Universal destabilization and slowing of spin transfer functions by a bath of spins

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We investigate the effect of a spin bath on the spin transfer functions of a permanently coupled spin system. When each spin is coupled to a separate environment, the effect on the transfer functions in the first excitation sector is amazingly simple: the group velocity is slowed down by a factor of two, and the fidelity is destabilized by a modulation of $|\cos Gt|$, where $G$ is the mean square coupling to the environment.

**Introduction:** Recently suggested protocols\(^1\)\(^2\)\(^3\) give a new perspective to the physics of strongly coupled spin systems. They demonstrate that the coherent transfer of spin flips can be used to transfer unknown quantum states and entanglement, a task of paramount importance in any quantum information application\(^4\)\(^5\). Generally, the relevant quantities determining the performance of the mentioned protocols are the time dependent transition amplitudes of local spin flips in a ferromagnetic ground state. We will refer to these amplitudes as “spin transfer functions”. The same functions also occur in the charge and energy transfer dynamics in molecular systems\(^6\)\(^7\) and in continuous time random walks\(^8\) to which our results equally apply.

It is both important and interesting to ask how these transfer functions change if the intended couplings between the spins are accompanied by unwanted couplings to environmental spins which do not take part in the transport. It is well known from the theory of open quantum systems\(^9\)\(^10\) that this can lead to dissipation and decoherence, which also means that quantum information is lost. Here we consider a model where the system is coupled to a spin environment through an exchange interaction because the same type of coupling is also responsible for the transport of the information through the system. Moreover, this coupling offers the unique opportunity of an analytic solution of our problem without any approximations regrading the strength of system-environment coupling (in most treatments of the effect of an environment on the evolution of a quantum system, the system-environment coupling is assumed to be weak) and allows us to include inhomogeneous interactions of the bath spins with the system. For such coupling, decoherence is possible for mixed (thermal) initial bath states\(^11\). However if the system and bath are both initially cooled to their ground states, is there still a non-trivial effect of the environment on the spin transfer functions? In this paper we find that there are two important effects: the spin transfer functions are slowed and a destabilized due to the environment. This has both positive and negative implications for the use of strongly coupled spin systems as quantum communication channels.

**Model:** We choose to start with a specific spin system, i.e. an open spin chain of arbitrary length $N$, with a Hamiltonian given by

$$H_S = -\frac{1}{2} \sum_{\ell=1}^{N-1} J_{\ell} (X_{\ell}X_{\ell+1} + Y_{\ell}Y_{\ell+1}), \quad (1)$$

where $J_{\ell}$ are some arbitrary couplings and $X_{\ell}$ and $Y_{\ell}$ are the Pauli-X and Y matrices for the $\ell$th spin. Towards the end of the paper we will however show that our results hold for any system where the number of excitations is conserved during dynamical evolution. In addition to the chain Hamiltonian, each spin $\ell$ of the chain interacts with an independent bath of $M_{\ell}$ environmental spins (see Fig. 1) via an inhomogeneous Hamiltonian,

$$H_{I,\ell} (\ell) = -\frac{1}{2} \sum_{k=1}^{M_{\ell}} g^{(\ell)}_{k} (X_{\ell}X^{(\ell)}_{k} + Y_{\ell}Y^{(\ell)}_{k}). \quad (2)$$

**Figure 1:** A spin chain of length $N = 5$ coupled to independent baths of spins.

In the above expression, the Pauli matrices $X_{\ell}$ and $Y_{\ell}$ act on the $\ell$th spin of the chain, whereas $X^{(\ell)}_{k}$ and $Y^{(\ell)}_{k}$ act on the $k$th environmental spin attached to the $\ell$th spin of the chain. We denote the total interaction Hamiltonian by

