0 & \cos \theta_R^2
\end{bmatrix} \begin{bmatrix}
\cos \theta_R^3 & \sin \theta_R^3 & 0 \\
-\sin \theta_R^3 & \cos \theta_R^3 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
e^{i \delta_R}
\end{bmatrix} \begin{bmatrix}
\cos \theta_R^2 & 0 & \sin \theta_R^2 \\
0 & 1 & 0 \\
-\sin \theta_R^2 & 0 & \cos \theta_R^2
\end{bmatrix} \begin{bmatrix}
\cos \theta_R^3 & \sin \theta_R^3 & 0 \\
-\sin \theta_R^3 & \cos \theta_R^3 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
e^{-i \delta_R}
\end{bmatrix}.

This phase, however, is not a physical phase since in this paper we adopt the condition that \( \theta_1^R \) equals to zero; the phase \( \delta^R \) in the most left and the most right matrices in the above expression can be absorbed into \( \kappa_1 \) and \( \kappa_2 \), respectively. Therefore, we can concentrate our attentions on four CP phases, \( \epsilon_i \) and \( \kappa_i \). Next, the effect of embedding \( \epsilon_i \) and \( \kappa_i \) into \( \mathcal{M} \) corresponds to the substitutions,

\[
m_1 \rightarrow \tilde{m}_1 = e^{i \epsilon_1} m_1 \quad m_3 \rightarrow \tilde{m}_3 = e^{i \kappa_3} m_3
\]

\[ M_1 \rightarrow \tilde{M}_1 = e^{-2i \epsilon_2} M_1 \quad M_3 \rightarrow \tilde{M}_3 = e^{-2i \kappa_2} M_3. \]

Using these substitutions, we can write down approximated expression of \( \mathcal{M} = (\tilde{\mathcal{M}}_D) U_R (\tilde{\mathcal{M}}_R)^{-1} U_R^T (\tilde{\mathcal{M}}_D) \) referring to Eq.(2.7) as

\[ \mathcal{M} \simeq \frac{\tilde{m}_1^2}{M_1} \times \begin{bmatrix}
1 \\
\frac{-m_2}{m_1} \theta_R^3 \frac{m_2^2}{m_1^2} (\theta_R^3)^2 + \frac{\tilde{M}_3}{M_2} \\
\frac{-\tilde{m}_3}{m_1} \theta_R^2 \frac{m_2 \tilde{m}_3}{m_1^2} \theta_R^3 \frac{m_2^2 \tilde{m}_3}{m_1^2} (\theta_R^3)^2 + \frac{\tilde{M}_3}{M_2}
\end{bmatrix}. \quad (3.1)

When we diagonalize this matrix along to the same way in previous section, the conditions to satisfy \( \Theta_2 = 0 \) after rotation around x-axis (\( \Theta_1 \)) are

\[ \theta_2^R = -(\tan \Theta_1) \frac{m_2}{m_3} \theta_3^R \quad \frac{M_2}{M_3} = \frac{m_2^2}{\tilde{m}_3^2} \quad (3.2)
\]

To maintain the statement that \( m_i, M_i \) and \( \theta_i^R \) are defined by real numbers, we set \( \kappa_1 = -\kappa_2 \) and regard \( \kappa_1 \) as a CP phase which embedded in mixing matrix \( O_x (\Theta_1) \), i.e. \( e^{\pm i \kappa_1} \sin \Theta_1 \). Thus, we can achieve correct diagonalization of \( \mathcal{M}_\nu \) maintaining \( m_i, M_i \) and \( \theta_i^R \) real numbers. This phase, \( \kappa_1 \), has a physical meaning as seen below. The diagonalized mass matrix is written as

\[ \mathcal{M}_\nu = e^{2i(\epsilon_1 + \epsilon_2)} \frac{\tilde{m}_1^2}{M_1} \times V_{\text{MNS}}^* \begin{bmatrix}
\tilde{\rho}_1 & 0 & 0 \\
0 & \tilde{\rho}_2 & 0 \\
0 & 0 & \tilde{\rho}_3
\end{bmatrix} V_{\text{MNS}}^T, \]
and MNS matrix can be written as

\[ V_{MNS} = V_{CKM}^\dagger \times O \]

\[ = V_{CKM}^\dagger \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Theta_1 & e^{i\kappa_1} \sin \Theta_1 \\ 0 & -e^{-i\kappa_1} \sin \Theta_1 & \cos \Theta_1 \end{bmatrix} \begin{bmatrix} \cos \Theta_3 & \sin \Theta_3 & 0 \\ -\sin \Theta_3 & \cos \Theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} 1 & 1 \\ e^{-i\kappa_1} & 1 \\ e^{i\kappa_1} & e^{-i\kappa_1} \end{bmatrix} \begin{bmatrix} 1 \\ V_{CKM}^\dagger \\ 1 \end{bmatrix} \]

\[ \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Theta_1 \sin \Theta_1 \\ 0 & -\sin \Theta_1 \cos \Theta_1 \end{bmatrix} \begin{bmatrix} \cos \Theta_3 & \sin \Theta_3 & 0 \\ -\sin \Theta_3 & \cos \Theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 1 \\ 1 \\ e^{i\kappa_1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\lambda^2}{2} & -\lambda \mathcal{O}(\lambda^3) \cdot e^{-i\kappa_1} \\ \lambda & 1 - \frac{\lambda^2}{2} - A\lambda^2 \cdot e^{-i\kappa_1} \\ \mathcal{O}(\lambda^3) \cdot e^{i\kappa_1} A\lambda^2 \cdot e^{i\kappa_1} & 1 \end{bmatrix}, \]

Rewriting

\[ \begin{bmatrix} 1 & 1 \\ e^{i\kappa_1} & e^{-i\kappa_1} \end{bmatrix} \begin{bmatrix} 1 \\ V_{CKM}^\dagger \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \]

the effect of \( \kappa_1 \) is always suppressed by \( \mathcal{O}(\lambda^2) \), so that we can follow the same procedure with Eq.\(^{(2.12)}\) up to \( \mathcal{O}(\lambda) \). Then, the right phase term shifts the arguments of \( \tilde{\rho}_3 \) as \( \arg(\tilde{\rho}_3) \rightarrow \arg(\tilde{\rho}_3) - 2\kappa_1 \). This phase has no effect on the absolute value of 3rd neutrino mass but appears in some phenomena in which observables are relevant to the Majorana phases, e.g. neutrinoless double beta decay mentioned latter in this paper.

Eventually, we can write the correct diagonal mass matrix \( M_\nu \) as

\[ M_\nu = e^{2i(\epsilon_1 + \epsilon_2) \frac{m^2}{M_1}} \times \begin{bmatrix} 1 & \rho_1 & 0 & 0 \\ \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & e^{-2i\kappa_1} \rho_3 \end{bmatrix} \begin{bmatrix} \tilde{\rho}_1 \end{bmatrix} \]

\[ \leftarrow \text{rephasing} \quad \frac{m^2}{M_1} \times V_{MNS}^* \begin{bmatrix} \tilde{\rho}_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \rho_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\rho}_3 \end{bmatrix} \]

\[ \tilde{\rho}_1 = 1 + \frac{\tan \Theta_3}{\cos \Theta_1} \bar{x} \]

\[ \tilde{\rho}_2 = 1 - \frac{\cot \Theta_3}{\cos \Theta_1} \bar{x} \]
\[ \tilde{\rho}_3 = 1 - 2 \left( \cot \frac{2 \Theta_3}{\cos \Theta_1} \right) \tilde{x} - \frac{1}{(\cos \Theta_1)^2} x^2 \]  
(3.6)

\[ \tilde{x} \overset{\text{def}}{=} e^{-i\epsilon_1} x \]  
(3.7)

Note that there remain three physical parameters, \( \epsilon_1, \kappa_1 \) and \( x \), and the phase \( \epsilon_2 \) does not appear explicitly, since it is not an independent parameter through \( \tilde{\rho}_3 \), i.e. \( \arg(\tilde{\rho}_3) = -2(\epsilon_1 + \epsilon_2) \), as seen in Eq. (2.9).

Furthermore, setting \( \Theta_1 \) equals to 45°, the absolute values of Eqs. (3.4), (3.5), (3.6) are

\[ |\tilde{\rho}_1|^2 = 1 + 2\sqrt{2}(\tan \Theta_3)(\cos \epsilon_1)x + 2(\tan \Theta_3)^2 x^2 \]  
(3.8)

\[ |\tilde{\rho}_2|^2 = 1 - 2\sqrt{2}(\cot \Theta_3)(\cos \epsilon_1)x + 2(\cot \Theta_3)^2 x^2 \]  
(3.9)

\[ |\tilde{\rho}_3|^2 = 1 - 4\sqrt{2}(\cot 2\Theta_3)(\cos \epsilon_1)x + 4 \left( 2(\cot 2\Theta_3)^2 - \cos 2\epsilon_1 \right) x^2 \]

\[ + 8\sqrt{2}(\cot 2\Theta_3)(\cos \epsilon_1)x^3 + 4x^4 \]  
(3.10)

Setting \( \Theta_3 = 41.15^\circ \) and substituting these into \( |\tilde{\rho}_3|^2 - |\tilde{\rho}_2|^2 = \Delta^{-1}(|\tilde{\rho}_2|^2 - |\tilde{\rho}_2|^1) = 0 \), we can easily solve this equation analytically. The result is shown in Fig. 2.