$$H_I \equiv \sum_{\ell=1}^{N} H_{I,\ell} (\ell). \quad (3)$$

The total Hamiltonian is given by $H = H_S + H_I$, where it is important to note that $[H_S, H_I] \neq 0$. The ground state of the system is given by the fully polarized state $|0,0\rangle$, with all chain and bath spins aligned along the $z$-axis. The above Hamiltonian describes an extremely complex and disordered system with a Hilbert space of dimension $2^{N+NM}$. In the context of state transfer however, only the dynamics of the first excitation sector is relevant. We proceed by mapping this sector to a much simpler system\(^9\)\(^10\). For $\ell = 1, 2, \ldots, N$ we define the states

$$|\ell, 0\rangle \equiv \sigma^{\dagger}_{\ell} |0,0\rangle \quad (4)$$

This allows us to include inhomogeneous interactions of the bath spins with the system.
\begin{align}
|0, \ell\rangle & \equiv \frac{1}{G_\ell} \sum_{k=1}^{M_\ell} g_k^{(\ell)} \sigma_k^{+}(\ell) |0, 0\rangle \\
G_\ell & = \sqrt{\frac{M_\ell}{\sum_{k=1}^{M_\ell} (g_k^{(\ell)})^2}}.
\end{align}

It is easily verified that
\begin{align}
H_S |\ell, 0\rangle & = -J(1 - \delta_{\ell 1}) |\ell - 1, 0\rangle - J(1 - \delta_{\ell N}) |\ell + 1, 0\rangle \\
H_S |0, \ell\rangle & = 0,
\end{align}

and
\begin{align}
H_I |\ell, 0\rangle & = -G_\ell |0, \ell\rangle \\
H_I |0, \ell\rangle & = -G_\ell |\ell, 0\rangle.
\end{align}

Hence these states define a $2N$-dimensional subspace that is invariant under the action of $H$. This subspace is equivalent to the first excitation sector of a system of $2N$ spin 1/2 particles, coupled as it is shown in Fig. 2. Our main assumption is that the bath couplings are in effect the same, i.e. $G_\ell = G$ for all $\ell$. Note however that the individual number of bath spins $M_\ell$ and bath couplings $g_k^{(\ell)}$ may still depend on $\ell$ and $k$ as long as their means square average is the same. Also, our analytic solution given in the next paragraph relies on this assumption, but numerics show that our main result [Equation (11)] remains a good approximation if the $G_\ell$ slightly vary and we take $G \equiv \langle G_\ell \rangle$.

\textbf{Results:}—In this paragraph, we solve the Schrödinger equation for the model outlined above and discuss the spin transfer functions. Firstly, let us denote the orthonormal eigenstates of $H_S$ alone by
\begin{align}
H_S |\psi_k\rangle = \epsilon_k |\psi_k\rangle & \quad (k = 1, 2, \ldots, N)
\end{align}

with
\begin{align}
|\psi_k\rangle = \sum_{\ell=1}^{N} a_{k\ell} |x, 0\rangle.
\end{align}

For what follows, it is not important whether analytic expressions for the eigensystem of $H_S$ can be found. Our result holds even for models that are not analytically solvable, such as the randomly coupled chains considered in [2]. We now make an ansatz for the eigenstates of the full Hamiltonian, motivated by the fact that the states
\begin{align}
|\phi_k^n\rangle & \equiv \frac{1}{\sqrt{2}} \left(|\ell, 0\rangle + (-1)^n |0, \ell\rangle\right) \quad (n = 1, 2)
\end{align}

are eigenstates of $H_I^{(\ell)}$ with the corresponding eigenvalues $\pm G$ [this follows directly from Eq. (8)]. Define the vectors
\begin{align}
|\Psi_k^n\rangle & \equiv \sum_{\ell=1}^{N} a_{k\ell} |\phi_k^n\rangle
\end{align}

with $k = 1, 2, \ldots, N$ and $n = 0, 1$. The $|\Psi_k^n\rangle$ form an orthonormal basis in which we express the matrix elements of the Hamiltonian. We can easily see that
\begin{align}
H_I |\Psi_k^n\rangle & = -(-1)^n G |\Psi_k^n\rangle
\end{align}

and
\begin{align}
H_S |\Psi_k^n\rangle & = \epsilon_k \sum_{x=1}^{N} a_{kx} |x, 0\rangle = \frac{\epsilon_k}{2} \left(|\Psi_k^0\rangle + |\Psi_k^1\rangle\right).
\end{align}