![Figure 2](image_url)

**FIG. 2:** Allowed values of \( x \) for \( \epsilon_1 \) in case of \( \Delta m_{atm}^2 > 0 \).

Using above analytical solution, we can also estimate the absolute values of neutrino
masses, $|\tilde{\mu}_1|, |\tilde{\mu}_2|$ and $|\tilde{\mu}_3|$ by using the following equation,

$$|	ilde{\mu}_i| = \frac{\left|\tilde{\rho}_i\right|}{\sqrt{|\tilde{\rho}_2|^2 - |\tilde{\rho}_1|^2}} \times \sqrt{\Delta m^2_{sol}},$$  \hspace{1cm} (3.11)$$

and the obtained results are shown in Fig.3. We find that there exist allowed values of $|\tilde{\mu}_i|$ for any values of $\epsilon_1$ in case of $\Delta m^2_{atm} > 0$ while we cannot find any allowed values of $|\tilde{\mu}_i|$ in case of $\Delta m^2_{atm} < 0$. We can find that the result shown in Fig.3 has complicated structure in certain region of $\epsilon_1$, i.e. $1.39 < \epsilon_1 < 1.75$ (the shaded area corresponds to this region). This is because as seen in Fig.2 there are two or three solutions of $x$ for $\epsilon_1$, there are also multi solutions of $|\tilde{\mu}_i|$ for $\epsilon_1$ in this region. We discriminate these solutions to three parts in Fig.4.

Next, we deduce the Majorana mass responsible for the neutrinoless double beta decay experiments [19]. Defining $\text{arg}(\tilde{\rho}_i) = \alpha_i$ and Using Eq.(3.3), we can write $\mathcal{M}_\nu$ as

$$\mathcal{M}_\nu \propto V_{MNS}^* \begin{bmatrix} e^{i\alpha_1} |\tilde{\mu}_1| & 0 & 0 \\ 0 & e^{i\alpha_2} |\tilde{\mu}_2| & 0 \\ 0 & 0 & e^{i\alpha_3-2i\kappa_1} |\tilde{\mu}_3| \end{bmatrix} V_{MNS}^\dagger = U^* \begin{bmatrix} |\tilde{\mu}_1| & 0 & 0 \\ 0 & |\tilde{\mu}_2| & 0 \\ 0 & 0 & |\tilde{\mu}_3| \end{bmatrix} U^\dagger$$  \hspace{1cm} (3.12)$$

$$U = V_{MNS} \cdot \text{diag} \left(1, e^{\frac{1}{2}(\alpha_1-\alpha_2)}, e^{\frac{1}{2}(\alpha_1-\alpha_3+2\kappa_1)} \right),$$

FIG. 3: The allowed values of $|\tilde{\mu}_i|$ for $\epsilon_1$. 

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FIG. 4: The allowed values of $|\tilde{\mu}_i|$ for $\epsilon_1$ ($1.39 < \epsilon_1 < 1.75$). These figures corresponds to the shaded area in Fig. 3.

where we neglect the overall phase. The two remaining phases, $\alpha_1 - \alpha_2$ and $\alpha_1 - \alpha_3 + 2\kappa_1$, are known as "Majorana phases". The definition of Majorana mass responsible for the neutrinoless double beta decay experiments is

$$\left| m_{ee} \right| = \left| \sum_{i=1}^{3} |\tilde{\mu}_i|(U^*_{1i})^2 \right| = \left| |\tilde{\mu}_1| \cos^2 \theta_{12} \cos^2 \theta_{13} + |\tilde{\mu}_2| \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i(\alpha_2 - \alpha_1)} + |\tilde{\mu}_3| \sin^2 \theta_{13} e^{i(\alpha_3 - \alpha_1 - 2\kappa_1)} \right|. \tag{3.13}$$

In general, this quantity has two independent CP phases and especially for the case of normal mass spectrum, $|m_{ee}|$ can take zero accidentally in certain region of $|\tilde{\mu}_1|$, $1.88 \times 10^{-3}$ eV < $|\tilde{\mu}_1| < 5.97 \times 10^{-3}$ eV. We, however, already found in Fig. 2 that the phase $\epsilon_1$ depends on $x$ in this model and this means that one Majorana phase, $\alpha_1 - \alpha_2$, is a function of $|\tilde{\mu}_1|$ shown in Fig. 4. On the other hand, the other Majorana phase, $\alpha_3 - \alpha_1 - 2\kappa_1$, is completely independent phase reflecting the uncertainly of $\kappa_1$. Therefore, we expect that the uncertainly of $|m_{ee}|$ by these two phases are strongly suppressed. The numerical results of $|m_{ee}|$ are shown in Fig. 5. We find in this figure that the allowed region of $|m_{ee}|$ is suppressed considerably compared with the general case.

This figure can be understood as follows. In general, $|m_{ee}|$ can be written

$$|m_{ee}| = \left| A + Be^{i(\alpha_2 - \alpha_1)} + Ce^{i(\alpha_3 - \alpha_1)} \right| = \left| A - Be^{i(\alpha_2 - \alpha_1 + \pi)} + Ce^{i(\alpha_3 - \alpha_1)} \right|$$
FIG. 5: Majorana mass $|m_{ee}|$ in case of $\Delta m_{atm}^2 > 0$. The dotted curve lines correspond to the general two independent Majorana phases case and solid curve lines correspond to the numerical results of our model. Note that in our model there exists a lower limit of $|\tilde{\mu}_1|$, i.e. $|\tilde{\mu}_1| \geq 0.59 \times 10^{-2}$ eV (the vertical dashed line), as seen in Fig.3.

$$
A = |\tilde{\mu}_1| \times \cos^2 \theta_{12} \cos^2 \theta_{13}
$$

$$
B = \sqrt{|\tilde{\mu}_1|^2 + \Delta m_{sol}^2} \times \sin^2 \theta_{12} \cos^2 \theta_{13}
$$

$$
C = \sqrt{|\tilde{\mu}_1|^2 + \Delta m_{sol}^2 + \Delta m_{atm}^2} \times \sin^2 \theta_{13},
$$

and the schematic view of the relation between these quantities are shown in Fig.6. The necessary condition to minimize $|m_{ee}|$ is

$$
\alpha_1 - \alpha_2 = \arccos \left( \frac{C^2 - A^2 - B^2}{2AB} \right) \quad (1.88 \times 10^{-3} \text{eV} \leq |\tilde{\mu}_1| \leq 5.97 \times 10^{-3} \text{eV})
$$

$$
= \pi \quad \text{(else).} \quad (3.14)
$$

We, however, can find in Fig.7 that the maximal values of $\alpha_1 - \alpha_2$ are significantly smaller than the general case in most of the region.
IV. CONCLUSION

In this paper, we discuss the extra rotations induced by additional diagonalization of Majorana mass matrix and derive the absolute values of three neutrino masses and Majorana mass responsible for the neutrinoless double beta decay experiment only invoking to the seesaw mechanism collaborated by the unification of neutrino Dirac mass matrix with that of up-type quarks and the left-right symmetry based on $SO(10)$ GUT.

We specify these extra rotations to bi-maximal rotations around $x$- and $z$-axes and find that these extra rotations can explain the interesting and nontrivial relations between CKM and MNS matrices. In this analysis, we find the specific value of $\theta_{13}$, which does not conflict with the experimental data at $3\sigma$ C.L.

In Sec. II we ignore CP phases for simplicity and find that the absolute values of neutrino masses satisfy $m_1 : m_2 : m_3 \approx 1 : 2 : 8$, i.e. neutrino masses have hierarchical structure, and that there is no solution in the case of inverted mass spectrum.

In Sec. III we reanalyze the absolute values of neutrino masses and Majorana mass responsible for the neutrinoless double beta decay experiment, by including all CP phases. In this analyses, we find that only two CP phases, $\epsilon_1$ and $\kappa_1$, remain as independent degrees of freedom. The former phase has a physical meaning both in the analyses of the absolute values of neutrino masses and Majorana mass responsible for the neutrinoless double beta decay.
decay experiment, while the latter phase appears only in the analysis of Majorana mass responsible for the neutrinoless double beta decay experiment. In the analysis of the absolute values of neutrino masses, we cannot decide them uniquely but find these quantities have well-defined lower bounds though we cannot find any solutions in the case of inverted mass spectrum. In the analysis of Majorana mass responsible for the neutrinoless double beta decay experiment, we find that one Majorana phase is a function of $|\tilde{\mu}_1|$ and that this reduces the allowed region of $|m_{ee}|$ considerably.

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[17] C. Albright et. al., hep-ex/0008064