Therefore the matrix elements of the full Hamiltonian $H = H_S + H_I$ are given by
\begin{align}
\langle \Psi_k^n | H | \Psi_k^n \rangle & = \delta_{kk'} \left(-(-1)^n G \delta_{nn'} + \frac{\epsilon_k}{2}\right).
\end{align}

The Hamiltonian is not diagonal in the states of Eq. (13). But $H$ is now block diagonal consisting of $N$ blocks of size 2, which can be easily diagonalized analytically. The orthonormal eigenstates of the Hamiltonian are given by
\begin{align}
|E_k^n\rangle & = c_k^{-1} \left\{((-1)^n \Delta_k - 2G) |\Psi_k^0\rangle + \epsilon_k |\Psi_k^1\rangle\right\}
\end{align}

with the eigenvalues
\begin{align}
E_k^n & = \frac{1}{2} \left(\epsilon_k + (-1)^n \Delta_k\right)
\end{align}

and the normalization
\begin{align}
c_{kn} & \equiv \sqrt{((-1)^n \Delta_k - 2G)^2 + \epsilon_k^2},
\end{align}

where
\begin{align}
\Delta_k & = \sqrt{4G^2 + \epsilon_k^2}.
\end{align}

Note that the ansatz of Eq. (13) that put $H$ in block diagonal form did not depend on the details of $H_S$ and $H_I^{(\ell)}$. The methods presented here can be applied to a much larger class of systems, including the generalized spin star systems (which include an interaction within the bath) discussed in [10].
After solving the Schrödinger equation, let us now turn to quantum state transfer. The relevant quantity is given by the transfer function

\[ f_{N,1}(t) = \langle N,0 \vert \exp \{-iHt\} \vert 1,0 \rangle = \sum_{k,n} \exp \{-iE_k^n t\} \langle E_{k}^n 1,0 \rangle \langle N,0\vert E_k^n \rangle. \]

The modulus of \( f_{N,1}(t) \) is between 0 (no transfer) and 1 (perfect transfer) and fully determines the fidelity of state transfer. Since

\[ \langle \ell,0\vert E_k^n \rangle = c_{kn}^{-1} \{((-1)^n \Delta_k - 2G) \langle \ell,0\vert \Psi_k^0 \rangle + \epsilon_k \langle \ell,0\vert \Psi_k^1 \rangle \} = \frac{c_{kn}^{-1}}{\sqrt{2}} ((-1)^n \Delta_k - 2G + \epsilon_k) a_{k\ell} \]

we get

\[ f_{N,1}(t) = \frac{1}{2} \sum_{k,n} e^{-\frac{i}{\hbar} (\epsilon_k + (-1)^n \Delta_k) ((-1)^n \Delta_k - 2G + \epsilon_k)^2} \rangle \langle \ell,0\vert \Psi_k^0 \rangle + \epsilon_k \langle \ell,0\vert \Psi_k^1 \rangle \rangle \frac{a_{k1} a_{kN}^*}{\epsilon_k}. \]

Eq. (21) is the main result of this article, fully determining the transfer of quantum information and entanglement in the presence of the environments. In the limit \( G \to 0 \), we have \( \Delta_k \approx \epsilon_k \) and \( f_{N,1}(t) \) approaches the usual result [1, 2, 3] without an environment,

\[ f_{0N,1}(t) = \sum_{k} \exp \{-i\epsilon_k t\} a_{k1} a_{kN}^*. \] (22)

In fact, a series expansion of Eq. (21) yields that the first modification of the transfer function is of the order of \( G^2 \),

\[ G^2 \sum_{k} \exp \{-i\epsilon_k t\} \left[ \exp \left\{ -i\epsilon_k t \right\} \left( -\frac{1}{\epsilon_k} + \frac{1}{\epsilon_k} \right) \right]. \]

Hence we the effect is small for very weakly coupled baths. However, as the chains get longer, the lowest lying energy \( \epsilon_1 \) usually approaches zero, so the changes become more significant (scaling as \( 1/\epsilon_k \)). For intermediate \( G \), we evaluated Eq. (21) numerically and found that the first peak of the transfer function generally becomes slightly lower, and gets shifted to higher times (Figures 2 and 3). A numeric search in the coupling space \( \{J_\ell, \ell = 1, \ldots, N-1\} \) however also revealed some rare examples where an environment can also slightly improve the peak of the transfer function (Fig 3). In the strong coupling regime \( G \gg \epsilon_k/2 \), we can approximate Eq. (21) by \( \Delta_k \approx 2G \). Inserting it in Eq. (21) then becomes

\[ f_{N,1}(t) \approx \frac{1}{2} e^{-igt} \sum_{k} \exp \left\{ -i\epsilon_k t \right\} a_{k1} a_{kN}^* + \]

\[ + \frac{1}{2} e^{igt} \sum_{k} \exp \left\{ -i\epsilon_k t \right\} a_{k1} a_{kN}^* \]

\[ = \cos(Gt) f_{0N,1}(t/2). \]

This surprisingly simple result consists of the normal transfer function, slowed down by a factor of 1/2, and modulated by a quickly oscillating term (Figures 3 and 4). Our derivation actually did not depend on the indexes of \( f(t) \) and we get for the transfer from the \( n \)th to the \( m \)th spin of the chain that

\[ f_{n,m}(t) \approx \cos(Gt) f_{n,m}(t/2). \] (25)

It may look surprising that the matrix \( f_{n,m} \) is no longer unitary. This is because we are considering the dynamics
Figure 5: A weakly coupled bath may even improve the transfer function for some specific choices of the $J_k$. This plot shows the transfer function $|f_{N,1}(t)|$ for $N = 10$. The couplings $J_k$ were found numerically.

of the chain only, which is an open quantum system. A heuristic interpretation of Eq. (25) is that the excitation oscillates back and forth between the chain and the bath (hence the modulation), and spends half of the time trapped in the bath (hence the slowing). If the time of the maximum of the transfer function $|f_{n,m}(t)|$ for $G = 0$ is a multiple of $\pi/2G$ then this maximum is also reached in the presence of the bath.

Finally, we want to stress that Eq. (25) is universal for any spin Hamiltonian that conserves the number of excitations, i.e. with $[H_S, \sum_{\ell} Z_{\ell}] = 0$. Thus our restriction to chain-like topology and exchange couplings for $H_S$ is not necessary. In fact the only difference in the whole derivation of Eq. (25) for a more general Hamiltonian is that Eq. (7) is replaced by

$$H_S(\ell, 0) = \sum_{\ell'} h_{\ell'}|\ell', 0\rangle.$$  

The Hamiltonian can still be formally diagonalized in the first excitation sector as in Eq. (11), and the states of Eq. (17) will still diagonalize the total Hamiltonian $H_S + H_I$. Also, rather than considering an exchange Hamiltonian for the interaction with the bath, we could have considered a Heisenberg interaction, but only for the special case where all bath couplings $g_k^{(\ell)}$ are all the same [11]. Up to some irrelevant phases, this leads to the same results as for the exchange interaction.

Conclusion:— We found a surprisingly simple and universal scaling law for the spin transfer functions in the presence of spin environments. In the context of quantum state transfer [1,2,3], this result is double-edged: on one hand, it shows that even for very strongly coupled baths quantum state transfer is possible, with the same fidelity and only reasonable slowing. On the other hand, it also shows that the fidelity as a function of time becomes destabilized with a quickly oscillating modulation factor. In practice, this factor will restrict the time-scale in which one has to be able read the state from the system. This demonstrates that even though a bath coupling need not introduce decoherence or dissipation to the system, there are other dynamical processes such as destabilization it may cause that can be problematic for quantum information processing.

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